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SOLVABILITY FOR A DIFFERENTIAL SYSTEMS VIA PHI-CAPUTO APPROACH

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Abstract. *In this paper, we discuss the existence and uniqueness of solutions for the coupled system of Phi-Caputo fractional differential equations. An illustrative example is included to show the applicability of our results.*

Keywords: *phi-Caputo derivative; existence of solution; fixed point.*

1. INTRODUCTION

The fractional calculus has a proved to be very important in various scientific fields, see for instance [1-4]. However, most of these research works have been considered by applications of the following fractonal approaches: Riemann–Liouville, Hadamard, Katugampola, and Caputo. But, the approch using ”functions with respect to another function” seem to be absent in the above studies. Such approach can be found, it is different from all the others since its kernel is considered as term of another function ϕ. Recently, some results related to this pproch have been considered in [5-8].

In most of the present articles, Schauder, Krasnoselskii, Darbo, or M*o*nch theories have been used to prove existence of solutions of nonlinear fractional differential equations with some restrictive conditions [9-11].

To cite a some works that have contributed to motivate our present work, we begin by recalling the paper [12], A. Benzidane and Z. Dahmani have considered the following class of nonlinear equations of Lane Emden type:

$$\begin{cases} \mathrm{D}^{\beta_1}\left(\mathrm{D}^{\alpha_1}+g_1(t)\right)u(t)+f_1(t,u(t),v(t),\mathrm{D}^{\delta_1}u(t),\mathrm{D}^{\delta_2}v(t))=h_1(t,u(t),v(t)) \\ \mathrm{D}^{\beta_2}\left(\mathrm{D}^{\alpha_2}+g_2(t)\right)v(t)+f_2(t,u(t),v(t),\mathrm{D}^{\delta_1}u(t),\mathrm{D}^{\delta_2}v(t))=h_2(t,u(t),v(t)) \\ u(0)=a_1, v(0)=a_2, u(1)=v_1, v(1)=b_2, t\in J, \end{cases}$$

where $J=[0,1]$, $0<\alpha_k,\beta_k<1, 0<\delta_k<\alpha_k<1, k=1,$; the functions $f_k:[0,1]\times\mathrm{R}^4\to\mathrm{R}$, $k=1,2$ are continuous, $g_k:(0,1]\to[0,+\infty)$ are continuous functions, singular at $t=0$, and $\lim\limits_{t\to0^+} g_k(t)=\infty$; the operators $\mathrm{D}^{\beta_k},\mathrm{D}^{\alpha_k}$ and D^{δ_k} $k=1,2$ are the derivatives in the sense of Caputo and the constants ak, bk are reals. The authors have studied the existence and

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uniqueness of solutions and the Ulam stability for the considered class.

In [13], A. Taïeb and Z. Dahmani established new existence and uniqueness results for the following problem:

$$\begin{cases} \mathbf{D}^{\alpha}u(t)+\sum_{i=}^{m} f_i(t,u(t),v(t),\mathbf{D}^{\delta_1}u(t),\mathbf{D}^{\delta_2}v(t))=0, \\ \mathbf{D}^{\beta}v(t)+\sum_{i=}^{m} g_i(t,u(t),v(t),\mathbf{D}^{\delta_1}u(t),\mathbf{D}^{\delta_2}v(t))=0, \\ t\in J, u(0)=u_0^*, v(0)=v_0^*, \\ u'(0)=u''(0)=v'(0)=v''(0)=0, \\ u'''(0)=\mathbf{I}^r u(z), v'''(0)=\mathbf{I}^e v(z), r>0, e>0 \end{cases}$$

where $\alpha,\beta\in(3,4),\delta_1,\delta_2\in(0,3), z,w\in(0,1),\mathbf{D}^{\alpha},\mathbf{D}^{\beta},\mathbf{D}^{\delta_1}$ and $\mathbf{D}^{\delta_2}$ denote the Caputo fractional derivatives and $\mathbf{I}^r,\mathbf{I}^e$ denote the Riemann-Liouville fractional integrals, $J:=[0,1]$, $u_0^*,v_0^*\in \mathrm{R}$. For each $i=1,...,m,$ f_i and g_i : $J\times \mathrm{R}^4\to \mathrm{R}$ are specific functions.

In this work we discuss the existence and uniqueness of solutions for the coupled system of $\phi-$ Caputo fractional differential equations,

$$\begin{cases} {}^{c}\mathbf{D}_{a^+}^{\alpha_1;\phi}\left({}^{c}\mathbf{D}_{a^+}^{\alpha_2;\phi}+\mu_1\right)u(t)=f(t,u(t),v(t)) \\ {}^{c}\mathbf{D}_{a^+}^{\alpha_3;\phi}\left({}^{c}\mathbf{D}_{a^+}^{\alpha_4;\phi}+\mu_2\right)v(t)=g(t,u(t),v(t)) \end{cases}, t\in J=[a,b] \tag{1.1}$$

with the initial conditions

$$\begin{cases} u(a)=u_a,\ \ u(b)=A\sum_{i=1}^{n}u(\zeta_i), \\ v(a)=v_a,\ \ v(b)=B\sum_{i=1}^{n}v(\zeta_i), \\ \mu_1,\mu_2>0,\ \ 0\le a<\zeta_i<b<\infty, \text{ and } \phi(b)-\phi(a)=M>0. \end{cases} \tag{1.2}$$

where ${}^{c}\mathbf{D}_{a^+}^{\alpha_i;\phi}, i=\overline{1,4}$ are the $\phi-$ Caputo fractional derivative of orders α_i, and $0<\alpha_i<1, i=\overline{1,4}\ \mu_1,\mu_2,A,B\in\mathrm{R}_+^*,u_a,v_a\in {}'^{m}, m\in\mathrm{N}^*$, and $\phi: J\to\mathrm{R}$ be an increasing function with $\phi'(t)\neq 0$, for all $t \in J$, to be defined later， $g,f : J\times{}'^{m}\times{}'^{m}\to{}'^{m}$ is a given function satisfying some assumptions that will be specified later.

