ORIGINAL PAPER

SOME PROPERTIES OF THE PLASTIC CONSTANT

ORHAN DİŞKAYA[1], HAMZA MENKEN[2]

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Abstract. *In this article, we construct the plastic number in the three-dimensional space. We examine the nested radicals and continued fraction expansions of the plastic ratio. In addition, we give some properties and geometric interpretations of the plastic constant.*

Keywords: *Padovan numbers; plastic ratio; golden ratio.*

1. INTRODUCTION

There are so many studies in the literature about the special number sequences such as Fibonacci, Lucas, Pell, Jacobsthal, Tribonacci, Padovan, and Perrin [1-11]. The most known of these are the Fibonacci numbers. One of the reasons that make Fibonacci numbers important is the golden ratio. Although the golden ratio appears in nature, art and architecture, the golden ratio is defined as the ratio of two consecutive Fibonacci numbers. The other important special numbers are the Padovan numbers.

The Padovan sequence is named after Richard Padovan who attributed its discovery to Dutch architect Hans van der Laan in his 1994 essay Dom. In [12], the Padovan sequence $\{P_n\}_{n\geq 0}$ is defined by

$$P_0 = P_1 = P_2 = 1 \text{ and } P_{n+3} = P_{n+1} + P_n. \tag{1.1}$$

Here, P_n is the nth Padovan number. First few terms of this sequence are 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21. Similarly, the ratio of successive the Padovan number converges to the plastic constant. Firstly, let us compute the ratios $\frac{P_{n+1}}{P_n}$ of the first 20 Padovan numbers, and then examine them for a possible pattern (Table 1). As n gets larger and larger, it appears that $\frac{P_{n+1}}{P_n}$ approaches a limit, namely, 1,32471795724474602596...

Table 1. Ratios of consecutive Padovan numbers.

n	P_{n+1}/P_n	n	P_{n+1}/P_n
0	1/1≈1.000000	10	16/12≈1.333333
1	1/1≈1.000000	11	21/16≈1.312500
2	2/1≈2.000000	12	28/21≈1.333333
3	2/2≈1.000000	13	37/28≈1.321428
4	3/2≈1.500000	14	49/37≈1.324324
5	4/3≈1.333333	15	65/49≈1.326530

[1] Mersin University, Graduate School of Natural and Applied Sciences, Mersin, Turkey.
E-mail: orhandiskaya@mersin.edu.tr.
[2] Mersin University, Department of Mathematics, Mersin, Turkey. E-mail: hmenken@mersin.edu.tr.

n	P_{n+1}/P_n	n	P_{n+1}/P_n
6	5/4≈1.250000	16	86/65≈1.323076
7	7/5≈1.400000	17	114/86≈1.325581
8	9/7≈1.285714	18	151/114≈1.324561
9	12/9≈1.333333	19	200/151≈1.324503

The plastic number p (also known as the plastic constant, the plastic ratio, the platin number and the minimal Pisot number) is a mathematical constant which is the unique real solution of the cubic equation 1.2.

$$x^3 - x - 1 = 0. \tag{1.2}$$

It has the exact value

$$p = \sqrt[3]{\frac{9+\sqrt{69}}{18}} + \sqrt[3]{\frac{9-\sqrt{69}}{18}} \approx 1{,}324718$$

that was firstly defined in 1924 by Gerard Cordonnier. He described applications to architecture and illustrated the use of the plastic constant in many buildings (for the details see [13]). Its decimal expansion begins with

$$\alpha \approx 1.3247 = p = \text{Plastic Ratio}.$$

The other unreal roots of the equation are as follows

$$\beta \approx -0.66236 - 0.56228i,\ \ \gamma \approx -0.66236 + 0.56228i.$$

Then, the following relations can be derived

$$\alpha + \beta + \gamma = 0,\ \ \alpha\beta + \alpha\gamma + \beta\gamma = -1,\ \ \alpha\beta\gamma = 1.$$

