ORIGINAL PAPER

SOME RESEARCH NOTES ON LIFTS OF THE *HSU* – (4,2) STRUCTURE ON COTANGENT AND TANGENT BUNDLE

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Abstract. *There are a lot of structures in tangent and cotangent bundle. One of them is the Hsu – (4,2) structure have been defined and studied by Yano, Hough and Chen [1] and the complete and horizontal lifts of Hsu – (4,2) structure extended in M^n to cotangent bundle by R. Nivas and M. Saxena [2]. Hsu-structure had been defined firstly by Prof Mishra [3]. This paper consists of two main sections. In the first part, we find the integrability conditions by calculating Nijenhuis tensors of the complete and horizontal lifts of Hsu – (4,2) structure. Later, we get the results of Tachibana operators applied to vector and covector fields according to the complete and horizontal lifts of Hsu – (4,2) structure and the conditions of almost holomorfic vector fields in cotangent bundle $T^*(M^n)$. Finally, we have studied the purity conditions of Sasakian metric with respect to the lifts of Hsu – (4,2) structure. In the second part, all results obtained in the first section were investigated according to the complete and horizontal lifts of the Hsu – (4,2) structure in tangent bundle $T^*(M^n)$.*

Keywords: *Integrability conditions; Tachibana operators; horizontal lift; complete lift; Sasakian metric; cotangent bundle.*

1. INTRODUCTION

The investigation for the integrability of tensorial structures on manifolds and extension to the tangent or cotangent bundle, whereas the defining tensor field satisfies a polynomial identity has been an actively discussed research topic in the last 50 years, initiated by the fundamental works of Kentaro Yano and his collaborators, see for example [4]. There are a lot of structures in tangent and cotangent bundle. One of them is the *Hsu – (4,2)* structure have been defined and studied by Yano, Hough and Chen [1] and the complete and horizontal lifts of *Hsu – (4,2)* structure extended in M^n to cotangent bundle by R. Nivas and M. Saxena [2].

Hsu-structure had been defined firstly by Prof Mishra [3]. In addition, a differentiable structure $F^{2v+4}+F^2=0, (F\neq 0, v\neq 0)$ studyed by K.K. Dube [5] and Upadhyay and Gupta have obtained some integrability conditions of $F(K,-(K-2)-$ structure, satisfying $F^K+F^{K-2}=0, (F$ is a tensor field of type (1,1)) [6]. This paper consists of two main sections. In the first part, we find the integrability conditions by calculating Nijenhuis tensors of the complete and horizontal lifts of *Hsu – (4,2)* structure.

Later, we get the results of Tachibana operators applied to vector and covector fields according to the complete and horizontal lifts of *Hsu – (4,2)* structure and the conditions of almost holomorfic vector fields in cotangent bundle $T^*(M^n)$. Finally, we have studied the purity conditions of Sasakian metric with respect to the lifts of *Hsu – (4,2)* structure . In the

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second part, all results obtained in the first section were investigated according to the complete and horizontal lifts of the *Hsu – (4,2)* structure in tangent bundle $T^*(M^n)$.

Let M^n be a differentiable manifold of class C^∞ and of dimension *n* and let $T^*(M^n)$ denote the cotangent bundle of M. Then $T^*(M^n)$ is also a differentiable manifold of class C^∞ and dimension 2n.

The following are notations and conventions that will be used in this paper.

1. $\mathfrak{I}_s^r(M^n)$ denotes the set of the tensor fields C^∞ and of type (r,s) on M^n. Similarly, $\mathfrak{I}_s^r(T^*(M^n))$ denotes the set of such tensor fields in $T^*(M^n)$.
2. The map π is the projection of $T^*(M^n)$ onto M^n.
3. Vector fields in M^n are denoted by X, Y, Z,…and Lie differentiation by L_X. The Lie product of vector fields X and Y is denoted by [X,Y].
4. Suffixes a, b, c,…, h, i, j…take the values *1* to *n* and $\bar{\imath} = i + n$. Suffixes A, B, C,...take the values *1* to *2n.*

If A is point in M^n, then $\pi^{-1}(A)$ is fiber over A. Any point $\rho \in \pi^{-1}(A)$ is denoted by the ordered pair (A, p_A), where ρ is 1 – form in M^n and p_A is the value of p at A. Let U be a coordinate neighborhood in M^n such that $A \in U$. Then U induces a coordinate neighborhood $p^{-1}(U)$ in $T^*(M^n)$ and $p \in \pi^{-1}(A)$.

1.1 THE COMPLETE LIFT OF $F^4 - \lambda^r F^2 = 0$ ON $T^*(M^n)$

Let M^n be an $n -$ dimensional differentiable manifold of class C^∞. Suppose there exist on M^n a tensor field $F(\neq 0)$ of type (1,1) satisfying

$$F^4 - \lambda^r F^2 = 0 \tag{1.1}$$

Where λ is complex number not equal to zero and *r* some finite integer. In such a manifold M^n, let us put

$$l = F^2/\lambda^r \, and \; m = I - F^2/\lambda^r \tag{1.2}$$

Where I denotes the unit tensor field. Then it is easy to show

$$l^2 = l, m^2 = m, l + m = I, lm = ml = 0 \tag{1.3}$$

Thus, the operators l and *m* when applied to the tangent space of M^n at a point are complementary projection operators. Hence there exist complementary distributions L^* and M^* corresponding to the projection operators l and *m* respectively. If the rank of *F* is constant everywhere and equal *r*, the dimensions of L^* and M^* are *r* and $(n - r)$ respectively. Let us call such a structure as *Hsu – (4,2)* structure of rank *r*.

Let F_i^h be the component of *U* at *A* in the coordinate neighbourhood *U* of M^n. Then the complete lift F^C of *F* is also a tensor field of type (1,1) in $T^*(M^n)$ whose components $\tilde{F}_B^A$ in $\pi^{-1}(U)$ are given by

$$\tilde{F}_i^h = F_i^h, \tag{1.4}$$

$$\tilde{F}_i^h = 0 \tag{1.5}$$

$$\tilde{F}_i^{\bar{h}} = p_a\left[\partial F_h^a/\partial x^i - \partial F_i^a/\partial x^h\right] \tag{1.6}$$

and

$$\tilde{F}_{\bar{i}}^{\bar{h}} = F_h^i, \tag{1.7}$$

where $(x^1, x^2, x^3, \dots, x^n)$ are coordinates of A in U and p_A has components $(p_1, p_2, p_3, \dots p_n)$. Thus we can write

$$F^C = \left(\tilde{F}_B^A\right) = \begin{bmatrix} F_i^h & 0 \\ p_a(\partial_i F_h^a - \partial_h F_i^a) & F_h^i \end{bmatrix} \tag{1.8}$$

where $\partial_i = \partial/\partial x^i$.
If we put

$$\partial_i F_h^a - \partial_h F_i^a = 2\partial[iF_h^a], \tag{1.9}$$

then the equation (1.8) can be written as

$$F^C = (\tilde{F}_B^A) = \begin{bmatrix} F_i^h & 0 \\ 2p_a\partial[iF_h^a] & F_h^i \end{bmatrix} \tag{1.10}$$

$$(F^C)^2 = \begin{bmatrix} F_i^h & 0 \\ 2p_a\partial[iF_h^a] & F_h^i \end{bmatrix} \begin{bmatrix} F_j^i & 0 \\ 2p_t\partial[jF_i^t] & F_i^j \end{bmatrix} \tag{1.11}$$

$$= \begin{bmatrix} F_i^h F_j^i & 0 \\ L_{hj} & F_i^j F_h^i \end{bmatrix}$$

Squaring (1.11) again we get

$$(F^C)^4 = \lambda^r \begin{bmatrix} F_t^h F_\lambda^t & 0 \\ L_{h\lambda} & F_t^\lambda F_h^t \end{bmatrix} \tag{1.12}$$

$$(F^C)^4 - \lambda^r (F^C)^2 = 0 \tag{1.13}$$

Thus the complete lift F^C of F also has $Hsu-(4,2)$ structure in the cotangent bundle $T*(M^n)$.

