ORIGINAL PAPER

ON OBLIQUE WAVE SOLUTIONS OF SOME SPACE-TIME FRACTIONAL MODIFIED KDV EQUATIONS

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Manuscript received: 16.06.2021; Accepted paper: 22.10.2021;
Published online: 30.12.2021.

Abstract. *The modified Kudryashov approach along with the conformable derivative is used to find a variety of askew wave solutions, with some free parameters, of the space-time fractional modified KdV equations. We study the wave solutions of the aforesaid mKdV equations that are obliquely propagated to consider the behaviour of physical issues in water waves and other fluids. The graphical depiction of these solutions is given via Mathematica for better understanding. Moreover, apart from the physical implication, these solutions may be helpful for an upgraded understanding of numerical solvers to compare the accuracy of their results and performances of wave dynamics as observed in science and engineering.*

Keywords: *conformable derivatives; space-time-fractional mKdV equations; modified Kudryashov approach; oblique wave solutions.*

1. INTRODUCTION

The dynamics of shallow water waves in various places like sea beaches are governed by the KdV and Boussinesq Equations. The idea of a soliton for the Korteweg-de Varies equation in water wave dynamics has been provided in [1-4]. The Korteweg-de Vries (KdV), Boussinesq, Kadomtsev-Petviashvili, and Whitham-Broer-Kaup (WBK) equations are the well established integrable models that describe the propagation of shallow water [5-7]. It is well stimulating to discuss the topic of solitary waves on water. The KdV equation has also an impingement in modelling blood pressure pulses [8-10]. Likewise, Wazwaz [11] announced the nonlinear $(3+1)$-dimensional modified KdV equations and scrutinize their exact soliton and kink solutions. Specifically, in [12-13], the soliton solutions for the following conformable fractional $(3+1)$-dimensional modified KdV equations are explored.

$$D_t^{\eta} v + 6 D_x^{\eta} v^3 + D_{xyz}^{3\eta} v = 0, \ \ 0 < \eta \le 1, \tag{1}$$

$$D_t^{\eta} v + 6 D_y^{\eta} v^3 + D_{xyz}^{3\eta} v = 0, \ \ 0 < \eta \le 1, \tag{2}$$

$$D_t^{\eta} v + 6 D_z^{\eta} v^3 + D_{xyz}^{3\eta} v = 0, \ \ 0 < \eta \le 1. \tag{3}$$

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There are various mathematical approaches to evaluate the analytical solutions of nonlinear evolution equations (NLEEs) for studying the solitary waves not only on water but also on plasmas, optical fibers, stochastic processes etc. [14-20]. Some of the commonly used approaches are studied in [21-34]. Furthermore, the generalized Kudryashov method [35-37], a modified form of Kudryashov and functional variable methods have been employed for solving discrete and fractional order partial differential equations (PDEs) as well [38-41]. Moreover, the exact solitary wave solutions to PDEs and FPDEs have also been found in [42-46]. In particular, authors have only focused on the solutions to the nonlinear FPDEs without obliqueness. The oblique wave propagation can be debated, in any varied instances, a wave that is inclined to the flow direction. The generalized $exp(-\Phi(\xi))$-expansion method [47] has been investigated for finding the oblique wave solutions to the conformable time-fractional extended Zakharov-Kuznetsov equation. Also, the modified Kudryashov approach has been utilized to find oblique closed-form solutions for some fractional order evolution equations [48]. Being prompt driven from the above facts and according to our best knowledge that no one has been employed the modified Kudryashov method for constructing the obliquely propagating wave solutions of space-time conformable fractional (3+1)-dimensional mKdV equations. In recent years, many evaluations in fractional order derivatives, Like Caputo, Hilfer, Riemann-Liouville, form and so on, have been made in the literature [49-52]. The well-known product, quotient and chain rules were the setbacks of one definition or another. Therefore the most engrossing definition of the fractional derivative with some of its properties are given in [53-54].