We confirm this observation. Let

$$p = \lim_{x\to\infty} \frac{P_{n+1}}{P_n} \approx 1.324717957244...$$

By the Padovan recurrence, we then have

$$\begin{aligned}
\lim_{x\to\infty} \frac{P_{n+1}}{P_n} &= \lim_{x\to\infty} \frac{P_{n-1}}{P_n} + \lim_{x\to\infty} \frac{P_{n-2}}{P_n} \\
&= \frac{1}{\lim\limits_{x\to\infty} \frac{P_n}{P_{n-1}}} + \frac{1}{\lim\limits_{x\to\infty} \frac{P_n}{P_{n-2}}} \\
&= \frac{1}{\lim\limits_{x\to\infty} \frac{P_n}{P_{n-1}}} + \frac{1}{1 + \frac{1}{\lim\limits_{x\to\infty} \frac{P_{n-2}}{P_{n-3}}}}
\end{aligned}$$

$$p = \frac{1}{p} + \frac{1}{1+\frac{1}{p}}$$

Thus p satisfies the cubic equation $p^3 - p - 1 = 0$.

1.1. A GEOMETRIC INTERPRETATION OF THE PLASTIC RATIO

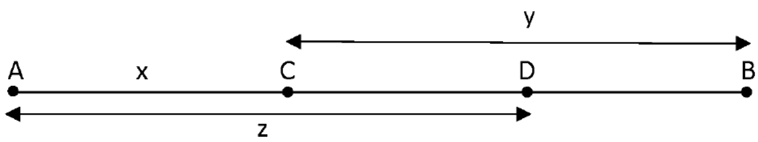

Figure 1. A geometric interpretation.

In [14], the golden ratio can be calculated by sectioning the segment AB in two parts AC and BC. Let $AC = x$ and $CB = y$ (Fig. 1). Then the equation $\psi = \frac{AC}{CB} = \frac{AB}{AC}$ yields

$$\frac{x}{y} = \frac{x+y}{x} \Rightarrow \frac{x}{y} = 1 + \frac{1}{\frac{x}{y}}$$

that is, $\left(\frac{x}{y}\right)^2 - \frac{x}{y} - 1 = 0$. So, $\frac{x}{y}$ satisfies the familiar quadratic equation $k^3 - k - 1 = 0.$ When the CB part of segment AB separate, we get segment AB in three parts. Let $AD = z$ and $CB = y$. The plastic number p is defined with

$$p = \frac{AC}{AD} = \frac{AD}{CB} = \frac{CB}{AC} = \frac{AC}{CD} = \frac{CD}{BD}$$

and follows

$$\left(\frac{z}{y}\right)^3 - \frac{z}{y} - 1 = 0$$

Thus, $\frac{z}{y}$ satisfies the familiar cubic equation $t^3 - t - 1 = 0.$ Look at [13, 15].

1.2. SOME SHAPES OF THE PLASTIC RATIO

The shape created by coinciding the sides of three equilateral triangles side by side, and the shape formed by coinciding the other equilateral triangles spirally counterclockwise with one side of the other (Fig. 2).

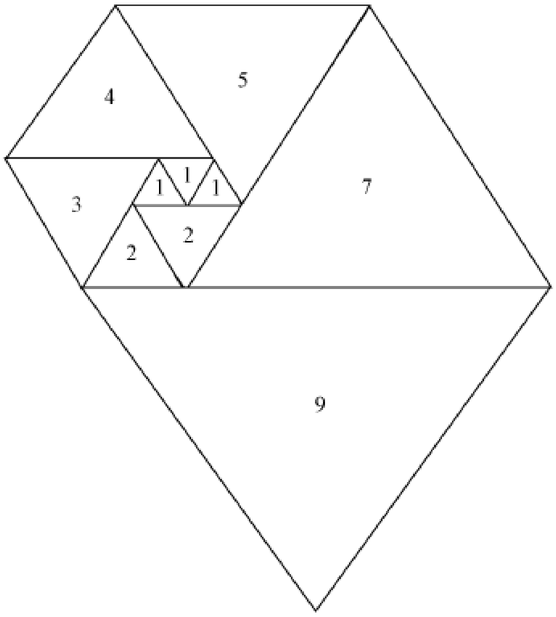

Figure 2. Shape-1 of Padovan numbers obtained in spiral form with triangles.