1.2. THE HORIZONTAL LIFT OF $F^4 - \lambda^r F^2 = 0$ ON $T^*(M^n)$

Let F and G be two tensor fields of type (1,1) on the manifold M^n. If F^H denotes the horizontal lift of F, we have [7]

$$F^H G^H + G^H F^H = (FG + GF)^H. \tag{1.14}$$

Taking F and G identical, we get

$$(F^H)^2 = (F^2)^H. \tag{1.15}$$

Squaring the above equation both sides and making use of the equation (1.14) we get

$$(F^H)^4 = (F^4)^H. \tag{1.16}$$

Since F gives $Hsu - (4,2)$ structure on M^n, we have

$$F^4 - \lambda^r F^2 = 0 \tag{1.17}$$

Taking horizontal lift in the above equation we get

$$(F^H)^4 - \lambda^r (F^H)^2 = 0. \tag{1.18}$$

Thus the horizontal lift F^H of F also admits $Hsu - (4,2)$ structure in the cotangent bundle.

2. RESULTS

2.1. THE NIJENHUIS TENSORS OF THE STRUCTURE $(F^C)^4 - \lambda^r (F^C)^2 = 0$ ON COTANGENT BUNDLE

Definition 1. Let F be a tensor field of type (1,1) admitting $F(4,2) -$ structure in M^n. The Nijenhuis tensor of a (1,1) tensor field F of M^n is given by

$$N_F = [FX, FY] - F[X, FY] - F[FX, Y] + F^2[X, Y] \tag{2.1}$$

for any $X, Y \in \mathfrak{I}_1^1(M^n)$ [8,9,10]. The condition of $N_F(X, Y) = N(X, Y) = 0$ is essential to integrability condition in these structures.
The Nijenhuis tensor N_F is defined local coordinates by

$$N_{ij}^k \partial_k = \left(F_i^s \partial_s^k F_j^k - F_j^l \partial_l F_i^k - \partial_i F_j^l F_l^k + \partial_j F_i^s F_s^k\right) \partial_k$$

where $X = \partial_i, Y = \partial_j, F \in \mathfrak{I}_1^1(M^n)$.

Proposition 1. If $X, Y \in \mathfrak{I}_0^1(M^n)$, $\omega, \theta \in \mathfrak{I}_1^0(M^n)$ and $F, G \in \mathfrak{I}_1^1(M^n)$, then [4]

$$\begin{gathered}[\omega^V, \theta^V] = 0, [\omega^V \gamma F] = (\omega \cdot F)^V, [\gamma F, \gamma G] = \gamma[F, G], \\ [X^C, \omega^V] = (L_X \omega)^v, [X^C, \gamma F \gamma] = \gamma(L_X F), [X^C Y^C] = [X, Y]^C\end{gathered} \tag{2.2}$$

where $\omega \cdot F$ is a $1 -$ form defined by $(\omega \cdot F)(Z) = \omega(FZ)$ for any $Z \in \mathfrak{I}_0^1(M^n)$ and L_X the operator of Lie derivation with respect to X.

Theorem 1. The Nijenhuis tensor $N(X^C,\omega^V)$ of the complete lift of F^4 vanishes if the Lie derivative of the tensor field F^2 with respect to the X is zero and F acts as Hsu −structure operator on M.

Proof: The Nijenhuis tensor $N(X^C,\omega^V)$ for the complete lift of F^4 is given by

$$N_{(F^4)^C(F^4)^C}(X^C\omega^V) = [(F^4)^C X^C,(F^4)^C\omega^V] - (F^4)^C[(F^4)^C X^C,\omega^V] \\ -(F^4)^C[X^C,(F^4)^C\omega^V] + (F^4)^C(F^4)^C[X^C,\omega^V]$$

$$[(\lambda^r F^2)^C X^C,(\lambda^r F^2)^C\omega^V] - (\lambda^r F^2)^C[(\lambda^r F^2)^C\omega^V] \\ -(\lambda^r F^2)^C[X^C,(\lambda^r F^2)^C\omega^V] + (\lambda^r F^2)^C(\lambda^r F^2)^C[X^C,\omega^V]$$

$$= \lambda^{2r}\{[(F^2)^C X^C,(F^2)^C\omega^V] - (F^2)^C[(F^2)^C X^C\omega^V] \\ -(F^2)^C[X^C,(F^2)^C\omega^V] + (F^2)^C(F^2)^C[X^C,\omega^V]\}$$

If we put the equation of $(F^2)^C X^C = (F^2X)^C + \gamma(L_X F^2)$ (see [4], pp. 243)

$$N_{(F^4)^C(F^4)^C}(X^C,\omega^V) = \lambda^{2r}\{[(F^2X)^C + \gamma L_X F^2,(\omega\cdot F^2)^V] \\ -(F^2)^C[(F^2X)^C + \gamma L_X F^2,\omega^V] - (F^2)^C[X^C,(\omega\cdot F^2)^V] \\ +(F^4)^C(L_X\omega)^V\}$$

$$= \lambda^{2r}\{[(F^2X)^C,(\omega\cdot F^2)^V] - [(\omega\cdot F^2)^V,\gamma L_X F^2] \\ -(F^2)^C[(F^2X)^C,\omega^C] + (F^2)^C[\omega^V,\gamma L_X F^2] \\ -(F^2)^C\left(L_X(\omega\cdot F^2)\right)^V + \left((L_X\omega)\cdot F^4\right)^V\Big\}$$

Let us now suppose that $L_X F^2 = 0$ then the equation takes the form

$$N_{(F^4)^C(F^4)^C}(X^C,\omega^V) = \lambda^{2r}\Big\{(L_{F^2X}(\omega\cdot F^2))^V - (F^2)^C(L_{(F^2X)}\omega)^V \\ -(F^2)^C(L_X(\omega\cdot F^2)^V + \left((L_X\omega)\cdot F^4\right)^V\Big\}$$

Let us now suppose that F acts as Hsu −structure on M [11]. Then $F^2 = \lambda^r I$.
Thus the equations becomes

$$N_{(F^4)^C(F^4)^C}(X^C,\omega^V) = \lambda^{4r}\{(L_X\omega)^V - ((L_X\omega)^V) + (L_X\omega)^V\} = 0,$$

Where $\omega \in \Im_1^0(M^n)$.

Theorem 2. The Nijenhuis tensor $N(\omega^V\theta^V)$ of the comple lift of F^4 vanishes.

Proof: The Nijenhuis tensor $N(\omega^V\theta^V)$ for the complete lift of F^4 is given by

$$N_{(F^4)^C(F^4)^C}(\omega^C,\theta^V) = [(F^4)^C\omega^V,(F^4)^C\theta^V] - (F^4)^C\omega^V,\theta^V \\ -(F^4)^C[\omega^V,(F^4)^C\theta^V] + (F^4)^C(F^4)^C[\omega^V,\theta^V]$$

$$= \lambda^{2r}\{[(\omega\cdot F^2)^V,(\theta\cdot F^2)^V] - (F^2)^C[(\omega\cdot F^2)^V,\theta^V] \\ -(F^2)^C[\omega^V,(\theta\cdot F^2)^V] + (F^2)^C(F^2)^C[\omega^V,\theta^V]\}.$$

Because of $[\omega^V \theta^V] = 0$ and $\omega \cdot F^2 \in \Im_1^0(M^n)$ on $T^*(M^n)$, the equation becomes $N_{(F^4)^C (F^4)^C}(\omega^V \theta^V) = 0.$

The theorem is completed.

2.2. TACHIBANA OPERATORS APPLIED TO VECTOR AND COVECTOR FIELDS ACCORDING TO LIFTS OF $(F^C)^4 - \lambda^r (F^C)^2 = 0$ ON COTANGENT BUNDLE

Definition 2. Let $\varphi \in \Im_1^1(M^n)$, and $\Im(M^n) = \sum_{r,s=0}^{\infty} \Im_1^1(M^n)$ be a tensor alebra over R. A map $\emptyset_\varphi\big|_{r+s>0}: \overset{*}{\Im}(M^n) \to \Im(M^n)$ is called as Tachibana operatör or $\emptyset_\varphi$ operatör on M^n if

a) $\emptyset_\varphi$ is linear with respect to constant coefficient,

b) $\emptyset_\varphi \left(K \overset{c}{\otimes} L \right) = \left(\emptyset_\varphi K\right)\emptyset_\varphi : \overset{*}{\Im}(M^n) \to \Im_{s+1}^{r}(M^n)$ for all r and s,

c) $\otimes L + K \otimes \emptyset_\varphi L$ for all $K, L \in \overset{*}{\Im}(M^n)$,

d) $\emptyset_{\varphi X} Y = -(L_Y \varphi) X$ for all $X, Y \in \Im_0^1(M^n)$ where L_Y is the Lie derivation with respect to Y (see [2, 4, 7]),

e) $\left(\emptyset_{\varphi X} \eta\right) Y = \left(d(\iota_Y \eta)\right)(\varphi X) - \left(d\left(\iota_Y(\eta o \varphi)\right)\right) X + \eta\left((L_Y \varphi) X\right)$
$= \emptyset X(\iota_Y \eta) - X\left(\iota_{\varphi Y} \eta\right) + \eta\left((\iota_Y \varphi) X\right)$

for all $\eta \in \Im_0^1(M^n)$ and $X, Y \in \Im_0^1(M^n)$, where $\iota_Y \eta = \eta(Y) = \overset{c}{\otimes} Y$, $\overset{*}{\Im}{}_s^r(M^n)$ the module of all pure tensor fields of type (r,s) on M^n with respect to the affinor field, $\overset{c}{\otimes}$ is a tensor product with a contraction C [8, 9, 12] (see [10] for applied to pure tensor field).