The conformable derivative of order $\mu \in (0,1]$ can be defined as

$$u_t^{(\mu)}(x,t) = \lim_{\varepsilon \to 0} \frac{u(x,t+\varepsilon t^{1-\mu}) - u(x,t)}{\varepsilon},\ u : \mathbb{R} \times (0,\infty) \to \mathbb{R}, \text{for all } t > 0\,. \tag{4}$$

This paper focuses to explore the conformable space-time fractional modified KdV equations of $(3+1)$-dimension for oblique wave exact solutions via the modified Kudryashov approach using conformable derivative by introducing a new travelling wave variable.

2. MAIN STEP FOR MODIFIED KUDRYASHOV APPROACH

Let us consider a nonlinear conformable space-time FDE that can be accessible in the form

$$\Omega(v, D_t^{\eta} v, D_x^{\eta} v, D_{tt}^{2\eta} v, D_{xx}^{2\eta} v, ...) = 0\,. \tag{5}$$

This FDE (5) can be reformulated into the following nonlinear ODE of integer order

$$\Lambda(f, f', f'', ...,) = 0\,, \tag{6}$$

with the use of the following wave variable and some properties of conformable derivative [50-51].

$$v(x,t) = f(\epsilon), \epsilon = k\frac{x^{\eta}}{\eta} + q\frac{y^{\eta}}{\eta} + m\frac{z^{\eta}}{\eta} - l\frac{t^{\eta}}{\eta}, k^2 + q^2 + m^2 = 1, \tag{7}$$

where l is nonzero arbitrary constant and the prime reveals the conformable differentiation with respect to ϵ. The current section provides a transient elucidation of the modified Kudryashov approach [25-26] that fabricates the new oblique wave exact solutions of conformable space-time fractional modified KdV equations. Consider the following solution for Eq. (6) that can be expressed as a finite series of the form

$$f(\epsilon) = \sum_{n=0}^{N} a_n Y^n(\epsilon), \tag{8}$$

where a_n, $n = 0,1,...,N(a_N \neq 0)$ are unknowns to be calculated, and $Y(\epsilon)$ satisfies a first-order nonlinear equation as

$$Y'(\epsilon) = Y(\epsilon)(Y(\epsilon)-1)\ln(a); a \neq 0,1. \tag{9}$$

It is obvious that Eq. (9) has a solution of the form:

$$Y(\epsilon) = \frac{1}{1+da^{\epsilon}}. \tag{10}$$

It should be noted that the positive integer N in Eq. (8) is computed by the use of the homogeneous balance principle. Inserting Eq. (8) and its necessary derivatives in Eq. (6) gives

$$P(Y(\epsilon)) = 0, \tag{11}$$

a polynomial in $Y(\epsilon)$. Considering the coefficients of each power $Y(\epsilon)$ in Eq. (11) are to be zero, we will extend to a system of nonlinear algebraic equations whose solution generates solutions for the original Eq. (5).

3. THE SPACE-TIME FRACTIONAL MKDV EQUATIONS AND THEIR OBLIQUE WAVE SOLUTIONS

In this section, we look for the oblique wave exact solutions of conformable fractional $(3+1)$-dimensional modified KdV Eqs. (1)-(3) via modified Kudryashov approach [35-37]. Consider the Eq. (1) and use the transformation

$$v(x,y,z,t) = f(\epsilon), \epsilon = \cos(\gamma)\sin(\delta)\frac{x^{\eta}}{\eta} + \sin(\gamma)\sin(\delta)\frac{y^{\eta}}{\eta} + \cos(\delta)\frac{z^{\eta}}{\eta} - l\frac{t^{\eta}}{\eta}, \tag{12}$$

where $(\cos(\gamma)\sin(\delta))^2 + (\sin(\gamma)\sin(\delta))^2 + \cos(\delta)^2 = 1$ ana we are left with

$$-lf' + (\cos(\gamma)\sin(\delta))(\sin(\gamma)\sin(\delta))\cos(\delta) f''' + 6(\cos(\gamma)\sin(\delta))(f^3)' = 0. \tag{13}$$

New integrating (13) once w.r.t. ϵ and taking zero constant of integration, we get

$$-lf + (\cos(\gamma)\sin(\delta))(\sin(\gamma)\sin(\delta))\cos(\delta) f'' + 6(\cos(\gamma)\sin(\delta)) f^3 = 0. \tag{14}$$