2. PRELIMINARIES ON PHI-CAPUTO DERIVATIVES

Let us consider the Banach space $X=\mathbf{C}\times\mathbf{C}$ with its norm:

$$\|(u,v)\|_X=\|u\|_\infty+\|v\|_\infty. \tag{2.1}$$

where, C be the Banach space of all continuous functions from J *into* $\mathbb{R}^m$, and

$$\|u\|_\infty = \sup_{t\in[a,b]} |u(t)|, \text{ and } \|v\|_\infty = \sup_{t\in[a,b]} |v(t)|.$$

Definition 1: ([14]) For $\alpha > 0$, let $n \in \mathbb{N}$ and let $\phi, u \in C^n(J)$ be two functions such that $\phi'(t) > 0$, for all $t \in J$.

•) The left-sided $\phi-$ Riemann Liouville fractional integral of order α for an integrable function $u : J \to R$ with respect to another function $\phi : J \to R$ is defined by

$$\mathtt{I}_{a^+}^{\alpha;\phi} u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \phi'(s)(\phi(t)-\phi(s))^{\alpha-1} u(s) ds, \tag{2.2}$$

•) The left-sided $\phi-$ Riemann Liouville fractional derivative of a function u of order α is defined by

$$\begin{aligned} \mathtt{D}_{a^+}^{\alpha;\phi} u(t) &= \left(\frac{1}{\phi'(t)}\frac{d}{dt}\right)^n \mathtt{I}_{a^+}^{n-\alpha;\phi} u(t) \text{ where } n = [\alpha]+1 \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\phi'(t)}\frac{d}{dt}\right)^n \int_a^t \phi'(s)(\phi(t)-\phi(s))^{n-\alpha-1} u(s) ds, \end{aligned} \tag{2.3}$$

•) The left-sided $\phi-$ Caputo fractional derivative of a function u *of* order α is defined by

$$^{c}\mathtt{D}_{a^+}^{\alpha;\phi} u(t) = I_{a^+}^{n-\alpha;\phi} \left(\frac{1}{\phi'(t)}\frac{d}{dt}\right)^n u(t), \tag{2.4}$$

where $n = [\alpha]+1,$ for $\alpha \notin \mathbb{N},$ $n = \alpha$ for $\alpha \in \mathbb{N}.$

•) Briefly, the definition (2.3) and (2.4), becomes as follows,,

$$^{c}\mathtt{D}_{a^+}^{\alpha;\phi} u(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^t \phi'(s)(\phi(t)-\phi(s))^{n-\alpha-1} u_\phi^{[n]}(s) ds, & \text{if } \alpha \notin \mathbb{N}, \\ u_\phi^{[n]}(t) & \text{if } \alpha \in \mathbb{N}. \end{cases} \tag{2.5}$$

where

$$u_\phi^{[n]}(t) = \left(\frac{1}{\phi'(t)}\frac{d}{dt}\right)^n u(t).$$

Remark

1) Γ is the gamma function. Equation (2.2) is the generalization of the Riemann Liouville and Hadamard fractional integrals when $\phi(t) = t$ and $\phi(t) = \ln t$, respectively.

2) Equation (2.5) is the generalization of the Caputo fractional derivative operator when $\phi(t) = t$. Moreover, for $\phi(t) = \ln t$, it gives the Caputo Hadamard fractional derivative.

Lemma 1. ([14]). Let $\alpha, \beta > 0$, and $u \in L^1(J)$. Then

$$\mathbf{I}_{a^+}^{\alpha;\phi}\mathbf{I}_{a^+}^{\beta;\phi}u(t) = \mathbf{I}_{a^+}^{\alpha+\beta;\phi}u(t), \quad \text{a.e. } t \in J.$$

In particular, if $u \in C(J)$. Then $\mathbf{I}_{a^+}^{\alpha;\phi}\mathbf{I}_{a^+}^{\beta;\phi}u(t) = \mathbf{I}_{a^+}^{\alpha+\beta;\phi}u(t), \quad t \in J.$

Next, we recall the property describing the composition rules for fractional ϕ -integrals and ϕ -derivatives.