Remark 1. If $r = s = 0$, then from c), d) and e) of Definition 2 we have $\emptyset_{\varphi X}(\iota_Y \eta) = \emptyset X(\iota_Y \eta) - X\left(\iota_{\varphi Y} \eta\right)$ for $\iota_Y \eta \in \Im_0^0(M^n)$, which is not well-defined $\emptyset_\varphi -$ operator. Different choices of Y and η leading to same function $f = \iota_Y \eta$ do get the same values. Consider $M^n = R^2$ with standard coordinates x,y. Let $\varphi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Consider the function $f = 1$. This may be written in many different ways as $\iota_Y \eta$. Indeed taking $\eta = dx$, we may choose $Y = \frac{\partial}{\partial_x}$ or $Y = \frac{\partial}{\partial_x} + x \frac{\partial}{\partial_y}$. Nov the right-hand side of $\emptyset_{\varphi X}(\iota_Y \eta) = \emptyset X(\iota_Y \eta) - X\left(\iota_{\varphi Y} \eta\right)$ is $(\emptyset X)1 - 0 = 0$ in the first case, and $(\emptyset X)1 - Xx = -Xx$ in the second case. For $X = \frac{\partial}{\partial_x}$, the latter expression is $-1 \neq 0$. Therefore, we put $r + s > 0$ [9].

Remark 2. From d) of Definition 2 we have

$$\emptyset_{\varphi X} Y = [\varphi X, Y] - \varphi[X, Y] \tag{2.3}$$

By virtue of

$$[fX, gY] = fg[X, Y] + f\left(X_g\right)Y - g\left(Y_f\right)X \tag{2.4}$$

for any $f, g \in \Im_0^0(M^n)$, we see that $\emptyset_{\varphi X} Y$ is linear in X, but not Y [9].

Theorem 3. Let $(F^4)^C$ be a tensor field of type (1,1) on $T^*(M^n)$ defined by (1.12). If the Tachibana operator $\emptyset_\varphi$ applied to vector and covector fields according to complete lifts of (F^4) on $T^*(M^n)$, then we get the following results:

i. $\emptyset_{(F^4)^C X^C} Y^C = -\lambda^r \left\{\left((L_Y F^2)X\right)^C + \gamma L_X L_Y F^2\right\}$,

ii. $\emptyset_{(F^4)^C X^C} \omega^V = -\lambda^r \left\{\left(L_X(\omega \cdot F^2)\right)^V - (L_{F^2 X}\omega)^V\right\}$,

iii. $\emptyset_{(F^4)^C \omega^V} X^C = -\lambda^r \left(\omega \cdot (L_X F^2)\right)^V$,

iv. $\emptyset_{(F^4)^C \omega^V} \theta^V = 0$

where complete lifts $X^C, Y^C \in \Im_0^1\left(T^*(M^n)\right)$ of $X, Y \in \Im_0^1(M^n)$ and the vertical lift $\omega^V \theta^V \in \Im_0^1\left(T^*(M^n)\right)$ of $\omega, \theta \in \Im_0^1(M^n)$ are given, respectively.

Proof:

i.
$$\begin{aligned}\emptyset_{(F^4)^C X^C} Y^C &= -\left(L_{Y^C}(F^4)^C\right)X^C = -L_{Y^C}(F^4)^C X^C + (F^4)^C L_{Y^C} X^C \\ &= \lambda^r\{-\left((L_Y F^2)X\right)^C - (F^2 L_Y X)^C - \gamma L_Y L_X F^2 \\ &\quad +\left(F^2(L_Y X)\right)^2 + \gamma L_Y L_X F^2 - \gamma L_X L_Y F^2\} \\ &= -\lambda^r \left\{\left((L_Y F^2)X\right)^C + \gamma L_X L_Y F^2\right\}\end{aligned}$$

ii.
$$\begin{aligned}\emptyset_{(F^4)^C X^C} \omega^V &= -\left(L_{\omega^V}(F^4)^C\right)X^C = -L_{\omega^V}(F^4)^C X^C + (F^4)^C L_{\omega^V} X^C \\ &\quad -L_{\omega^V}(\lambda^r F^2)X^C + (\lambda^r F^2)^C\left(-(L_X\omega)\right)^V \\ &= -\lambda^r \left\{L_{\omega^V}(F^2 X)^C + L_{\omega^V}\gamma(L_X F^2) + \left((L_X\omega)\cdot F^2\right)^V\right\} \\ &= -\lambda^r \left\{\left(L_X(\omega \cdot F^2)\right)^V - (L_{F^2 X}\omega)^V\right\}\end{aligned}$$

iii.
$$\begin{aligned}\emptyset_{(F^4)^C \omega^V} X^C &= -\left(L_{X^C}(F^4)^C\right)\omega^V = -L_{X^C}(F^4)^C \omega^V + (F^4)^C L_{X^C}\omega^V \\ &= -\lambda^r\{\left(L_{X^C}(F^2)^C\right)\omega^V - (F^2)^C (L_X\omega)^V\} \\ &= -\lambda^r \left\{(L_X\omega \cdot F^2)^V - \left((L_X\omega)\cdot F^2\right)^V\right\} \\ &= -\lambda^r\left(\omega \cdot (L_X F^2)\right)^V\end{aligned}$$

iv.
$$\begin{aligned}\emptyset_{(F^4)^C \omega^V} \theta^V &= -\left(L_{\theta^V}(F^4)^C\right)\omega^V = -L_{\theta^V}(F^4)^C \omega^V + (F^4)^C L_{\theta^V}\omega^V \\ &= -\lambda^r L_{\theta^V}(\omega \cdot F^2)^V \\ &= 0\end{aligned}$$

Theorem 4. If $L_Y F^2 = 0$ for $Y \in M^n$, then its complete lift Y^C to the cotangent bundle is an almost holomorfic vector field with respect to the structure $(F^4)^C - \lambda^r (F^2)^C = 0$.

Proof:

i.
$$\begin{aligned}L_{Y^C}((F^4)^C)X^C &= L_{Y^C}(F^4)^C X^C - (F^4)^C L_{Y^C} X^C \\ &= \lambda^r \left\{\left((L_Y F^2)X\right)^C + (F^2 L_Y X)^C + \gamma L_Y L_X F^2 \right. \\ &\quad \left. - \left(F^2(L_Y X)\right)^C - \gamma\gamma L_Y L_X F^2 + \gamma L_X L_Y F^2\right\}\end{aligned}$$

$$= \lambda^r \left\{ \left((L_Y F^2) X \right)^C + \gamma L_X L_Y F^2 \right\}$$

ii. $$\left(L_{Y^C} (F^4)^C \right) \omega^V = L_{Y^C} (F^4)^C \omega^V - (F^4)^C L_{Y^C} \omega^V$$
$$= \lambda^r \left\{ L_{Y^C} (F^2)^C \omega^V - (F^2)^C (L_Y \omega)^V \right\}$$
$$= \lambda^r \left\{ (L_Y \omega \cdot F^2)^V - \left((L_Y \omega) \cdot F^2 \right)^V \right\}$$
$$= \lambda^r \left(\omega \cdot (L_Y F^2) \right)^V$$

2.3. THE PURITY CONDITIONS OF SASAKIAN METRIC WITH RESPECT TO $(F^4)^C$ ON $T^*(M^n)$

Definition 3. A Sasakian metric s_g is defined on $T^*(M^n)$ by the three equations

$$s_g(\omega^V, \theta^V) = \left(g^{-1}(\omega, \theta) \right)^V = g^{-1}(\omega, \theta) \cdot \pi \tag{2.5}$$

$$s_g(\omega^V, Y^H) = 0 \tag{2.6}$$

$$s_g(X^H, Y^H) = \left(g(X, Y) \right)^V = g(X, Y) \cdot \pi. \tag{2.7}$$

For each $x \in M^n$ the scalar product $g^{-1} = \left(g^{ij} \right)$ is defined on the cotangent space $\pi^{-1}(x) = T_x^*(M^n)$ by

$$g^{-1}(\omega, \theta) = g^{ij} \omega_i \theta_j \tag{2.8}$$

where $X, Y \in \Im^1_0(M^n)$ and $\omega, \theta \in \Im^1_0(M^n)$. Since any tensor field of type (0,2) on $T^*(M^n)$ is completely determined by its action on vector fields of type X^H and ω^V (see [4], pp.280), it follows that s_g is completely determined by equations (2.5), (2.6) and (2.7).