The homogeneous balance between f'' and f^3 gives $N=1$, then the nontrivial solution (8) reduces to:

$$f(\epsilon) = a_0 + a_1 Y(\epsilon)\,. \tag{15}$$

By implanting the solution (15) in reduced equation Eq. (14) and equating the coefficients of each $Y(\epsilon)$ to zero, we procure a set of nonlinear algebraic equations

$$6a_0^3\cos(\gamma)\sin(\delta) - a_0 l = 0,$$

$$18a_0^2 a_1\cos(\gamma)\sin(\delta) + a_1\ln(a)^2\sin(\gamma)\cos(\gamma)\sin^2(\delta)\cos(\delta) - a_1 l = 0,$$

$$18a_0 a_1^2\cos(\gamma)\sin(\delta) - 3a_1\ln(a)^2\sin(\gamma)\cos(\gamma)\sin^2(\delta)\cos(\delta) = 0,$$

$$6a_1^3\cos(\gamma)\sin(\delta) + 2a_1\ln(a)^2\sin(\gamma)\cos(\gamma)\sin^2(\delta)\cos(\delta) = 0,$$

and its solution yields the following sets:

$$Set-1: \quad a_0 = -\frac{i\ln(a)\sqrt{\sin(\gamma)\sin(2\delta)}}{2\sqrt{6}}, a_1 = \frac{i\ln(a)\sqrt{\sin(\gamma)\sin(2\delta)}}{\sqrt{6}}, \quad l = -\frac{1}{4}\ln(a)^2\sin(2\gamma)\sin^2(\delta)\cos(\delta)$$

$$Set-2: \quad a_0 = \frac{i\ln(a)\sqrt{\sin(\gamma)\sin(2\delta)}}{2\sqrt{6}}, a_1 = -\frac{i\ln(a)\sqrt{\sin(\gamma)\sin(2\delta)}}{\sqrt{6}}, \quad l = -\frac{1}{4}\ln(a)^2\sin(2\gamma)\sin^2(\delta)\cos(\delta)$$

These sets give the following new exact oblique wave solutions:

from set -1.

$$f_1(\epsilon) = \frac{i\ln(a)\sqrt{\sin(\gamma)\sin(2\delta)}}{\sqrt{6}\left(da^{\epsilon}+1\right)} - \frac{i\ln(a)\sqrt{\sin(\gamma)\sin(2\delta)}}{2\sqrt{6}}, \tag{16}$$

from set -2,. $$f_2(\epsilon) = \frac{i\ln(a)\sqrt{\sin(\gamma)\sin(2\delta)}}{2\sqrt{6}} - \frac{i\ln(a)\sqrt{\sin(\gamma)\sin(2\delta)}}{\sqrt{6}\left(da^{\epsilon}+1\right)}, \tag{17}$$

where $\epsilon = \cos(\gamma)\sin(\delta)\frac{x^{\eta}}{\eta} + \sin(\gamma)\sin(\delta)\frac{y^{\eta}}{\eta} + \cos(\delta)\frac{z^{\eta}}{\eta} + \frac{1}{4}\ln(a)^2\sin(2\gamma)\sin^2(\delta)\cos(\delta)\frac{t^{\eta}}{\eta}$.

The obtained solutions of Eq. (1), given in Eq. (16), are graphed here for different η-values. For the sake of simplicity, we take $\gamma = \pi/3$, $\delta = \pi/6$, $a=4$, and $z=0, y=1$ for $3D$ -graphs appear in Figs. 1-3.