Lemma 2. ([14]). Let $\alpha > 0$ The following holds:

If $u \in C([a,b])$ then

$$^{c}\mathbf{D}_{a^+}^{\alpha;\phi}\mathbf{I}_{a^+}^{\alpha;\phi}u(t) = u(t), t \in [a,b].$$

If $u \in C^n(J)$, $n-1 < \alpha < n.$ Then

$$\mathbf{I}_{a^+}^{\alpha;\phi c}\mathbf{D}_{a^+}^{\alpha;\phi}u(t) = u(t) - \sum_{k=0}^{n-1}\frac{u_{\phi}^{[k]}(a)}{k\,!}\left[\phi(t)-\phi(a)\right]^k ,$$

for all $t \in [a,b]$. In particular, if $0 < \alpha < 1$, we have

$$\mathbf{I}_{a^+}^{\alpha;\phi c}\mathbf{D}_{a^+}^{\alpha;\phi}u(t) = u(t) - u(a).$$

Lemma 3. ([14,15]). Let $t > a$, $\alpha \geq 0$; and $\beta > 0$. Then

- $\mathbf{I}_{a^+}^{\alpha;\phi}\left[\phi(t)-\phi(a)\right]^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left[\phi(t)-\phi(a)\right]^{\beta+\alpha-1},$
- $^{c}\mathbf{D}_{a^+}^{\alpha;\phi}\left[\phi(t)-\phi(a)\right]^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left[\phi(t)-\phi(a)\right]^{\beta-\alpha-1},$
- $^{c}\mathbf{D}_{a^+}^{\alpha;\phi}\left[\phi(t)-\phi(a)\right]^{k} = 0,$ for all $k \in \{0,...,n-1\}, n \in \mathrm{N}$.

Lemma 4. ([15]**)**. Let $\alpha > 0, n \in \mathrm{N}$; such that $n-1 < q \leq n.$ Then:

- $^{c}\mathbf{D}_{a^+}^{q;\phi}\mathbf{I}_{a^+}^{\alpha;\phi}u(t) = {}^{c}\mathbf{D}_{a^+}^{q-\alpha;\phi}u(t);$ if $q > \alpha.$
- $^{c}\mathbf{D}_{a^+}^{q;\phi}\mathbf{I}_{a^+}^{\alpha;\phi}u(t)u(t) = \mathbf{I}_{a^+}^{\alpha-q;\phi}u(t);$ if $\alpha > q.$

Definition 2. ([15**])** A square matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(G)$ is strictly less than 1. In other words, this means that all the eigenvalues of G are in the open unit disc i.e. $|\lambda| < 1$; for every $\lambda \in \mathrm{C}$ with $\det(G - \lambda I) = 0$; where *I* denotes the unit matrix of $M_{m\times m}(\mathrm{R})$.

Theorem 2. ([16]) Let *X* be a generalized Banach space and $T : X \to X$ be a continuous and compact mapping. Then either,

(a) The set

$$\mathbf{A} = \{x \in X \ : \ x = \lambda T(x) \ \text{ for some } \ 0 < \lambda < 1\}$$

in unbounded, or

(b) The operator *T* has a fixed point.

Theorem 1. ([17]) Let $(X;d)$ be a complete generalized metric space and $T:X\to X$ a contractive operator with Lipschitz matrix G : Then T has a unique fixed point u_0 and for each $u\in X$ we have

$$d\left(T^k\left(u\right),u_0\right)\le M^k\left(M\right)^{-1}d(u,T\left(u\right)),\quad \text{for all } k\in \mathrm{N}.$$

Lemma 5. ([18]) Given a function $u\in C^n\left[a,b\right]$ and $0<q<1,$ we have

$$\left|\mathbf{I}_{a^+}^{q;\phi}u(x)-\mathbf{I}_{a^+}^{q;\phi}u(y)\right|\le\frac{2\|u\|}{\Gamma\left(q+1\right)}\left(\phi(x)-\phi(y)\right)^q.$$

Lemma 6. For a given $f,g\in L^1(J)$, the unique coupled pair of functions $(u;v)$ solution of the system

$$\begin{cases}{}^c\mathbf{D}_{a^+}^{\alpha_1;\phi}\left({}^c\mathbf{D}_{a^+}^{\alpha_2;\phi}+\mu_1\right)u(t)=f(t)\\ {}^c\mathbf{D}_{a^+}^{\alpha_3;\phi}\left({}^c\mathbf{D}_{a^+}^{\alpha_4;\phi}+\mu_2\right)v(t)=g(t)\end{cases},t\in J=\left(a,b\right) \tag{2.6}$$

and

$$\begin{cases}u(a)=u_a,\quad u(b)=A\sum_{i=1}^{n}u\left(\zeta_i\right),\\ v(a)=v_a,\quad v(b)=B\sum_{i=1}^{n}v\left(\zeta_i\right),\\ \mu_1,\mu_2>0,\quad 0\le a<\zeta_i<b<\infty,\ \text{and}\ \phi\left(b\right)-\phi(a)=M>0.\end{cases} \tag{2.7}$$

is given by

$$\begin{aligned}u(t)=&\mathbf{I}_{a^+}^{\alpha_1+\alpha_2;\phi}f\,(t)-\frac{(\phi(t)-\phi(a))^{\alpha_2}}{M^{\alpha_2}}\mathbf{I}_{a^+}^{\alpha_1+\alpha_2;\phi}f\,(b)\\ &-\mu_1\mathbf{I}_{a^+}^{\alpha_2;\phi}u(t)+\frac{\mu_1(\phi(t)-\phi(a))^{\alpha_2}}{M^{\alpha_2}}\mathbf{I}_{a^+}^{\alpha_2;\phi}u(b)\\ &+\frac{A\sum_{i=1}^{n}u\left(\zeta_i\right)}{M^{\alpha_2}}(\phi(t)-\phi(a))^{\alpha_2}-\frac{u_a(\phi(t)-\phi(a))^{\alpha_2}}{M^{\alpha_2}}+u_a,\end{aligned}$$