Theorem 5. Let $T^*(M^n), s_g$ be the cotangent bundle equipped with Sasakian metric s_g and a tensor field $(F^4)^C$ of type (1,1) defined by (1.12). Sasakian metric s_g is pure with respect to $(F^4)^C$ if acts as Hsu-structure operator $(F^2 = \lambda^r I)$ on M and $\nabla F^2 = 0$. (I = identity tensor field of type (1,1).

Proof: We put

$$S = (\tilde{X}, \tilde{Y}) = s_g\left((F^4)^C \tilde{X}, \tilde{Y} \right) - s_g\left(\tilde{X}, (F^4)^C \tilde{Y} \right)$$

If $S(\tilde{X}, \tilde{Y}) = O$, for all vector fields $\tilde{X}$ and $\tilde{Y}$ which are of the form $\omega^V \theta^V$ or X^H, Y^H, then $S = 0$. By virtue of $(F^4)^C - \lambda^r (F^2)^C = 0$ and (2.5), (2.6), (2.7), we get

i. $$S(\omega^V, \theta^V) = s_g((F^4)^C \omega^V, \theta^V) - s_g(\omega^V, (F^4)^C \theta^V)$$
$$= s_g((\lambda^r F^2)^c \omega^V, \theta^V) - s_g(\omega^V, (F^4)^C \theta^V)$$
$$= \lambda^r \left(s_g((\omega \cdot F^2)^V, \theta^V) - s_g(\omega^V, (\theta \cdot F^2)^V) \right.$$
$$= \lambda^r \left(\left(g^{-1}\left((\omega \cdot F^2) \theta \right) \right)^V - \left(g^{-1}\left(\omega, (\theta \cdot F^2) \right) \right)^V \right)$$

ii. $$S(X^H \theta^V) = s_g((F^4)^C X^H, \theta^H) - s_g(X^H, (F^4)^C \theta^V)$$

$$= s_g\left(((\lambda^r(F^2)^C X^H, \theta^V) - s_g\left(X^H, (\lambda^r(F^2))^C\theta^V\right)\right.$$
$$= \lambda^{rs} g((F^2X)^H, \theta^V) + \lambda^r\left(s_g((\rho[\nabla F^2]_X)^V\theta^V)\right)$$
$$= \lambda^r\left(g^{-1}\left((\rho[\nabla F^2]_X), \theta\right)\right)^V,$$

where $\nabla_X F + F(\nabla_X) - \nabla FX = [\nabla F]_X$ (see [4], pp.279).

iii. $$S(X^HY^H) = s_g((F^4)^C X^H, Y^H) - s_g(X^H, (F^4)^C Y^H)$$
$$= \lambda^r\left\{s_g((F^2X)^H, Y^H) + s_g\left((\rho[\nabla F^2]_X)\right)^V, Y^H\right)$$
$$\left. -s_g(X^H, (F^2Y)^H) - s_g\left(X^H, (\rho[\nabla F^2]_Y)\right)^V\right\}$$
$$= \lambda^r\left(\left(g\left((F^2X), Y\right)\right)^V - \left(g\left(X, (F^2Y)\right)\right)^V\right),$$

where $F^C X^H = (FX)^H + \gamma([\nabla F]_X)$ for all $X^H \in \Im_0^1\left(T^*(M^n)\right), F^C \in \Im_1^1\left(T^*(M^n)\right)$ and $[\nabla F]_X \in \Im_1^1(M^n)$ (see [4], pp.279).

2.4. THE STRUCTURE $(F^4)^H - \lambda^r(F^2)^H = 0$ ON COTANGENT BUNDLE

In this section, we find the integrability conditions by calculating Nijenhuis tensors of the horizontal lifts of Hsu – (4,2) structure. Later, we get the results of Tachibana operators applied to vector and covector fields according to the horizontal lifts of Hsu – (4,2) structure in cotangent bundle $T^*(M^n)$. Finally, we have studied the purity conditions of Sasakian metric with respect to the lifts of Hsu – (4,2) structure.

Theorem 6. The Nijenhuis tensor $N_{(F^4)^H(F^4)^H}(X^HY^H)$ of the horizontal lift F^4 vanishes if F acts as *Hsu*- structure on M.

Proof: The Nijenhuis tensor $N(X^HY^H)$ for the horizontal lift of F^4 is given by

$$N_{(F^4)^H(F^4)^H}(X^HY^H) = [(F^4)^H X^H, (F^4)^H Y^H] - (F^4)^H[(F^4)^H X^H, Y^H]$$
$$-(F^4)^H[X^H, (F^4)^H Y^H] + (F^4)^H(F^4)^H[X^H, Y^H]$$

$$= \lambda^{2r}\{[(F^2)^H X^H, (F^2)^H Y^H] - (F^2)^H[(F^2)^H X^H, Y^H]$$
$$-(F^2)^H[X^H, (F^2)^H Y^H] + (F^2)^H(F^2)^H[X^H, Y^H]\}$$

$$= \lambda^{2r}\left\{\{[F^2X, F^2Y] - F^2[(F^2X), Y] + F^2[X, F^2Y]\right.$$
$$+F^4[X, Y]\}^H + \gamma\left\{R(F^2X, F^2Y) - R\left((F^2X), Y\right)F^2\right.$$
$$\left.\left. -R(X, F^2Y)F^2 + R(X, Y)F^4\right\}\right\}$$

Let us suppose that F acts as Hsu – structure on M [11]. Then

$$F^2 = \lambda^r I. \tag{2.9}$$

Thus the equation becomes

$$N_{(F^4)^H(F^4)^H}(X^H Y^H) = \lambda^{4r}\{\{[X,Y]-[X,Y]+[X,Y]\}^H$$
$$+\gamma\{R(X,Y)-R(X,Y)-R(X,Y)+R(X,Y)\}.$$

Therefore, it follows

$$N_{(F^4)^H,(F^4)^H}(X^H Y^H) = 0$$

Theorem 7. The Nijenhuis tensor $N_{(F^4)^H(F^4)^H}(X^H\omega^V)$ of the horizontal lift F^4 vanishes if $\nabla F^2 = 0$.

Proof:

$$N_{(F^4)^H(F^4)^H}(X^H\omega^V) = [(F^4)^H X^H, (F^4)^H\omega^V] - (F^4)^H[(F^4)^H X^H, \omega^V]$$
$$-(F^4)^H[X^H, (F^4)^H\omega^V] + (F^4)^H(F^4)^H[X^H, \omega^V]$$

$$= \lambda^{2r}\Big\{\big(\nabla_{F^2X}(\omega\cdot F^2)\big)^V - \big((\nabla_{F^2X})\cdot F^2\big)^V$$
$$-\Big(\big(\nabla_X(\omega\cdot F^2)\big)\cdot F^2\Big)^V + \big((\nabla_X\omega)\cdot F^4\big)^V\Big\}$$

$$= \lambda^{2r}\{(\omega\cdot(\nabla_{F^2X}F^2) - (\omega\cdot(\nabla_X F^2)F^2\}^V$$

where $F\in\Im_1^1(M), X\in\Im_0^1(M), \omega\in\Im_1^0(M)$. The theorem is proved.

Theorem 8. The Nijenhuis tensor $N_{(F^4)^H(F^4)^H}(\omega^V,\theta^V)$ of the horizontal lift F^4 vanishes.