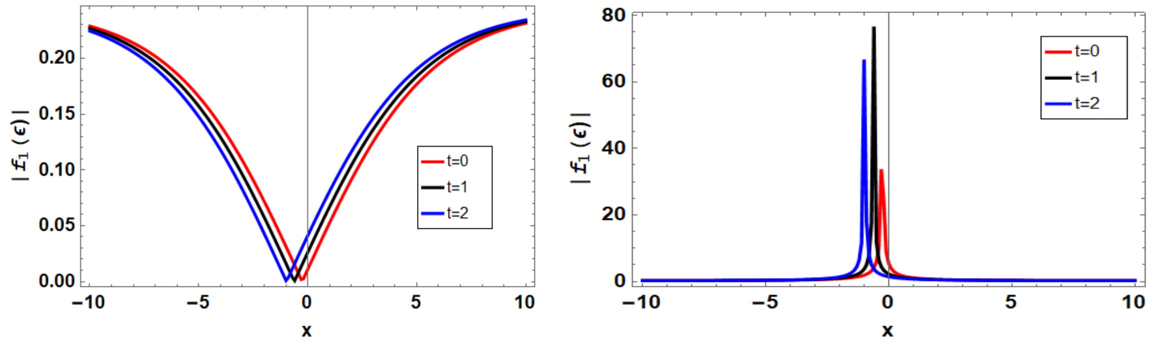

Figure 1. 3*D* oblique wave profile of f_1 appears in Eq. (16) is given corresponding (a) to *d*=0.6 and (b) to *d*=-0.6 for fixed *η*=1.

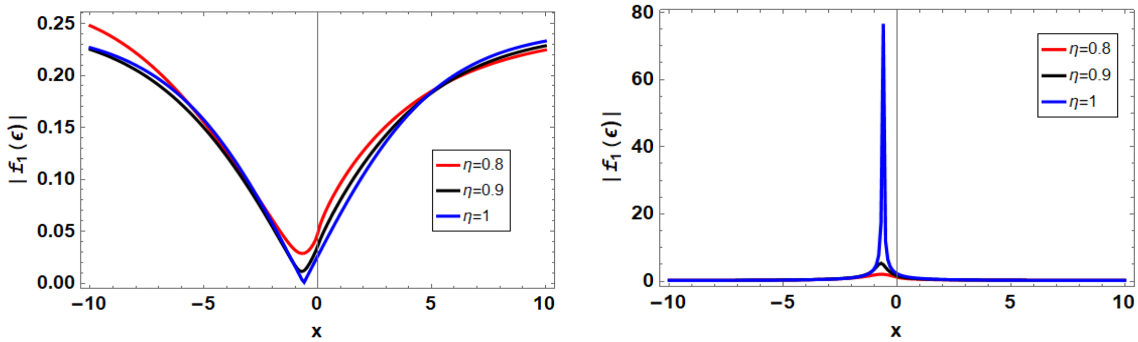

Figure 2. 3*D* oblique wave profile of f_1 appears in Eq. (16) is given corresponding (a) to *d*=0.6 and (b) to *d*=-0.6 for fixed *y*=1.

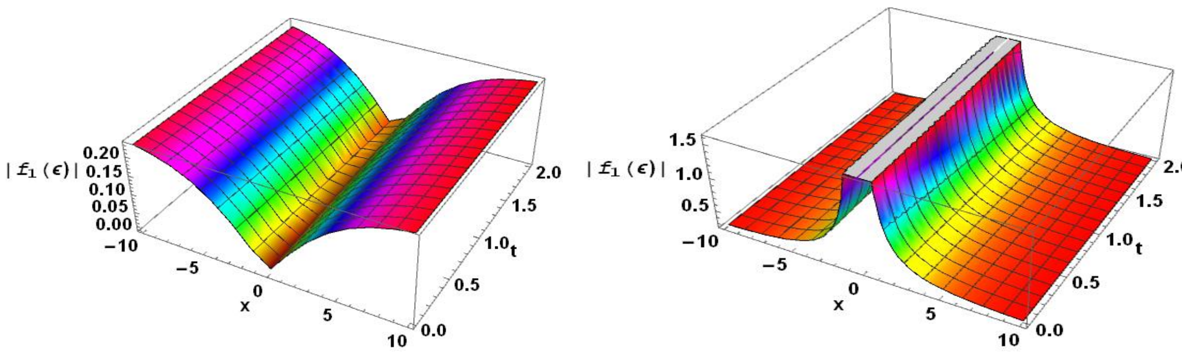

Figure 3. 3*D* oblique wave profile of f_1 appears in Eq. (16) is given corresponding (a) to *d*=0.6 and (b) to *d*=-0.6 for fixed *η*=1.