and

$$\begin{aligned}v(t)=&\mathbf{I}_{a^+}^{\alpha_3+\alpha_4;\phi}g\,(t)-\frac{(\phi(t)-\phi(a))^{\alpha_4}}{M^{\alpha_4}}\mathbf{I}_{a^+}^{\alpha_3+\alpha_{4'};\phi}g\,(b)\\ &-\mu_2\mathbf{I}_{a^+}^{\alpha_4;\phi}v(t)+\frac{\mu_2(\phi(t)-\phi(a))^{\alpha_4}}{M^{\alpha_4}}\mathbf{I}_{a^+}^{\alpha_4;\phi}v(b)\\ &+\frac{b\sum_{i=1}^{n}v\left(\zeta_i\right)}{M^{\alpha_4}}(\phi(t)-\phi(a))^{\alpha_4}-\frac{v_a(\phi(t)-\phi(a))^{\alpha_4}}{M^{\alpha_4}}+v_a\end{aligned}$$

Proof: For $0<\alpha_i<1, i=\overline{1,2}$, Lemma yields

$$u(t)=\mathrm{I}_{a^+}^{\alpha_1+\alpha_2;\phi}f(t)-\mu_1\mathrm{I}_{a^+}^{\alpha_2;\phi}u(t)+\mathrm{I}_{a^+}^{\alpha_2;\phi}c_1+c_2,$$

where $c_1,c_2\in R^m$, by conditions $u(a)=u_a$, we get

$$u(t)=\mathrm{I}_{a^+}^{\alpha_1+\alpha_2;\phi}f(t)-\mu_1\mathrm{I}_{a^+}^{\alpha_2;\phi}u(t)+\mathrm{I}_{a^+}^{\alpha_2;\phi}c_1+u_a$$

by integrating we find

$$u(t)=\mathrm{I}_{a^+}^{\alpha_1+\alpha_2;\phi}f(t)-\mu_1\mathrm{I}_{a^+}^{\alpha_2;\phi}u(t)+c_1\frac{(\phi(t)-\phi(a))^{\alpha_2}}{\Gamma(\alpha_2+1)}+u_a. \tag{2.8}$$

by conditions $\phi(b)-\phi(a)=M$, $u(b)=A\sum_{i=1}^{n}u(\zeta_i)$, we give

$$A\sum_{i=1}^{n}u(\zeta_i)=\mathrm{I}_{a^+}^{\alpha_1+\alpha_2;\phi}f(b)-\mu_1\mathrm{I}_{a^+}^{\alpha_2;\phi}u(b)+c_1\frac{M^{\alpha_2}}{\Gamma(\alpha_2+1)}+u_a$$

So,

$$c_1=\frac{A\Gamma(\alpha_2+1)}{M^{\alpha_2}}\sum_{i=1}^{n}u(\zeta_i)-\frac{\Gamma(\alpha_2+1)}{M^{\alpha_2}}\mathrm{I}_{a^+}^{\alpha_1+\alpha_2;\varphi}f(b)+\frac{\mu_1\Gamma(\alpha_2+1)}{M^{\alpha_2}}\mathrm{I}_{a^+}^{\alpha_2;\varphi}u(b)-\frac{u_a\Gamma(\alpha_2+1)}{M^{\alpha_2}}$$

Substituting the values of c_1, into (2.8), we find the solution.

3. MAIN RESULTS

Now, wee need to consider the following assumptions:

- H_1 There exists continuous functions $\phi_1,\phi_2,\phi_3,\phi_4 :[a,b]\to R^+$ such that

$$|f(t,u,v)-f(t,x,y)|\le\phi_1(t)|u-x|+\phi_2(t)|v-y|,$$

and

$$|g(t,u,v)-g(t,x,y)|\le\phi_3(t)|u-x|+\phi_4(t)|v-y|,$$

for all $t\in[a,b]$, $u,v,x,y\in R^m$.

- H_2 There exists continuous functions $\psi_1,\psi_2,\psi_3,\psi_4 :[a,b]\to R^+$ such that

$$|f(t,u,v)|\le\psi_1(t)|u|+\psi_2(t)|v|,$$

and

$$|g(t,u,v)|\le\psi_3(t)|u|+\psi_4(t)|v|,$$

for all $t \in [a,b]$, $u, v \in R^m$.

- H_3 For any bounded set $U \subset X$, the sets

$$\{f\ (t, u(t), v(t)), g(t, u(t), v(t)) : t \in J, \text{ and } (u, v) \in U\},$$

are equicontinuous in X .

Theorem 2. Under the hypotheses (H_1), the coupled system (1.1-1.2) has a unique solution. If the matrix

$$G=\begin{pmatrix} \frac{2M^{\alpha_1+\alpha_2}\|\varphi_1\|_\infty}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{2\mu_1 M^{\alpha_2}}{\Gamma(\alpha_2+1)} + nA & \|\varphi_2\|_\infty \\ \frac{2M^{\alpha_3+\alpha_4}\|\varphi_3\|_\infty}{\Gamma(\alpha_3+\alpha_4+1)} + \frac{2\mu_2 M^{\alpha_4}}{\Gamma(\alpha_4+1)} + nB & \|\varphi_4\|_\infty \end{pmatrix}$$

converges to 0.