Proof:

$$N_{(F^4)^H(F^4)^H}(\omega^V,\theta^V) = [(F^4)^H\omega^V, (F^4)^H\theta^V] - (F^4)^H[(F^4)^H\omega^H, \theta^V]$$
$$-(F^4)^H[\omega^V, (F^4)^H\theta^V] + (F^4)^H(F^4)^H[\omega^H, \theta^V]$$

$$= \lambda^{2r}\{[(\omega\cdot F^2)^V, (\theta\cdot F^2)^V] - (F^2)^H[(\omega\cdot F^2)^V, \theta^V]$$
$$-(F^2)^H[\omega^V, (\theta\cdot F^2)^V] + (F^2)^H(F^2)^H[\omega^V, \theta^V]$$

Because of $[\omega^V,\theta^V]=0$ and $\omega\cdot F^2\in\Im_0^1(M^n)$ on $T^*(M^n)$, the equation becomes

$$N_{(F^4)^H(F^4)^H}(\omega^V,\theta^V) = 0$$

Theorem 9. Let $(F^4)^H$ be a tensor field of type (1,1) on $T^*(M^n)$. If the Tachibana operator $\emptyset_\varphi$ applied to vector and covector fields according to horizontal lifts of F^4 defined by (1.16) on $T^*(M^n)$, then we get the following results.

i. $\emptyset_{(F^4)^H X^H} Y^H = \lambda^r\Big\{-\big((L_Y F^2)X\big)^H - \big(\rho R(Y,F^2X)\big)^V + \Big(\big(\rho R(Y,X)\big)F^2\Big)^V\Big\},$

ii. $\emptyset_{(F^4)^H X^H}\omega^V = \lambda^r\Big\{(\nabla_{F^2X}\omega)^V - \big((\nabla_X\omega)\cdot F^2\big)^V\Big\}$

iii. $\emptyset_{(F^4)^H\omega^V} X^H = -\lambda^r\big(\omega\cdot(\nabla_X F^2)\big)^V,$

iv. $\emptyset_{(F^4)^H\omega^V}\theta^V = 0,$

where horizontal lifts $X^H, Y^H \in \Im_0^1(T^*(M^n))$ of $X, Y \in \Im_0^1(M^n)$ and the vertical lift $\omega^V, \theta^V \in \Im_0^1(T^*(M^n))$ of $\omega, \theta \in \Im_0^1(M^n)$ are given, respectively.

Proof:

i.
$$\begin{aligned}\emptyset_{(F^4)^H X^H} Y^H &= -(L_Y H (F^4)^H) X^H \\ &= -L_{Y^H}(F^4)^H X^H - (F^4)^H L_{Y^H} X^H \\ &= \lambda^r \Big\{ -\big((L_Y F^2)X\big)^H - \big(\rho R(Y, F^2 X)\big)^V \\ &\quad + \Big(\big(\rho R(Y,X)\big)F^2\Big)^V \Big\},\end{aligned}$$

ii.
$$\begin{aligned}\emptyset_{(F^4)^H X^H} \omega^V &= -\big(L_{\omega^V}(F^4)^H\big) X^H \\ &\quad -L_{\omega^V}(F^4)^H X^H + (F^4)^H L_{\omega^V} X^H \\ &= -\lambda^r L_{\omega^V}(F^2 X)^H - \lambda^r (F^2)^H (\nabla_X \omega)^V \\ &= \lambda^r \Big\{ (\nabla_{F^2 X}\omega)^V - \big((\nabla_X \omega)\cdot F^2\big)^V \Big\},\end{aligned}$$

iii.
$$\begin{aligned}\emptyset_{(F^4)^H \omega^V} X^H &= -(L_{X^H}(F^4)^H)\omega^V \\ &= -\lambda^r \big(\nabla_X(\omega \cdot F^2)\big)^V + \lambda^r \big((\nabla_X \omega)\cdot F^2\big)^V \\ &= -\lambda^r \big(\omega \cdot (\nabla_X F^2)\big)^V\end{aligned}$$

iv.
$$\begin{aligned}\emptyset_{(F^4)^H \omega^V} \theta^V &= -\big(L_{\theta^V}(F^4)^H\big)\omega^V \\ &= -L_{\theta^V}(F^4)^H \omega^V + (F^4)^H L_{\theta^V}\omega^V \\ &= 0\end{aligned}$$

Theorem 10. Let $\big(T^*(M^n),^s g\big)$ be the cotangent bundle equipped with Sasakian metric s_g and a tensor field $(F^4)^H$ of type (1,1) defined by (1.16). Sasakian metric s_g is pure with respect to $(F^4)^H$ if $F^2 = \lambda^r I$ ($I =$ identity tensor field of type (1,1).

Proof: We put

$$S = (\tilde{X}, \tilde{Y}) =^s g\big((F^4)^H \tilde{X}, \tilde{Y}\big) -^s g\big(\tilde{X}, (F^4)^H \tilde{Y}\big).$$

If $S(\tilde{X}, \tilde{Y}) = 0$ for all vector fields $\tilde{X}$ and $\tilde{Y}$ which are of the form ω^V, θ^V or X^H, Y^H, then $S = 0$. By virtue of $(F^4)^H - \lambda^r (F^2)^H = 0$ and (2.5), (2.6),(2.7), we get

i.
$$\begin{aligned}S(\omega^V, \theta^V) &= s_g((F^4)^H \omega^V, \theta^V) - s_g(\omega^V, (F^4)^H \theta^V) \\ &= s_g((\lambda^r F^2)^H \omega^V, \theta^V) - s_g(\omega^V, (\lambda^r F^2)^H \theta^V) \\ &= \lambda^r \Big(s_g((\omega \cdot F^2)^V, \theta^V) - s_g(\omega^V, (\omega \cdot F^2)^V) \Big).\end{aligned}$$

ii.
$$\begin{aligned}S(X^H, \theta^V) &= s_g((F^4)^H X^H, \theta^V) - s_g(X^H, (F^4)^H \theta^V) \\ &= s_g((\lambda^r F^2)^H X^H, \theta^V) - s_g(X^H, (\lambda^r F^2)^H \theta^V) \\ &= \lambda^r \Big(s_g((F^2 X)^V, \theta^V) - s_g(X^H, (\omega \cdot F^2)^V) \Big) \\ &= 0.\end{aligned}$$

iii. $$S(X^H,Y^H) = s_g((F^4)^H X^H, Y^H) - s_g(X^H, (F^4)^H Y^H)$$
$$= s_g((\lambda^r F^2)^H X^H, Y^H) - s_g(X^H, (\lambda^r F^2)^H Y^H)$$
$$= \lambda^r \left(s_g((F^2X)^H, Y^H) - s_g(X^H, (F^2Y)^V)\right).$$

Thus, $F^2 = \lambda^r I$, then s_g is pure with respect to $(F^4)^H$.

2.5. THE STRUCTURE $(F^4)^C - \lambda^r (F^2)^C = 0$ *ON TANGENT BUNDLE* $T(M^n)$

Let M^n be an n – dimensional differentiable manifold of class C^∞. Suppose there exist on M^n a tensor field $F(\neq 0)$ of type (1,1) satisfying

$$F^4 - \lambda^r F^2 = 0,$$

Where λ is complex number not equal to zero and r some finite integer. In such a manifold M^n, let us put

$$l = F^2/\lambda^r \, and \; m = I - F^2/\lambda^r,$$

Where I denotes the unit tensor field. Then it is easy to show

$$l^2 = l, m^2 = m, l + m = I, lm = ml = 0.$$

Thus, the operators *l* and *m* when applied to the tangent space of M^n at a point are complementary projection operators. Hence there exist complementary distributions L^* and M^* corresponding to the projection operators *l* and *m* respectively. If the rank of *F* is constant everywhere and equal to *r*, the dimensions of L^* and M^* are *r* and $(n - r)$ respectively. Let us call such a structure as $Hsu - (4,2)$ structure of rank *r*.