Consider now the Eq. (2) along with the transformation (12), and integrating the obtained equation once w.r.t. ϵ by taking zero constant of integration, we approach

$$-lf + (\cos(\gamma)\sin(\delta))(\sin(\gamma)\sin(\delta))\cos(\delta) f'' + 6(\sin(\gamma)\sin(\delta)) f^3 = 0. \tag{18}$$

It is noted that if we replace $\sin(\gamma)$ within Eqs. (16)-(17), we obtain the oblique wave solution of Eq. (2) as follows.

$$f_3(\epsilon) = \frac{i\ln(a)\sqrt{\cos(\gamma)\sin(2\delta)}}{\sqrt{6}\left(da^{\epsilon}+1\right)} - \frac{i\ln(a)\sqrt{\cos(\gamma)\sin(2\delta)}}{2\sqrt{6}}, \tag{19}$$

$$f_4(\epsilon)=\frac{i\ln(a)\sqrt{\cos(\gamma)\sin(2\delta)}}{2\sqrt{6}}-\frac{i\ln(a)\sqrt{\cos(\gamma)\sin(2\delta)}}{\sqrt{6}\left(da^{\epsilon}+1\right)}, \tag{20}$$

where

$$\epsilon=\cos(\gamma)\sin(\delta)\frac{x^{\eta}}{\eta}+\sin(\gamma)\sin(\delta)\frac{y^{\eta}}{\eta}+\cos(\delta)\frac{z^{\eta}}{\eta}+\frac{1}{2}\ln(a)^2\sin(\gamma)\cos(\gamma)\sin^2(\delta)\cos(\delta)\frac{t^{\eta}}{\eta}$$

The obtained solutions of Eq. (2) are graphed here for different η-values and d-values. Figs. 4-6 reveal the solution f_3 given in Eq. (19) of Eq. (2) for different η and y values.

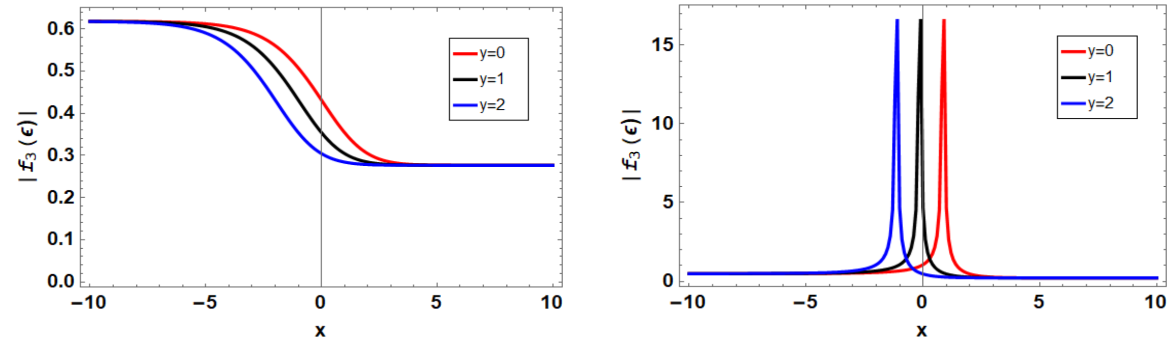

Figure 4. 3D oblique wave profile of f_3 appears in Eq. (19) is given corresponding (a) to d=1 and (b) to d=-1. Other values are γ = δ = π/4, z = -1, t = 2 and η = 1.