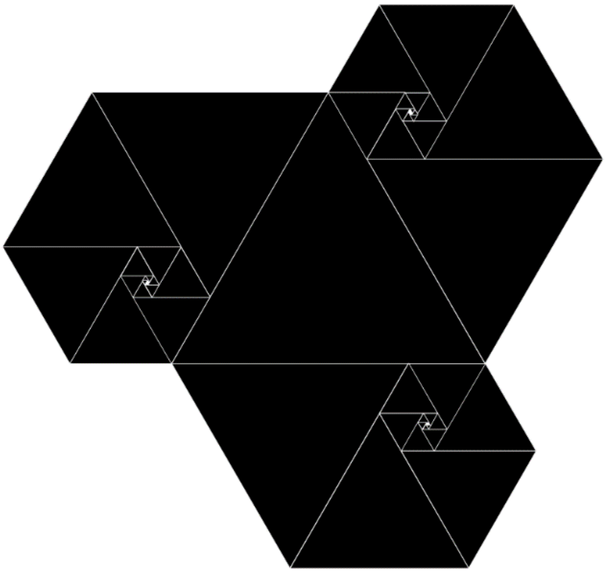

Figure 3. Shape-2 of Padovan numbers obtained in spiral form with triangles.

The different representation see Fig. 3. This shape is called the Padovan triangles [16].

The following figure (Fig. 4) is formed by bringing the co-squares anticlockwise side by side with an overlapping corner.

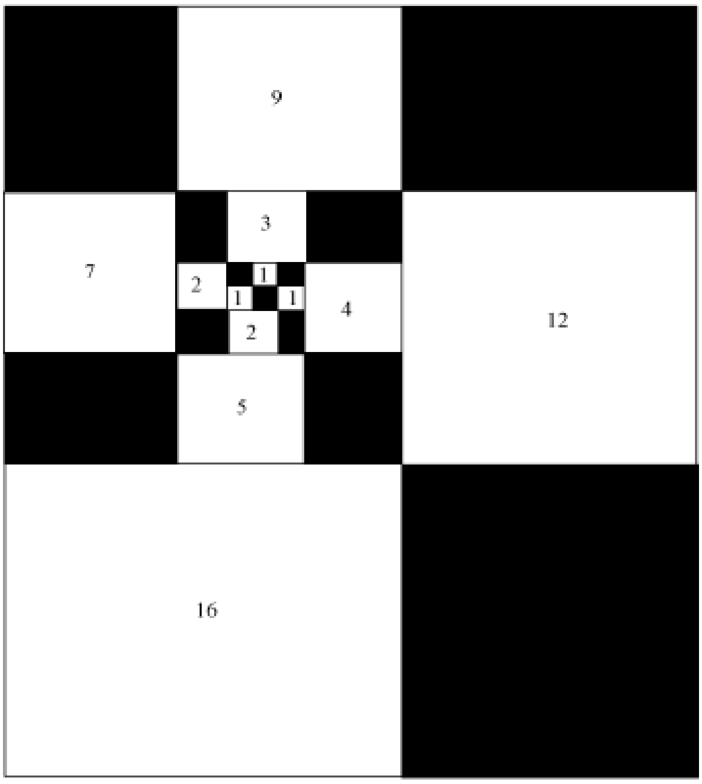

Figure 4. Padovan numbers obtained in spiral form with quadrangles.

2. NESTED RADICALS EXPANSIONS AND CONTINUED FRACTION OF THE PLASTIC CONSTANT

In [17], recall that p is reel root of the cubic equation $x^3 = x+1$. This equation has exactly one positive solution, namely, p. Since $x=\sqrt{1+\frac{1}{x}}$, it follows by iteration that