Let F_i^h be the component of *F* at *A* in the coordinate neighbourhood *U* of M^n. Then the complete lift F^C of *F* is also a tensor field of type (1,1) in $T^*(M^n)$ whose components $\tilde{F}_B^A$ in $\pi^{-1}(U)$ are given by

$$F^C = \begin{pmatrix} F_i^h & 0 \\ \partial F_i^h & F_i^h \end{pmatrix}. \tag{2.10}$$

Let $F, G \in \Im_1^1(M^n)$ then we have

$$(FG)^C = F^C G^C. \tag{2.11}$$

Putting $F = G$ we obtain

$$(F^2)^C = (F^C)^2. \tag{2.12}$$

Putting $G = F^2$ in (2.11) and making use of (2.12) we get

$$(F^3)^C = (F^C)^3. \tag{2.13}$$

Continuing the above process of replacing *G* in equation (2.11) by some higher degree of *F* we obtain

$$(F^4)^C = (F^C)^4. \tag{2.14}$$

Taking complete lift on both sides of equation (1.1) we get

$$(F^4)^C - \lambda^r (F^2)^C = 0. \tag{2.15}$$

which in view of the equation (2.14) gives

$$(F^4)^C - \lambda^r (F^2)^C = 0. \tag{2.16}$$

Thus the complete lift F^C of F also has $Hsu-(4,2)$ structure in the tangent bundle $T(M^n)$.

Definition 4 Let X and Y be any vector fields on a Riemannian manifold (M^n, g), we have [4]

$$[X^H, Y^V] = [X,Y]^H - (R(X,Y)u)^V,$$

$$[X^H, Y^V] = (\nabla_X Y)^V,$$

$$[X^H, Y^V] = 0,$$

Where R is the Riemannian curvature tensor of g defined by

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

In particular, we have the vertical spray u^V and the horizontal spray u^H on $T(M^n)$ defined by

$$u^V = u^i(\partial_i)^V = u^i\partial_i - u^H = u^i(\partial_i)^H = u^i\delta_i,$$

where $\delta_i = \partial_i - u^j \Gamma^{s}_{ji}\partial_s -$. u^V is also called the canonical or Liouville vector field on $T(M^n)$.

Theorem 11. The Nijenhuis tensor $N_{(F^4)^C(F^4)^C}(X^C Y^C)$ of the complete lift of F^4 vanishes if the Nijenhuis tensor of the F^2 is zero.

Proof: In consequence of Definition 1 the Nijenhuis tensor of $(F^4)^C$ is given by

$$\begin{aligned} N_{(F^4)^C(F^4)^C}(X^C Y^C) &= [(F^4)^C X^C, (F^4)^C Y^C] - (F^4)^C[(F^4)^C X^C Y^C] \\ &\quad -(F^4)^C[X^C, (F^4)^C Y^C] + (F^4)^C (F^4)^C [X^C, Y^C] \\ &= \lambda^{2r}\{[(F^2X)^C, (F^2Y)^C] - (F^2)^C[(F^2X)^C, Y^C] \\ &\quad -(F^2)^C[X^C, (F^2Y)^C] + (F^2)^C(F^2)^C[X^C, Y^C]\} \\ &= \lambda^{2r}\{[F^2X, F^2Y] - F^2[F^2X, Y] - F^2[X, F^2Y] + F^4[X,Y]\}^C \\ &= \lambda^{2r} N_{F^2F^2}(X,Y)^C \end{aligned}$$

Theorem 12. The Nijenhuis tensor $N_{(F^4)^C(F^4)^C}(X^C Y^V)$ of the complete lift of F^4 vanishes if the Nijenhius tensor F^2 is zero.

Proof:

$$N_{(F^4)^C(F^4)^C}(X^CY^V) = [(F^4)^C X^C, (F^4)^C Y^V] - (F^4)^C[(F^4)^C X^C Y^V] -(F^4)^C[X^C, (F^4)^C Y^V] + (F^4)^C (F^4)^C [X^C, Y^V]$$

$$= \lambda^{2r}\{[(F^2X)^C, (F^2Y)^V] - (F^2)^C[(F^2X)^C, Y^V] -(F^2)^C[X^C, (F^2Y)^V] + (F^4)^C[X,Y]^V\}$$

$$= \lambda^{2r}\{[F^2X, F^2Y] - (F^2[F^2X,Y])^V -(F^2[X,F^2Y])^V - (F^4[X,Y])^V\}$$

$$= \lambda^{2r} N_{F^2F^2}(X,Y)^V$$

Theorem 13. The Nijenhuis tensor $N_{(F^4)^C(F^4)^C}(X^VY^V)$ of the complete lift of F^4 vanishes.

Proof:

$$N_{(F^4)^C(F^4)^C}(X^VY^V) = [(F^4)^C X^V, (F^4)^C Y^V] - (F^4)^C[(F^4)^C X^V Y^V] -(F^4)^C[X^V, (F^4)^C Y^V] + (F^4)^C (F^4)^C [X^V, Y^V]$$

$$= \lambda^{2r}\{[(F^2X)^V, (F^2Y)^V] - (F^2)^C[(F^2X)^V, Y^V] -(F^2)^C[X^V, (F^2Y)^V] + (F^4)^C[X,Y]^V\}$$

$$= 0$$

2.6. THE PURITY CONDITIONS OF SASAKIAN METRIC WITH RESPECT TO $(F^4)^C$ ON $T(M^n)$

Definition 5. The Sasaki metric s_g is a (positive definite) Riemannian metric on the tangent bundle $T(M^n)$ which is derived from the given Riemannian metric on M as follows:

$$s_g(X^H, Y^H) = g(X,Y), \tag{2.17}$$

$$s_g(X^H, Y^H) = s_g(X^V, Y^H) = 0$$

$$s_g(X^V, Y^V) = g(X,Y)$$

for all $X, Y \in \Im_0^1(M^n)$.

Theorem 14 The Sasaki metric s_g is pure with respect to $(F^4)^C$ if $\nabla F^2 = 0$ and $F^2 = \lambda^r I$, where I = identity tensor field of type (1,1).

Proof: $S(\tilde{X}, \tilde{Y}) = {}^s g((F^4)^C \tilde{X}, \tilde{Y}) - {}^s g(\tilde{X}, (F^4)^C \tilde{Y})$ if $S(\tilde{X}, \tilde{Y}) = 0$ for all vector fields $\tilde{X}$ and $\tilde{Y}$ which are of the form X^V, Y^V or X^H, Y^H then $S = 0$.

i. $$S(X^V, Y^V) = s_g((F^4)^C X^V, Y^V) - s_g(X^V, (F^4)^C Y^V)$$
$$= \lambda^r\{s_g((F^2X)^V, Y^V) - s_g(X^V, (F^2Y)^V)\}$$
$$= \lambda^r\left\{\left(g(F^2X,Y)\right)^V - \left(g(X,F^2Y)\right)^V\right\}$$

ii. $S(X^V, Y^H) = s_g((F^4)^C X^V, Y^H) - s_g(X^V, (F^4)^C Y^H)$
$= \lambda^r s_g\big((X^V, (F^2Y)^H) + (\nabla_r F^2)Y^H\big)$
$= -\lambda^r s_g(X^V, (\nabla_r F^2)Y^H)$
$= -\lambda^r s_g\left(X^V, \left(\big((\nabla F^2)u\big)y\right)^V\right)$
$= -\lambda^r \left(g\big(X, \big((\nabla F^2)u\big)Y\big)^V\right)$

iii. $S(X^H, Y^H) = s_g((F^4)^C X^H, Y^H) - s_g(X^H, (F^4)^C Y^H)$
$= \lambda^r s_g((F^2)^C X^H, Y^H) - \lambda^r s_g(X^H, (F^2)^C Y^H)$
$= \lambda^r s_g((F^2X)^H + (\nabla_r F^2)X^H, Y^H)$
$-\lambda^r s_g(X^H, (F^2Y)^H + (\nabla_r F^2)Y^H)$
$= \lambda^r \left\{g\big((F^2X), Y\big)^V - g\big(X, (F^2Y)\big)^V\right\}$

Theorem 15. Let ϕ_φ be the Tachibana operator and the structure $(F^4)^C - \lambda^r (F^2)^C = 0$ defined by Definition 2 and (2.16), respectively. If $L_Y F^2 = 0$, then all results with respect to $(F^4)^C$ is zero, where $X, Y \in \Im_0^1(M)$, the complete lifts $X^C, Y^C \in \Im_0^1\big(T(M)\big)$, and the vertical lift $X^V, Y^V \in \Im_0^1\big(T(M)\big)$.