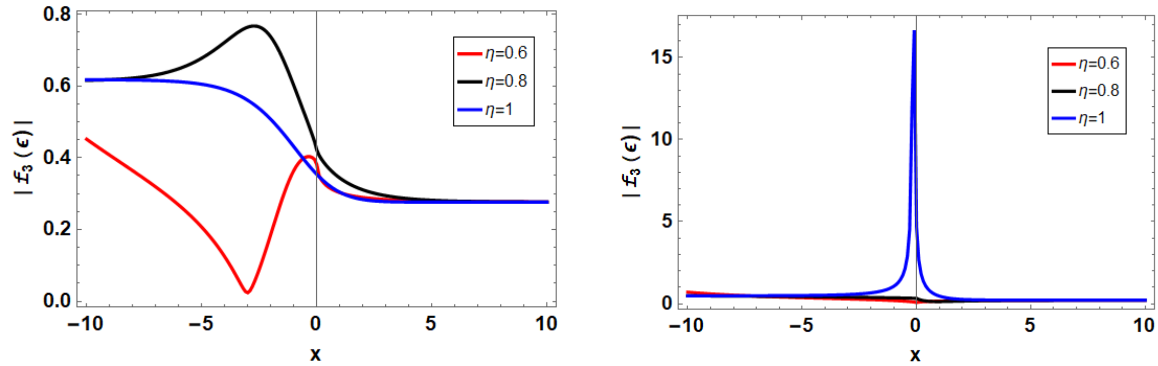

Figure 5. 3D oblique wave profile of f_3 appears in Eq. (19) is given corresponding (a) to d=1 and (b) to d=-1. Other values are γ = δ = π/4, t = 2 and z = -1.

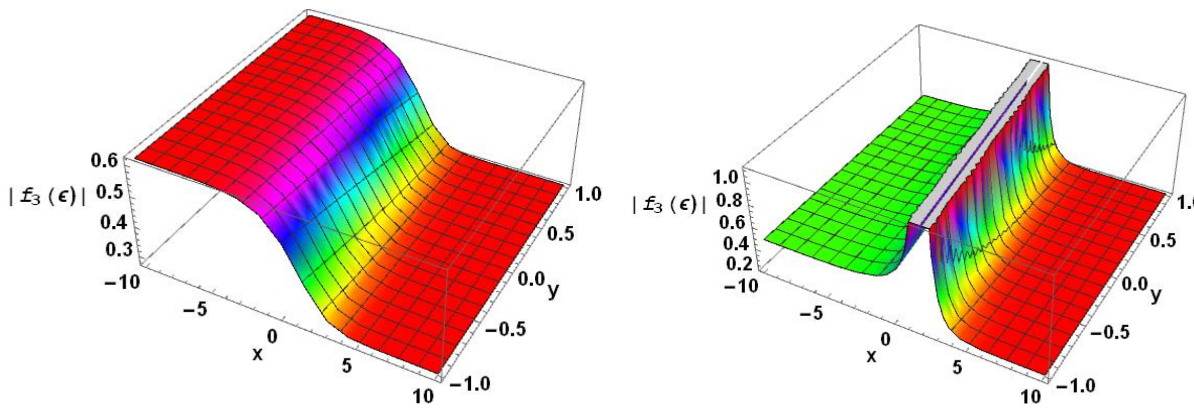

Figure 6. 3D oblique wave profile of f_3 appears in Eq. (19) is given corresponding (a) to d=1 and (b) to d=-1. Other values are γ = δ = π/4, z = -1, t = 2 and η = 1.

3.1. THE OBLIQUE WAVE EXACT SOLUTIONS OF THE EQ. (3)

In this section, we look for the solutions of $(3+1)$-dimension mKdV Eq. (3) via the modified Kudryashov approach. Consider the Eq. (3) and using the transformation (12), we are with the following form:

$$-lf' + (\cos(\gamma)\sin(\delta))(\sin(\gamma)\sin(\delta))\cos(\delta) f''' + 6\cos(\delta)(f^3)' = 0. \tag{21}$$

New integrating (21) once w.r.t. ϵ and taking zero constant of integration, we get

$$-lf + (\cos(\gamma)\sin(\delta))(\sin(\gamma)\sin(\delta))\cos(\delta) f'' + 6\cos(\delta) f^3 = 0. \tag{22}$$