Proof: We define the operator $T_i : X \to X$ $(i = 1, 2)$ as follow:

$$\begin{aligned}(T_1(u,v))(t) &= \mathrm{I}_{a^+}^{\alpha_1+\alpha_2;\phi} f(t,u(t),v(t)) - \frac{(\phi(t)-\phi(a))^{\alpha_2}}{M^{\alpha_2}} \mathrm{I}_{a^+}^{\alpha_1+\alpha_2;\phi} f(b) \\ &\quad -\mu_1 \mathrm{I}_{a^+}^{\alpha_2;\phi} u(t) + \frac{\mu_1(\phi(t)-\phi(a))^{\alpha_2}}{M^{\alpha_2}} \mathrm{I}_{a^+}^{\alpha_2;\phi} u(b) \\ &\quad + \frac{A\sum_{i=1}^{n} u(\zeta_i)}{M^{\alpha_2}} (\phi(t)-\phi(a))^{\alpha_2} - \frac{u_a(\phi(t)-\phi(a))^{\alpha_2}}{M^{\alpha_2}} + u_a,\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}(T_2(u,v))(t) &= \mathrm{I}_{a^+}^{\alpha_3+\alpha_4;\phi} g(t,u(t),v(t)) - \frac{(\phi(t)-\phi(a))^{\alpha_4}}{M^{\alpha_4}} \mathrm{I}_{a^+}^{\alpha_3+\alpha_4;\phi} g(b) \\ &\quad -\mu_2 \mathrm{I}_{a^+}^{\alpha_4;\phi} v(t) + \frac{\mu_2(\phi(t)-\phi(a))^{\alpha_4}}{M^{\alpha_4}} \mathrm{I}_{a^+}^{\alpha_4;\phi} v(b) \\ &\quad + \frac{b\sum_{i=1}^{n} v(\zeta_i)}{M^{\alpha_4}} (\phi(t)-\phi(a))^{\alpha_4} - \frac{v_a(\phi(t)-\phi(a))^{\alpha_4}}{M^{\alpha_4}} + v_a,\end{aligned} \tag{3.2}$$

where if $e \in \{\alpha_2, \alpha_1+\alpha_2, \alpha_3, \alpha_3+\alpha_4\}, x \in \{t, b\}$, then

$$\mathrm{I}_{a^+}^{e;\phi} u(x) = \frac{1}{\Gamma(e)} \int_a^x \phi'(s)(\phi(x)-\phi(s))^{e-1} u(s) ds,$$

$$\mathrm{I}_{a^+}^{e;\phi} v(x) = \frac{1}{\Gamma(e)} \int_a^x \phi'(s)(\phi(x)-\phi(s))^{e-1} v(s) ds,$$

$$\mathtt{I}_{a^+}^{e;\phi} f(x, u(x), v(x)) = \frac{1}{\Gamma(e)} \int_a^x \phi'(s)(\phi(x) - \phi(s))^{e-1} f(s, u(s), v(s)) ds,$$

and

$$\mathtt{I}_{a^+}^{e;\phi} g(x, u(x), v(x)) = \frac{1}{\Gamma(e)} \int_a^x \phi'(s)(\phi(x) - \phi(s))^{e-1} g(s, u(s), v(s)) ds.$$

Consider the operator $T : X \to X$ as follow:

$$\left(T(u,v)\right)(t) = \left(\left(T_1(u,v)\right)(t), \left(T_2(u,v)\right)(t)\right) \tag{3.3}$$

Clearly, (u_0, v_0) fixed points of the operator $T \Rightarrow (u_0, v_0)$ are solutions of the system (1.1)-(1.2).

For any $(u,v),(x,y) \in X$ and $t \in J$, we have

$$\begin{aligned}
&\left\|\left((T_1(u,v) - (T_1(x,y)\right)(t)\right\| \\
&\le \frac{1}{\Gamma(\alpha_1+\alpha_2)} \int_a^t \phi'(s)(\phi(t) - \phi(s))^{\alpha_1+\alpha_2-1} \left|f(t, u(s), v(s)) - f(t, x(s), y(s))\right| ds \\
&+ \frac{(\phi(t) - \phi(a))^{\alpha_2}}{M^{\alpha_2}\Gamma(\alpha_1+\alpha_2)} \int_a^b \phi'(s)(\phi(b) - \phi(s))^{\alpha_1+\alpha_2-1} \left|f(t, u(s), v(s)) - f(t, x(s), y(s))\right| ds \\
&+ \frac{\mu_1}{\Gamma(\alpha_2)} \int_a^t \phi'(s)(\phi(t) - \phi(s))^{\alpha_2-1} \left|u - x\right|(s) ds \\
&+ \frac{\mu_1\left(\phi(t) - \phi(a)\right)^{\alpha_2}}{M^{\alpha_2}\Gamma(\alpha_2)} \int_a^b \phi'(s)(\phi(b) - \phi(s))^{\alpha_2-1} \left|u - x\right|(s) ds \\
&+ \frac{A \sum_{i=1}^{n} \left|u - x\right|(\zeta_i)}{M^{\alpha_2}} \left(\phi(t) - \phi(a)\right)^{\alpha_2} + \\
&\le \left(\frac{2M^{\alpha_1+\alpha_2}\|\varphi_1\|_\infty}{\Gamma(\alpha_1+\alpha_2+1)} + \frac{2\mu_1 M^{\alpha_2}}{\Gamma(\alpha_2+1)} + nA\right)\|u - x\|_\infty + \|\varphi_2\|_\infty \|v - y\|_\infty .
\end{aligned}$$