$$p=\sqrt{1+\frac{1}{\sqrt{1+\frac{1}{\sqrt{1+\frac{1}{\ddots}}}}}}$$

Next, consider the equation $x^3 = x+1$. Since, $x=\sqrt[3]{x+1}$, we also have

$$p=\sqrt[3]{1+\sqrt[3]{1+\sqrt[3]{1+\cdots}}}\,.$$

Next, consider the equation $x=\sqrt[4]{1+x+\frac{1}{x}}$, we have

$$p=\sqrt[4]{1+\sqrt[4]{1+\sqrt[4]{1+\ddots+\frac{1}{\ddots}}+\frac{1}{\sqrt[4]{1+\ddots+\frac{1}{\ddots}}}}+\frac{1}{\sqrt[4]{1+\sqrt[4]{1+\ddots+\frac{1}{\ddots}}+\frac{1}{\sqrt[4]{1+\ddots+\frac{1}{\ddots}}}}}}\,.$$

Next, consider the equation $x=\sqrt[5]{2+x+\frac{1}{x}}$, we have

$$p=\sqrt[5]{2+\sqrt[5]{2+\sqrt[5]{2+\ddots+\frac{1}{\ddots}}+\frac{1}{\sqrt[5]{2+\ddots+\frac{1}{\ddots}}}}+\frac{1}{\sqrt[5]{2+\sqrt[5]{2+\ddots+\frac{1}{\ddots}}+\frac{1}{\sqrt[5]{2+\ddots+\frac{1}{\ddots}}}}}}\,.$$

Next, consider the equation $x=\sqrt[6]{2+2x+\frac{1}{x}}$.

So, if we continue, we have

$$x=\sqrt[7]{3+2x+\frac{2}{x}},$$

$$x=\sqrt[8]{4+3x+\frac{2}{x}},$$

$$x=\sqrt[9]{5+4x+\frac{3}{x}},$$

...

$$x=\sqrt[n]{P_{n-2}+P_{n-3}x+\frac{P_{n-4}}{x}}.$$

Thus, we get

$$x=\sqrt[n]{P_{n-2}+P_{n-3}\sqrt[n]{P_{n-2}+P_{n-3}\ddots+\frac{P_{n-4}}{\ddots}}+\frac{P_{n-4}}{\sqrt[n]{P_{n-2}+P_{n-3}\ddots+\frac{P_{n-4}}{\ddots}}}}.$$

In [18, 19], the simply continued fraction expansion of the plastic constant is

$$p=1+\cfrac{1}{3+\cfrac{1}{12+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\ddots}}}}}$$

The continued fraction expansion of plastic number (the positive root of $x^3=x+1$) is [1, 3, 12, 1, 1, 3, 2, 3, 2, 4, 2, 141, 80, 2, 5, 1, 2, 8, 2, 1, 1, 3, 1, 8, 2, 1, 1, 14, 1, 1, 2, 1, 1, ...](the On-Line Encyclopedia of Integer Sequences (OEIS) A072117).

3. GATTEI-LIKE DISCOVERY OF THE PLASTIC RATİO

When P. Gattei was at Queen Elizabeth's Grammar School in Blackburn, England, he stumbled upon a problem involving the inverse f of a real-valued function f [20]. Accidentally, he dropped the minus sign and ended up taking the derivative f' of f .

Theorem 3.1: Let $f(x)=Ax^{n^2}$, where A is the positive real number. p is plastic constant.

$$f'(x)=\left(f^{-1}(x)\right)^n \Rightarrow n=p. \tag{3.1}$$

Proof: Then $f'(x)=An^2x^{n^2-1}$ and $f^{-1}(x)=\left(\frac{x}{A}\right)^{\frac{1}{n^2}}$.

By equation (3.1), this implies $An^2x^{n^2-1}=\left(\left(\frac{x}{A}\right)^{\frac{1}{n^2}}\right)^n$; that is $A^{n+1}n^{2n}x^{n^3-n-1}=1$. So $n^3-n-1=0$ and $A^{n+1}n^{2n}=1$. Then $n=\alpha,\ \beta$ or δ are one of the roots in (1.2). n cannot be β and γ. Because A must be a positive real number. Therefore $n=\alpha=p$.

4. GEOMETRIC SHAPES OF THE PLASTIC RATIO

4.1. GRAPHICS OF THE PLASTIC CONSTANT

It is a bit easier to solve the corresponding equation in x: $x^3 - x - 1 = 0$ (Fig. 5).