i. $\emptyset_{(F^4)^C X^C} Y^C = -\lambda^r \big((L_Y F^2)X\big)^C$

ii. $\emptyset_{(F^4)^C X^C} Y^V = -\lambda^r \big((L_Y F^2)X\big)^V$

iii. $\emptyset_{(F^4)^C X^V} Y^C = -\lambda^r \big((L_Y F^2)X\big)^V$

iv. $\emptyset_{(F^4)^C X^V} Y^V = 0$

Proof:

i. $\emptyset_{(F^4)^C X^C} Y^C = -\big(L_{Y^C}(F^4)^C\big)X^C$
$= \lambda^r \big\{-L_{Y^C}(F^2X)^C + (F^2)^C L_{Y^C} X^C\big\}$
$= -\lambda^r \big((L_Y F^2)X\big)^C$

ii. $\emptyset_{(F^4)^C X^C} Y^V = -(L_{Y^V}(F^4)^C)X^C$
$= -L_{Y^V}(F^4)^C X^C + (F^4)^C L_{Y^V} X^C$
$= \lambda^r \{-L_{Y^V}(F^2X)^C + (F^2)^C L_{Y^V} X^C\}$
$= -\lambda^r \big((L_Y F^2)X\big)^V$

iii. $\emptyset_{(F^4)^C X^V} Y^C = -\big(L_{Y^C}(F^4)^C\big)X^V$
$= -L_{Y^C}(F^4)^C X^V + (F^4)^C L_{Y^C} X^V$
$= \lambda^r \big\{-L_{Y^C}(F^2X)^V + (F^2)^C L_{Y^C} X^V\big\}$
$= -\lambda^r \big((L_Y F^2)X\big)^V$

iv. $\emptyset_{(F^4)^C X^V} Y^V = -(L_{Y^V}(F^4)^C)X^V$
$= -L_{Y^V}(F^4)^C X^V + (F^4)^C L_{Y^V} X^V$
$= 0$

Theorem 16. If $L_Y F^2 = 0$ for $Y \in M$, then its complete lift Y^C to the tangent bundle is an almost holomorfic vector field with respect to the structure $(F^4)^C - \lambda^r (F^2)^C = 0$.

Proof:

i. $$\left(L_{Y^C}(F^4)^C\right)X^C = L_{Y^C}(F^4)^C X^C - (F^4)^C L_{Y^C} X^C$$
$$= \lambda^r\left\{L_{Y^C}(F^2X)^C - (F^2)^C L_{Y^C} X^C\right\}$$
$$= \lambda^r\left((L_Y F^2)X\right)^C$$

ii. $$(L_{Y^V}(F^4)^C)X^V = L_{Y^C}(F^4)^C X^V - (F^4)^C L_{Y^C} X^V$$
$$= \lambda^r\left\{L_{Y^C}(F^2X)^V - (F^2)^C L_{Y^C} X^V\right\}$$
$$= \lambda^r\left((L_Y F^2)X\right)^V$$

2.7. THE STRUCTURE $(F^4)^H - \lambda^r (F^2)^H = 0$ ON TANGENT BUNDLE $T(M^n)$.

Let F_i^h be the component of F at A in the coordinate neighbourhood U of M^n. Then the horizontal lift F^H of F is also a tensor field of type (1,1) in $T(M^n)$ whose components $\tilde{F}_B^A$ in $\pi^{-1}(U)$ are given by

$$F^H = F^C - \gamma(\nabla F) = \begin{pmatrix} F_i^h & 0 \\ -\Gamma_t^h F_i^t + \Gamma_i^t F_t^h & F_i^h \end{pmatrix}. \qquad (2.18)$$

Let F, G be two tensor fields of type (1,1) on the manifold M. If F^H denotes the horizontal lift of F, we have

$$(FG)^H = F^H G^H \qquad (2.19)$$

Taking F and G identical, we get

$$(F^H)^2 = (F^2)^H \qquad (2.20)$$

Multiplying both sides by F^H and making use of the same (2.20), we get

$$(F^H)^3 = (F^3)^H$$

and so on. Thus it follows that

$$(F^H)^4 = (F^4)^H \qquad (2.21)$$

Taking horizontal lift on both sides of equation $F^4 - \lambda^r F^2 = 0$ we get

$$(F^4)^H - \lambda^r (F^2)^H = 0 \qquad (2.22)$$

view of (2.21), we can write

$$(F^H)^4 - \lambda^r (F^H)^2 = 0 \qquad (2.23)$$

Thus the horizontal lift F^H of F also has $Hsu-(4,2)$ structure in the tangent bundle $T(M^n)$.

Theorem 17. The Nijenhuis tensor $N_{(F^4)^H(F^4)^H}(X^H,Y^H)$ of the horizontal lift of F^4 vanishes if the Nijenhuis tensor of the F^2 is zero and

$$\Big\{-\big(\hat{R}(F^2X,F^2Y)u\big)+\Big(F^2\big(\hat{R}(F^2X,Y)u\big)\Big)-(F^2)^2\big(\hat{R}(X,Y)u\big)\Big\}^V=0.$$

Proof:

$$\begin{aligned}N_{(F^4)^H(F^4)^H}(X^H,Y^H)&=[(F^4)^HX^H,(F^4)^HY^H]-(F^4)^H[(F^4)^HX^H,Y^H]\\&\quad-(F^4)^H[X^H,(F^4)^HY^H]+(F^4)^H(F^4)^H[X^H,Y^H]\\&=\lambda^r\{[(F^2X)^H,(F^2Y)^H]-(F^2)^H[(F^2X)^H,Y^H]\\&\quad-(F^2)^H[X^H(F^2Y)^H]+(F^2)^H(F^2)^H[X^H,Y^H]\}\\&=\lambda^{2r}\{([F^2X,F^2Y]-(F^2)[F^2X,Y]\\&\quad-(F^2)^H[X,F^2Y]-(F^2)(F^2)[X,Y]^H)\\&\quad-\big(\hat{R}(F^2X,F^2Y)u\big)^V+\Big(F^2\big(\hat{R}(F^2X,Y)u\big)\Big)^V\\&\quad+\Big(F^2\big(\hat{R}(X,F^2Y)u\big)\Big)^V-\Big((F^2)^2\Big(\hat{R}(X,Y)\Big)u\Big)^V\Big\}\\&=\lambda^{2r}\Big\{\big(N_{F^2F^2}(X,Y)\big)^H-\big(\hat{R}(F^2X,F^2Y)u\big)^V\\&\quad+\Big(F^2\big(\hat{R}(F^2X,Y)u\big)\Big)^V+\Big(F^2\big(\hat{R}(X,F^2Y)u\big)\Big)^V\\&\quad-\Big((F^2)^2\big(\hat{R}(X,Y)u\big)\Big)^V\Big\}.\end{aligned}$$

If $N_{F^2F^2}(X,Y)=0$ and

$$\Big\{-\big(\hat{R}(F^2X,F^2Y)u\big)+\Big(F^2\big(\hat{R}(F^2X,Y)u\big)\Big)+\Big(F^2\big(\hat{R}(X,F^2Y)u\big)\Big)-(F^2)^2\big(\hat{R}(X,Y)u\big)\Big\}^V=0.$$

then we get $N_{(F^4)^H(F^4)^H}(X^H,Y^H)=0$, where $\hat{R}$ denotes the curvature tensor of the affine connection $\widehat{\nabla}$ defined by $\widehat{\nabla}_XY=\widehat{\nabla}_XY+[X,Y]$ (see [4], pp.88-89).

Theorem 18. The Nijenhuis tensor $N_{(F^4)^H(F^4)^H}(X^H,Y^H)$ of the horizontal lift of F^4 vanishes if the Nijenhuis tensor of the F^2 is zero and $\nabla F^2=0$.

Proof:

$$\begin{aligned}N_{(F^4)^H(F^4)^H}(X^H,Y^V)&=[(F^4)^HX^H,(F^4)^HY^V]-(F^4)^H[(F^4)^HX^H,Y^V]\\&\quad-(F^4)^H[X^H,(F^4)^HY^V]+(F^4)^H(F^4)^H[X^H,Y^V]\\&=\lambda^{2r}\{[F^2X,F^2Y]^V-(F^2[F^2X,Y])^V-(F^2[X,F^2Y])^V\\&\quad+((F^2)^2[X,Y])^V+(\nabla_{F^2Y}F^2X)^V-\big(F^2(\nabla_YF^2X)\big)^V\\&\quad-\big(F^2(\nabla_{F^2Y}X)\big)^V+((F^2)^2\nabla_YX)^V\Big\}\end{aligned}$$

$$= \lambda^{2r}\left(N_{F^2F^2}(X,Y)\right)^V + \left((\nabla_{F^2Y}F^2)X\right)^V - \left(F^2\left((\nabla_Y F^2)X\right)\right)^V\Big\}$$

Theorem 19. The Nijenhuis tensor $N_{(F^4)^H(F^4)^H}(X^V,Y^V)$ of the horizontal lift of F^4 vanishes.