Also,

$$\begin{aligned}
&\left\|\left((T_2(u,v) - (T_2(x,y)\right)(t)\right\| \\
&\le \left(\frac{2M^{\alpha_3+\alpha_4}\|\varphi_3\|_\infty}{\Gamma(\alpha_3+\alpha_4+1)} + \frac{2\mu_2 M^{\alpha_4}}{\Gamma(\alpha_4+1)} + nB\right)\|u - x\|_\infty + \|\varphi_4\|_\infty \|v - y\|_\infty .
\end{aligned}$$

So,

$$d\left(T(u,v), T(x,y)\right) \le Md\left((u,v),(x,y)\right),$$

where

$$d\left((u,v),(x,y)\right)=\begin{pmatrix}\|u-x\|_\infty\\ \\ \|v-y\|_\infty\end{pmatrix}.$$

Hence, the matrix G converges to zero, then the system (1.1)-(1.2) has a unique solution, according to Theorem

Theorem 3. Assume that the hypotheses (H_1)-(H_3) hold. If $L<1$; then the coupled system (1.1)-(1.2) has at least one solution, where

$$L=\frac{2M^{\alpha_1+\alpha_2}\|\psi_1\|_\infty}{\Gamma(\alpha_1+\alpha_2+1)}+\frac{2M^{\alpha_3+\alpha_4}\|\psi_3\|_\infty}{\Gamma(\alpha_3+\alpha_4+1)}+\frac{2\mu_1M^{\alpha_2}}{\Gamma(\alpha_2+1)}$$
$$+\frac{2\mu_2M^{\alpha_4}}{\Gamma(\alpha_4+1)}+n\left(A+B\right)+\|\psi_2\|_\infty+\|\psi_4\|_\infty.$$

Proof:

First step: ***T*** is continuous, where *T* the operator defined in (3.3) we show that the operator *T* satisfies all conditions of Theorem.

Let $(u_n,v_n)_n$ be a sequence converges to $(u,v)\in X$, for any $t\in J$, we have

$$\left\|\left((T_1(u_n,v_n)-(T_1(u,v)\right)(t)\right\|$$
$$\le\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^t\phi'(s)(\phi(t)-\phi(s))^{\alpha_1+\alpha_2-1}\left|f(t,u_n(s),v_n(s))-f(t,u(s),v(s))\right|ds$$
$$+\frac{(\phi(t)-\phi(a))^{\alpha_2}}{M^{\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^b\phi'(s)(\phi(t)-\phi(s))^{\alpha_1+\alpha_2-1}\left|f(t,u_n(s),v_n(s))-f(t,u(s),v(s))\right|ds$$
$$+\frac{\mu_1}{\Gamma(\alpha_2)}\int_a^t\phi'(s)(\phi(t)-\phi(s))^{\alpha_2-1}\left|u_n-u\right|(s)ds$$
$$+\frac{\mu_1\left(\phi(t)-\phi(a)\right)^{\alpha_2}}{M^{\alpha_2}\Gamma(\alpha_2)}\int_a^b\phi'(s)(\phi(t)-\phi(s))^{\alpha_2-1}\left|u_n-u\right|(s)ds$$
$$+\frac{A\sum_{i=1}^{n}\left|u_n-u\right|\left(\zeta_i\right)}{M^{\alpha_2}}\left(\phi(t)-\phi(a)\right)^{\alpha_2}$$
$$\le\left(\frac{2M^{\alpha_1+\alpha_2}\|\varphi_1\|_\infty}{\Gamma(\alpha_1+\alpha_2+1)}+\frac{2\mu_1M^{\alpha_2}}{\Gamma(\alpha_2+1)}+nA\right)\|u_n-u\|_\infty+\|\varphi_2\|_\infty\|v_v-v\|_\infty.$$

Also,

$$\left\|\left((T_2(u_n,v_n)-(T_2(u,v)\right)(t)\right\|$$
$$\le\left(\frac{2M^{\alpha_3+\alpha_4}\|\varphi_3\|_\infty}{\Gamma(\alpha_3+\alpha_4+1)}+\frac{2\mu_2M^{\alpha_4}}{\Gamma(\alpha_4+1)}+nB\right)\|u_n-u\|_\infty+\|\varphi_4\|_\infty\|v_v-v\|_\infty.$$

By the Lebesgue dominated convergence theorem and f, g are continuous we get

$$\left\|\left((T(u_n,v_n)-(T(u,v))\right)(t)\right\|_X \to 0 \text{ as } n\to\infty.$$

Therefore T is continuous.