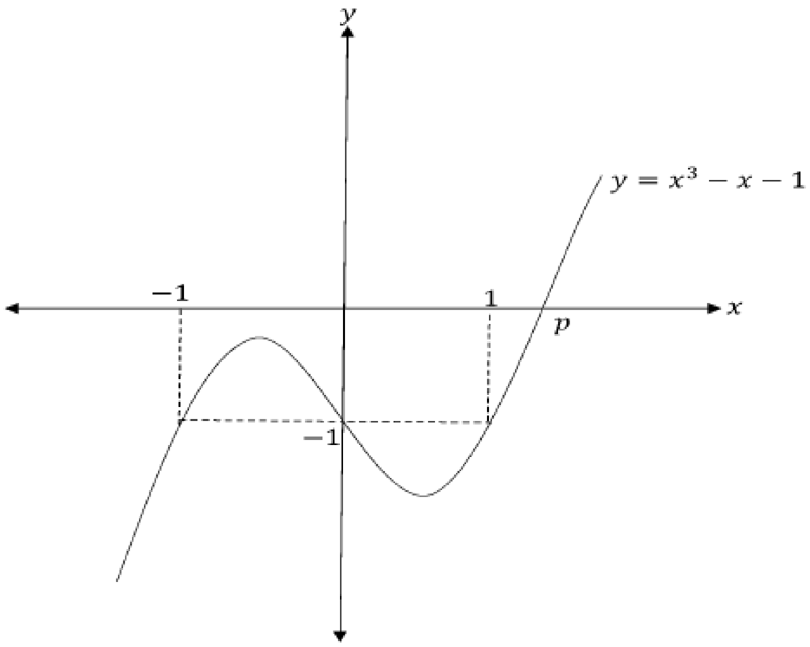

Figure 5. Graphic 1 of the Plastic Constant.

Using Cardano's formula, we have

$$t = \sqrt[3]{-\frac{n}{2} + \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}} + \sqrt[3]{-\frac{n}{2} - \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}}$$

for the real solution of the equation $t^3 - pt - q = 0$, the real value p of x is given by

$$p = \sqrt[3]{-\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{27}}} + \sqrt[3]{-\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{27}}} = \sqrt[3]{\frac{9+\sqrt{69}}{18}} + \sqrt[3]{\frac{9-\sqrt{69}}{18}} \approx 1{,}324718$$

The graph of cubic and linear functions is given in Fig. 6

The apse of the point where these graphics intersect gives the plastic constant. Because, the positive real root of $x^3 - x - 1 = 0$ is p.

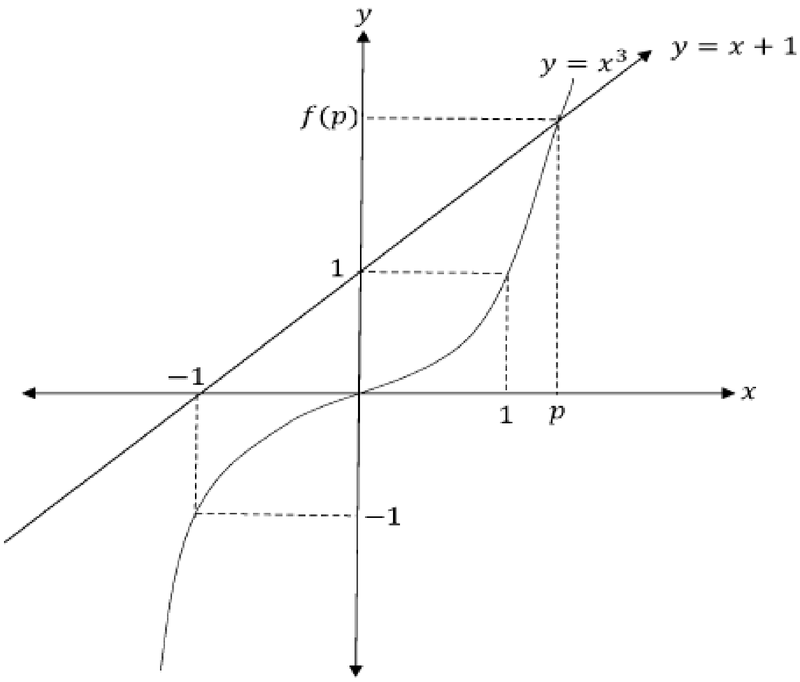

Figure 6. Graphic 2 of the Plastic Constant.