The homogeneous balance between f'' and f^3 gives $N=1$, then the nontrivial solution (8) reduces to:

$$f(\epsilon) = a_0 + a_1 Y(\epsilon). \tag{23}$$

By implanting the solution (15) in reduced equation Eq. (22) and equating the coefficients of each $Y(\epsilon)$ to zero, we procure a set of nonlinear algebraic equations

$$6a_0^3\cos(\delta) - a_0 l = 0,$$

$$a_1\ln(a)^2\sin(\gamma)\cos(\gamma)\sin^2(\delta)\cos(\delta) - a_1 l + 18a_0^2 a_1\cos(\delta) = 0,$$

$$18a_0 a_1^2\cos(\delta) - 3a_1\ln(a)^2\sin(\gamma)\cos(\gamma)\sin^2(\delta)\cos(\delta) = 0,$$

$$2a_1\ln(a)^2\sin(\gamma)\cos(\gamma)\sin^2(\delta)\cos(\delta) + 6a_1^3\cos(\delta) = 0,$$

and its solution yields the following sets:

$$Set-1: \begin{aligned} a_0 &= -\frac{i\ln(a)\sqrt{\sin(2\gamma)}\sin(\delta)}{2\sqrt{6}}, a_1 = \frac{i\ln(a)\sqrt{\sin(2\gamma)}\sin(\delta)}{\sqrt{6}}, \\ l &= -\frac{1}{4}\ln(a)^2\sin(2\gamma)\sin^2(\delta)\cos(\delta). \end{aligned}$$

$$Set-2: \begin{aligned} a_0 &= \frac{i\ln(a)\sqrt{\sin(2\gamma)}\sin(\delta)}{2\sqrt{6}}, a_1 = -\frac{i\ln(a)\sqrt{\sin(2\gamma)}\sin(\delta)}{\sqrt{6}}, \\ l &= -\frac{1}{4}\ln(a)^2\sin(2\gamma)\sin^2(\delta)\cos(\delta). \end{aligned}$$

These sets give the following new exact oblique wave solutions:

$$from\,set-1\;\; f_5(\epsilon)=\frac{i\ln(a)\sin(\delta)\sqrt{\sin(2\gamma)}}{\sqrt{6}\left(da^{\epsilon}+1\right)}-\frac{i\ln(a)\sin(\delta)\sqrt{\sin(2\gamma)}}{2\sqrt{6}}. \tag{24}$$

$$from\,set-2\;\; f_6(\epsilon)=\frac{i\ln(a)\sin(\delta)\sqrt{\sin(2\gamma)}}{2\sqrt{6}}-\frac{i\ln(a)\sin(\delta)\sqrt{\sin(2\gamma)}}{\sqrt{6}\left(da^{\epsilon}+1\right)}, \tag{25}$$

where $\epsilon=\cos(\gamma)\sin(\delta)\frac{x^{\eta}}{\eta}+\sin(\gamma)\sin(\delta)\frac{y^{\eta}}{\eta}+\cos(\delta)\frac{z^{\eta}}{\eta}+\frac{1}{4}\ln(a)^2\sin(2\gamma)\sin^2(\delta)\cos(\delta).$

The obtained solutions of Eq. (3) are graphed here for different η-values.

Figs. 7-9 reveal the solution f_5 given in Eq. (24) of Eq. (3) for t and η values.

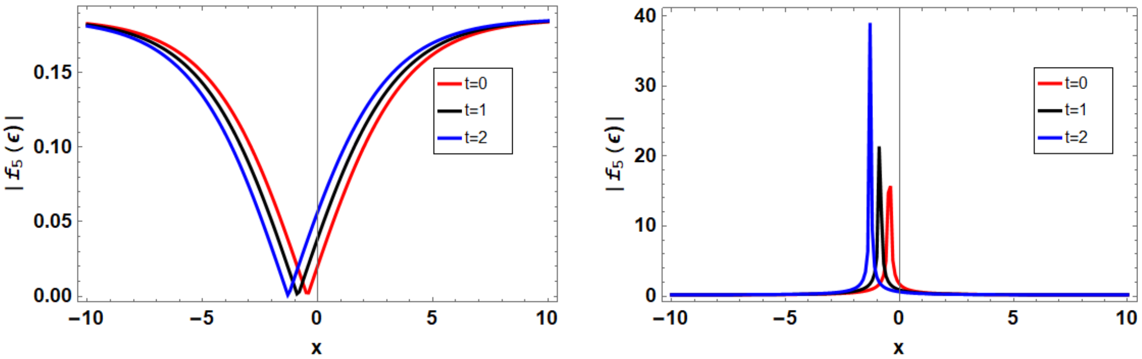

Figure 7. 3D oblique wave profile of f_5 appears in Eq. (24) is given corresponding (a) to d=0.2 and (b) to d=-0.2 for γ = π/3, δ = π/4, y = z = 1, t = 2 and η = 1.