Second step: ***T*** maps bounded sets into bounded sets in X:
Let $r>0$,

$$U_r=\left\{(x,v)\in X, \|x\|_\infty \le r, \|v\|_\infty \le r\right\}. \quad (3.4)$$

For any $(u,v)\in U_r$ and $t\in J$, we have

$$\begin{aligned}
&\left\|(T_1(u,v)(t)\right\| \\
&\le \frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^t \phi'(s)(\phi(t)-\phi(s))^{\alpha_1+\alpha_2-1}\left|f(t,u(s),v(s))\right|ds \\
&+\frac{(\phi(t)-\phi(a))^{\alpha_2}}{M^{\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^b \phi'(s)(\phi(t)-\phi(s))^{\alpha_1+\alpha_2-1}\left|f(t,u(s),v(s))\right|ds \\
&+\frac{\mu_1}{\Gamma(\alpha_2)}\int_a^t \phi'(s)(\phi(t)-\phi(s))^{\alpha_2-1}|u|(s)ds \\
&+\frac{\mu_1\left(\phi(t)-\phi(a)\right)^{\alpha_2}}{M^{\alpha_2}\Gamma(\alpha_2)}\int_a^b \phi'(s)(\phi(t)-\phi(s))^{\alpha_2-1}|u|(s)ds \\
&+\frac{A\sum_{i=1}^{n}|u|(\zeta_i)}{M^{\alpha_2}}\left(\phi(t)-\phi(a)\right)^{\alpha_2}+\frac{\|u_a\|(\phi(t)-\phi(a))^{\alpha_2}}{M^{\alpha_2}}+\|u_a\| \\
&\le\left(\frac{2M^{\alpha_1+\alpha_2}\|\psi_1\|_\infty}{\Gamma(\alpha_1+\alpha_2+1)}+\frac{2\mu_1M^{\alpha_2}}{\Gamma(\alpha_2+1)}+nA\right)\|u\|_\infty+\|\psi_2\|_\infty\|v\|_\infty \\
&\le\left(\frac{2M^{\alpha_1+\alpha_2}\|\psi_1\|_\infty}{\Gamma(\alpha_1+\alpha_2+1)}+\frac{2\mu_1M^{\alpha_2}}{\Gamma(\alpha_2+1)}+nA+\|\psi_2\|_\infty\right)r+2\|u_a\|=R_1.
\end{aligned}$$

Also,

$$\begin{aligned}
&\left\|(T_2(u,v)(t)\right\| \\
&\le\left(\frac{2M^{\alpha_3+\alpha_4}\|\psi_3\|_\infty}{\Gamma(\alpha_3+\alpha_4+1)}+\frac{2\mu_2M^{\alpha_4}}{\Gamma(\alpha_4+1)}+nB+\|\psi_4\|_\infty\right)r+2\|v_a\|=R_2.
\end{aligned}$$

So,

$$\left\|(T(u,v)(t)\right\|_X\le R_1+R_2=R.$$

Third step: T maps bounded sets into equicontinuous sets in X. Let U_r defined in (3.4), for $t_2,\ t_1\in J$ with $t_2\ge t_1$ and $(u,v)\in U_r$

$$\begin{aligned}
&\left((T_1(u,v)(t_2)-(T_1(u,v))(t_1)\right)\\
&=\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^{t_2}\phi'(s)(\phi(t_2)-\phi(s))^{\alpha_1+\alpha_2-1}\left|f(t,u(s),v(s))\right|ds\\
&-\frac{1}{\Gamma(\alpha_1+\alpha_2)}\int_a^{t_1}\phi'(s)(\phi(t_1)-\phi(s))^{\alpha_1+\alpha_2-1}\left|f(t,u(s),v(s))\right|ds\\
&+\frac{(\phi(t_2)-\phi(a))^{\alpha_2}-(\phi(t_1)-\phi(a))^{\alpha_2}}{M^{\alpha_2}\Gamma(\alpha_1+\alpha_2)}\int_a^{b}\phi'(s)(\phi(b)-\phi(s))^{\alpha_1+\alpha_2-1}\left|f(t,u(s),v(s))\right|ds\\
&+\frac{\mu_1}{\Gamma(\alpha_2)}\int_a^{t_2}\phi'(s)(\phi(t_2)-\phi(s))^{\alpha_2-1}\left|u\right|(s)ds-\frac{\mu_1}{\Gamma(\alpha_2)}\int_a^{t_1}\phi'(s)(\phi(t_1)-\phi(s))^{\alpha_2-1}\left|u\right|(s)ds\\
&+\frac{\mu_1(\phi(t_2)-\phi(a))^{\alpha_2}-(\phi(t_1)-\phi(a))^{\alpha_2}}{M^{\alpha_2}\Gamma(\alpha_2)}\int_a^{b}\phi'(s)(\phi(b)-\phi(s))^{\alpha_2-1}\left|u\right|(s)ds\\
&+\frac{A\sum_{i=1}^{n}\left|u\right|(\zeta_i)}{M^{\alpha_2}}\left((\phi(t_2)-\phi(a))^{\alpha_2}-(\phi(t_1)-\phi(a))^{\alpha_2}\right)\\
&-\frac{u_a\left((\phi(t_2)-\phi(a))^{\alpha_2}-(\phi(t_1)-\phi(a))^{\alpha_2}\right)}{M^{\alpha_2}}.
\end{aligned}$$