4.2. PLASTIC RATIO IN THE TRIANGLE

Let $DE//EC$ be parallel lines. The areas of the triangles are proportional to the squares of their vertices. Assume that the following triangle exists (Fig. 7).

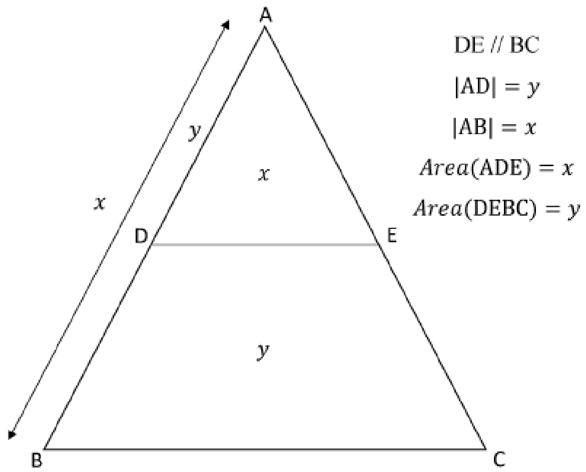

Figure 7. Plastic ratio in a triangle.

$$\left(\frac{y}{x}\right)^2 = \frac{x}{x+y}.$$

That is $\left(\frac{x}{y}\right)^3 - \left(\frac{x}{y}\right) - 1 = 0$. So $\frac{x}{y}$ satisfies the familiar cubic equation $t^3 - t - 1 = 0$.

4.3. PLASTIC RATIO IN UPRIGHT PRISMS

Let's consider the cube prism with a edge length. Remove a cube prism with b edge length of this cube prism. The volume of the resulting shape is equal to the volume of the square upright prism in the a and b side lengths as follows (Fig. 8):

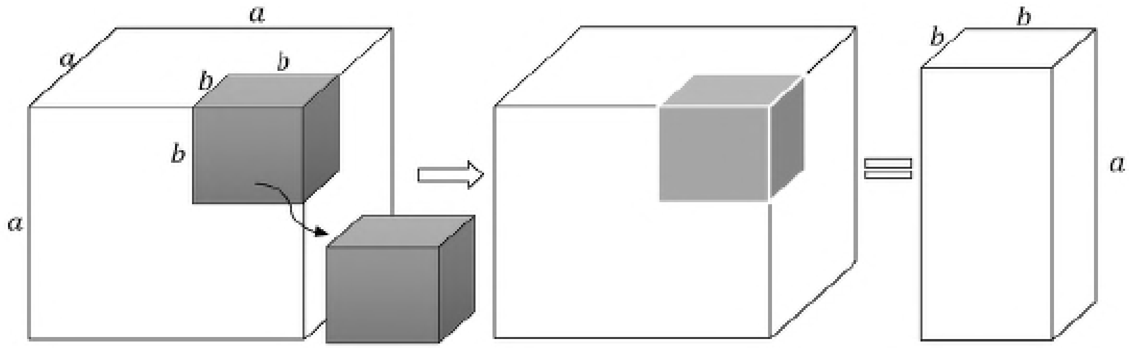

Figure 8. Plastic Ratio in Upright Prisms

Since the volume of the resulting shape equals that of the square upright prism, it follows that

$$a^3 - b^3 = ab^2$$

$$a^3 - ab^2 - b^3 = 0$$

$$\left(\frac{a}{b}\right)^3 - \frac{a}{b} - 1 = 0.$$

Hence, $\frac{a}{b} = p$ gives the plastic ratio.