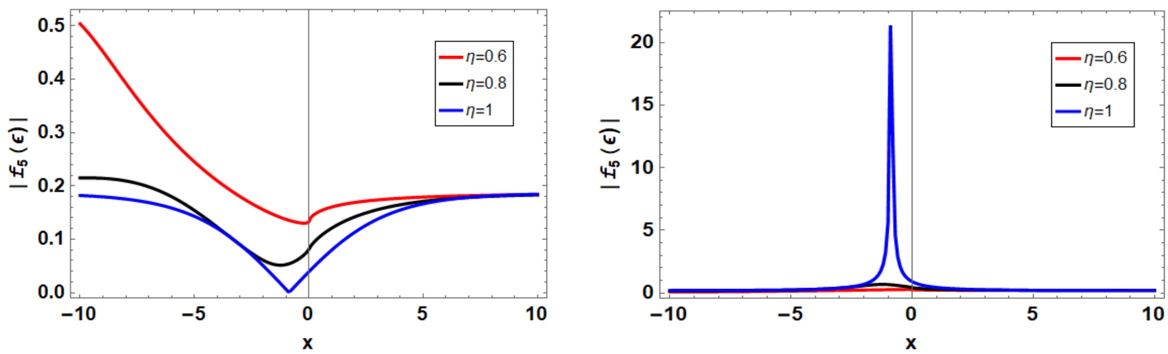

Figure 8. 3D oblique wave profile of f_5 appears in Eq. (24) is given corresponding (a) to d=0.2 and (b) to d=-0.2 for γ = π/3 and δ = π/4.

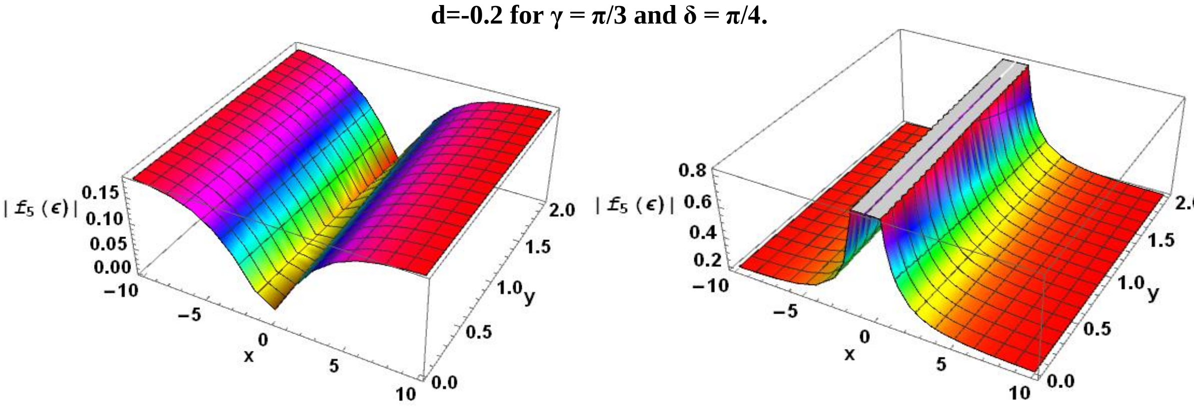

Figure 9. 3D oblique wave profile of f_5 appears in Eq. (24) is given corresponding (a) to d=0.2 and (b) to d=-0.2 for γ = π/3, δ = π/4, and η = 1.

4. CONCLUSIONS

This study has investigated the oblique wave exact solutions for the conformable space-time fractional mKdV equations. The modified Kudryashov approach has been employed to report the wave solutions for the aforementioned mKdV equations with the help of soft computations. It is important to mention that the obtained solutions are very helpful to know the internal nature of prorogating waves. Furthermore, the graphical representation of some solutions has been left over for a better understanding of the reader. The performance of the used method in this research shows its power, effectiveness, accuracy and explains the ability of the method for applying to other nonlinear fractional differential equations.

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