Proof:

$$N_{(F^4)^H(F^4)^H}(X^V,Y^V) = [(F^4)^H X^V,(F^4)^H Y^V] - (F^4)^H[(F^4)^H X^V, Y^V]$$
$$-(F^4)^H[X^V,(F^4)^H Y^V] + (F^4)^H(F^4)^H[X^V,Y^V]$$

$$= \lambda^{2r}\{[(F^2X)^V + (F^2Y)^V] - (F^2)^H[(F^2X)^V, Y^V]$$
$$-(F^2)^H[X^V,(F^2Y)^V] + ((F^2)^2)^H[X^V,Y^V]\}$$

Because of $[X^V,Y^V] = 0$ for $X,Y \in M$, we have $N_{(F^4)^H(F^4)^H}(X^V,Y^V) = 0$.

The theorem is proved.

Theorem 20. The Sasakian metric s_g is pure with respect to $(F^4)^H$ if $F^2 = \lambda^r I$, where I = identity tensor field of type (1,1).

Proof: $S(\tilde{X},\tilde{Y}) = {}^s g((F^4)^H\tilde{X},\tilde{Y}) - {}^s g(\tilde{X},(F^4)^H\tilde{Y})$ if $S(\tilde{X},\tilde{Y}) = 0$ for all vector fields $\tilde{X}$ and $\tilde{Y}$ which are of the form X^V, Y^V or X^H, Y^H then $S = 0$

i. $$S(X^V,Y^V) = s_g((F^4)^H X^V, Y^V) - s_g(X^V,(F^4)^H Y^V)$$
$$= \lambda^r\{s_g((F^2X)^V, Y^V) - s_g(X^V,(F^2Y)^V)\}$$
$$= \lambda^r\left\{\left(g(F^2X,Y)\right)^V - \left(g(X,F^2Y)\right)^V\right\}$$

ii. $$S(X^V,Y^H) = s_g((F^4)^H X^V, Y^H) - s_g(X^V,(F^4)^H Y^H)$$
$$= -\lambda^r g(X^V,(F^2Y)^H)$$
$$= 0$$

iii. $$S(X^H,Y^H) = s_g((F^4)^H X^H, Y^H) - s_g(X^H,(F^4)^H Y^H)$$
$$= \lambda^r\{\left(s_g(F^2X)^H, Y^H\right) - s_g(X^H,(F^2Y)^H)\}$$
$$= \lambda^r\left\{(g(F^2X),Y)^V - \left(g(X,(F^2Y)^H)\right)^V\right\}$$

Theorem 21. Let $\emptyset_\varphi$ be the Tachibana operator and the structure $(F^4)^H - \lambda^r(F^2)^H = 0$ defined by Definition 2 and (2.22), respectively. if $L_Y F^2 = 0$ and $F^2 = \lambda^r I$, then all results with respect to $(F^4)^H$ is zero, where $X,Y \in \Im_0^1(M)$, the horizontal lifts $X^H, Y^H \in \Im_0^1(T(M^n))$ and the vertical lift $X^V, Y^V \in \Im_0^1(T(M^n))$.

i. $\emptyset_{(F^4)^H X^H} Y^H = -\lambda^r\left\{-\left((L_Y F^2)X\right)^H + \left(\hat{R}(Y,F^2X)u\right)^V - \left(F^2\left(\hat{R}(Y,X)u\right)\right)^V\right\}$

ii. $\emptyset_{(F^4)^H X^H} Y^V = \lambda^r\left\{-\left((L_Y F^2)X\right)^V + \left((\nabla_Y F^2)X\right)^V\right\}$

iii. $\emptyset_{(F^4)^H X^V} Y^H = \lambda^r\left\{-\left((L_Y F^2)X\right)^V - (\nabla_{F^2X}Y)^V + \left(F^2(\nabla_X Y)\right)^V\right\}$,

iv. $\emptyset_{(F^4)^H X^H} Y^V = 0$,

Proof:

i. $$\emptyset_{(F^4)^H X^H} Y^H = -(L_{Y^H}(F^4)^H)X^H$$
$$= -L_{Y^C}(F^4)^H X^H + (F^4)L_{Y^H}X^H$$
$$= -\lambda^r[Y, F^2X]^H + \lambda^r \gamma \hat{R}[Y, F^2X]$$
$$+\lambda^r(F^2[Y, X])^H - \lambda^r(F^2)^H\left(\hat{R}(Y, X)u\right)^V$$
$$= -\lambda^r\left\{-\left((L_Y F^2)X\right)^H + \left(\hat{R}(Y, F^2X)u\right)^V - \left(F^2\left(\hat{R}(Y, X)u\right)\right)^V\right\}$$

ii. $$\emptyset_{(F^4)^H X^H} Y^V = -(L_{Y^V}(F^4)^H)X^H$$
$$= -L_{Y^V}(F^4X)^H + (F^4)^H L_{Y^V}X^H$$
$$= -\lambda^r[Y, F^2X]^V + \lambda^r(\nabla_Y F^2X)^V + \lambda^r(F^2[Y, X])^V - \lambda^r\left(F^2(\nabla_Y X)\right)^V$$
$$= \lambda^r\left\{-\left((L_Y F^2)X\right)^V + \left((\nabla_Y F^2)X\right)^V\right\}$$

iii. $$\emptyset_{(F^4)^H X^V} Y^H = -(L_{Y^H}(F^4)^H)X^V$$
$$= -L_{Y^H}(F^4X)^V + (F^4)^H L_{Y^H}X^V$$
$$= -\lambda^r[Y, F^2X]^V - \lambda^r(\nabla_{F^2X}Y)^V + \lambda^r(F^2[Y, X])^H + \lambda^r\left(F^2(\nabla_X Y)\right)^V$$
$$= \lambda^r\left\{-\left((L_Y F^2)X\right)^V - (\nabla_{F^2X}Y)^V + \left(F^2(\nabla_X Y)\right)^V\right\}$$

iv. $$\emptyset_{(F^4)^H X^H} Y^V = -(L_{Y^V}(F^4)^H)X^V$$
$$= -\lambda^r L_{Y^V}(F^2X)^V + \lambda^r(F^2)^H L_{Y^V}X^V$$
$$= 0$$

Theorem 22. If $L_Y F^2 = 0$ and $F^2 = \lambda^r I$ for $Y \in M$, then its horizontal lift Y^H to the tangent bundle is an almost holomorfic vector field with respect to $(F^4)^H$.

Proof:

i. $$(L_{Y^H}(F^4)^H)X^H = L_{Y^H}(F^4)^H X^H - (F^4)^H L_{Y^H}X^H$$
$$= \lambda^r[Y, F^2X]^H - \lambda^r\gamma\hat{R}(Y, F^2X)$$
$$-\lambda^r(F^2[Y, X])^H + \lambda^r(F^2)^H\left(\hat{R}(Y, X)u\right)^V$$
$$= \lambda^r\left\{\left((L_Y F^2)X\right)^H - \left(\hat{R}(Y, F^2X)u\right)^V\right\}$$

ii. $$(L_{Y^H}(F^4)^H)X^V = L_{Y^H}(F^4X)^V - (F^4)^H L_{Y^H}X^V$$
$$= \lambda^r[Y, F^2X]^V - \lambda^r(\nabla_{F^2X}Y)^V - \lambda^r(F^2[Y, X])^V - \lambda^r\left(F^2(\nabla_X Y)\right)^V$$
$$= \lambda^r\left\{\left((L_Y F^2)X\right)^H + (\nabla_{F^2X}Y)^V\left(F^2(\nabla_X Y)\right)^V\right\}$$

3. CONCLUSIONS

In this paper, firstly, obtained the integrability conditions by calculating Nijenhuis tensors of the complete and horizontal lifts of $Hsu - (4{,}2)$ structure, the results of Tachibana operators applied to vector and covector fields according to the complete and horizontal lifts of $Hsu - (4{,}2)$ structure and the conditions of almost holomorfic vector fields in cotangent bundle $T^*(M^n)$, the purity conditions of Sasakian metric with respect to the lifts of $Hsu -$

$(4{,}2)$ structure . In addition, all results obtained in the first section were investigated according to the complete and horizontal lifts of the $Hsu-(4{,}2)$ structure in tangent bundle $T^*(M^n)$.

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