4.4. PLASTIC RATIO IN CYLINDERS

Let's consider the cylinder with radii and height a. Remove a cylinder with radii b and height a of this cylinder. The volume of the resulting shape is equal to the volume of the cylinder with radii and height b as follows (Fig. 9):

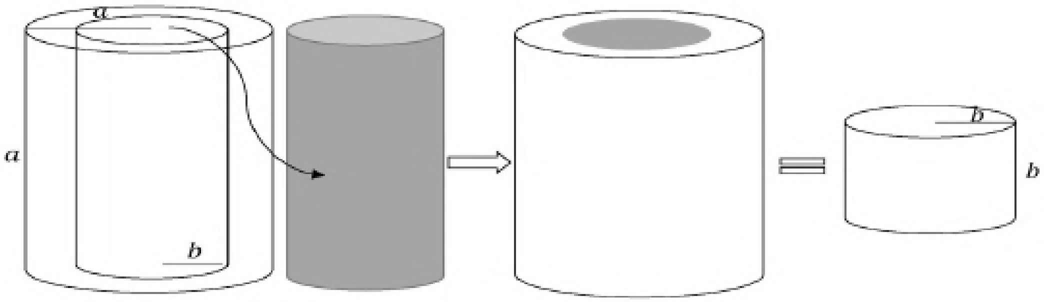

Figure 9. Plastic Ratio in Cylinders.

Since the volume of the resulting shape equals that of the cylinder, it follows that

$$a^3 - ab^2 = b^3$$

$$a^3 - ab^2 - b^3 = 0$$

$$\left(\frac{a}{b}\right)^3 - \frac{a}{b} - 1 = 0.$$

So, $\frac{a}{b} = p$, the plastic ratio.

4.5. PLASTIC RATIO IN PYRAMIDS

Let's consider the pyramid with base edges and height a. Remove a pyramid with base edges b and height a of this pyramid. The volume of the resulting shape is equal to the volume of the pyramid with base edges and height b as follows (Fig. 10):

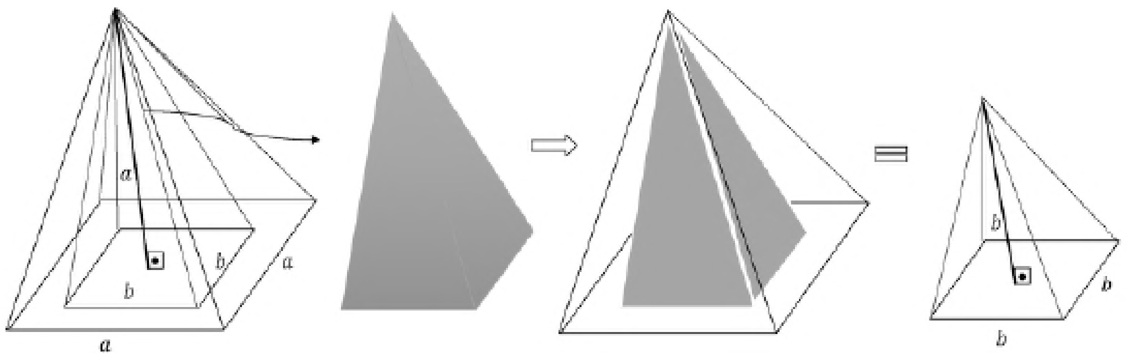

Figure 10. Plastic Ratio in Pyramids.

Since the volume of the resulting shape equals that of the pyramid, it follows that

$$\frac{1}{3}a^3 - \frac{1}{3}ab^2 = \frac{1}{3}b^3$$

$$a^3 - ab^2 - b^3 = 0$$

$$\left(\frac{a}{b}\right)^3 - \frac{a}{b} - 1 = 0.$$

So, $\frac{a}{b} = p$, the plastic ratio.

4.6. PLASTIC RATIO IN CONES

Let's consider the cone with radii and height a. Remove a cone with radii b and height a of this cone. The volume of the resulting shape is equal to the volume of the cone with radii and height b as follows (Fig. 11):

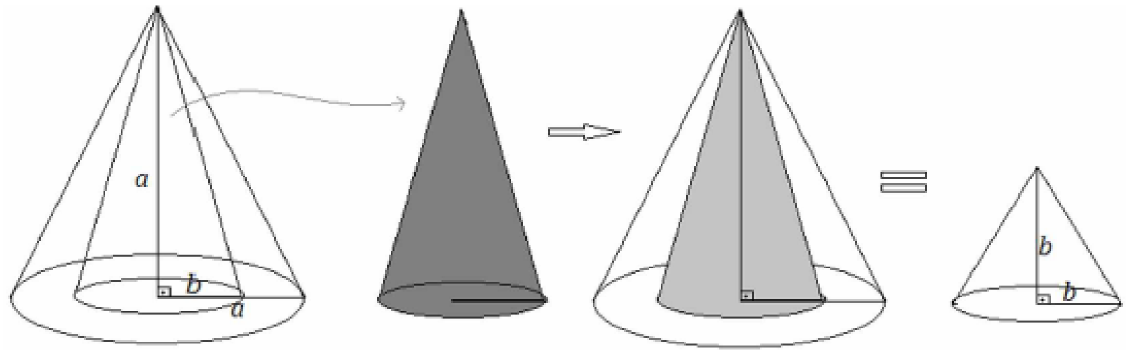

Figure 11. Plastic Ratio in Cones.

Since the volume of the resulting shape equals that of the cone, it follows that

$$\frac{1}{3}a^3\pi - \frac{1}{3}ab^2\pi = \frac{1}{3}b^3\pi$$

$$a^3 - ab^2 - b^3 = 0$$

$$\left(\frac{a}{b}\right)^3 - \frac{a}{b} - 1 = 0.$$

So, $\frac{a}{b} = p$, the plastic ratio.

4.7. PLASTIC RATIO IN THE SPHERE

In [16], consider a ball (globe-shaped volume) formed by two concentric spheres with radii a and b, where $a > b$. Inscribe an ellipse of the major axis $2a$ and minor axis $2b$, so it touches the outer and inner spheres (Fig. 12). Suppose the volume of the ellipse equals that of the ball. Compute the ratio $\frac{a}{b}$.

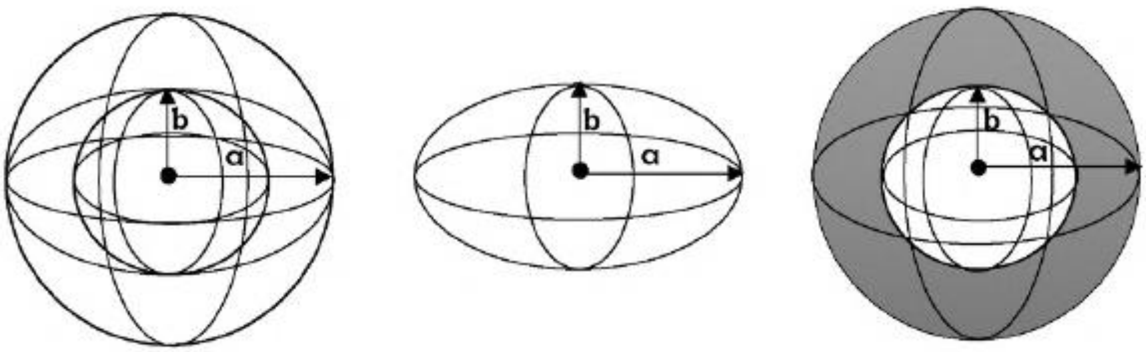

Figure 12. Plastic Ratio in the Sphere.

Since the volume of the ellipse equals that of the ball, it follows that

$$\frac{4}{3}\pi b^2 a=\frac{4}{3}\pi\left(a^3-b^3\right)$$
$$a^3-ab^2-b^3=0$$
$$\left(\frac{a}{b}\right)^3-\frac{a}{b}-1=0.$$

So, $\frac{a}{b}=p$, the plastic ratio.

5. CONCLUSION

The plastic ratio is a ratio that has the various features of the golden ratio. Therefore, various scientific studies about the golden ratio, which has an important place in nature, were examined and it was researched how these studies would give results for the plastic ratio. Thus, the results obtained for the plastic ratio are included in this study.

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