So,

$$\begin{aligned}
&\left\|\left((T_1(u,v)(t_2)-(T_1(u,v))(t_1)\right)\right\|\\
&\le\frac{\left(\|\psi_1\|_\infty+\|\psi_2\|_\infty\right)r}{\Gamma(\alpha_1+\alpha_2+1)}(\phi(t_2)-\phi(s))^{\alpha_1+\alpha_2}\\
&+\left(\frac{M^{\alpha_1}\left(\|\psi_1\|_\infty+\|\psi_2\|_\infty\right)r}{\Gamma(\alpha_1+\alpha_2+1)}+\frac{2r\mu_1}{\Gamma(\alpha_2+2)}+\frac{\|u_a\|+Anr}{M^{\alpha_2}}\right)(\phi(t_2)-\phi(t_1))^{\alpha_2}.
\end{aligned}$$

then

$$\left\|\left((T_1(u,v)(t_2)-(T_1(u,v))(t_1)\right)\right\|\to 0 \quad \text{as} \quad t_2\to t_1.$$

Also,

$$\begin{aligned}
&\left\|\left((T_2(u,v)(t_2)-(T_2(u,v))(t_1)\right)\right\|\\
&\le\frac{\left(\|\psi_3\|_\infty+\|\psi_3\|_\infty\right)r}{\Gamma(\alpha_3+\alpha_4+1)}(\phi(t_2)-\phi(s))^{\alpha_3+\alpha_4}\\
&+\left(\frac{M^{\alpha_1}\left(\|\psi_3\|_\infty+\|\psi_3\|_\infty\right)r}{\Gamma(\alpha_3+\alpha_4+1)}+\frac{2r\mu_2}{\Gamma(\alpha_4+2)}+\frac{\|v_a\|+Bnr}{M^{\alpha_4}}\right)(\phi(t_2)-\phi(t_1))^{\alpha_4}.
\end{aligned}$$

then $\left\|\left((T_2(u,v)(t_2)-(T_2(u,v))(t_1)\right)\right\|\to 0 \quad \text{as} \quad t_2\to t_1.$

Hence, the set $T(U_r)$ is equicontinuous in *X*.

As a consequence by the Arzela-Ascoli theorem, we conclude that *T* maps U_r into a precompact set in *X*.

Fourth step: Let E defined by

$$E=\left\{(u,v)\in X \ : (u,v)=\lambda T(u,v), 0<\lambda<1\right\}$$

E is bounded in X. Let $(u,v)\in X$ such that $(u,v)=\lambda T(u,v)$, then $u=\lambda T_1(u,v)$, and $v=\lambda T_2(u,v)$. Thus, for $t \in J$, we have

$$\begin{aligned}\|u\|_\infty &= \|T_1(u,v)\| \\ &\le \left(\frac{2M^{\alpha_1+\alpha_2}\|\psi_1\|_\infty}{\Gamma(\alpha_1+\alpha_2+1)}+\frac{2\mu_1 M^{\alpha_2}}{\Gamma(\alpha_2+1)}+nA\right)\|u\|_\infty+\left(\|\psi_2\|_\infty\right)\|v\|_\infty+2\|u_a\| \\ &\le \left(\frac{2M^{\alpha_1+\alpha_2}\|\psi_1\|_\infty}{\Gamma(\alpha_1+\alpha_2+1)}+\frac{2\mu_1 M^{\alpha_2}}{\Gamma(\alpha_2+1)}+nA\right)\|(u,v)\|_X+\left(\|\psi_2\|_\infty\right)\|(u,v)\|_X+2\|u_a\| \\ &\le \left(\frac{2M^{\alpha_1+\alpha_2}\|\psi_1\|_\infty}{\Gamma(\alpha_1+\alpha_2+1)}+\frac{2\mu_1 M^{\alpha_2}}{\Gamma(\alpha_2+1)}+nA+\|\psi_2\|_\infty\right)\|(u,v)\|_X+2\|u_a\|.\end{aligned}$$

Also,

$$\begin{aligned}\|v\|_\infty &= \lambda\|T_2(u,v)(t)\| \\ &\le \left(\frac{2M^{\alpha_3+\alpha_4}\|\psi_3\|_\infty}{\Gamma(\alpha_3+\alpha_4+1)}+\frac{2\mu_2 M^{\alpha_4}}{\Gamma(\alpha_4+1)}+nA+\|\psi_4\|_\infty\right)\|(u,v)\|_X+2\|v_a\|.\end{aligned}$$

So, we get,

$$\begin{aligned}&\|u\|_\infty+\|v\|_\infty \\ &\le \Bigg(\frac{2M^{\alpha_1+\alpha_2}\|\psi_1\|_\infty}{\Gamma(\alpha_1+\alpha_2+1)}+\frac{2M^{\alpha_3+\alpha_4}\|\psi_3\|_\infty}{\Gamma(\alpha_3+\alpha_4+1)}+\frac{2\mu_1 M^{\alpha_2}}{\Gamma(\alpha_2+1)}+\frac{2\mu_2 M^{\alpha_4}}{\Gamma(\alpha_4+1)}+ \\ &\quad n(A+B)+\|\psi_2\|_\infty+\|\psi_4\|_\infty\Bigg)\|(u,v)\|_X+2\left(\|u_a\|+\|v_a\|\right)\end{aligned}$$

So,

$$\begin{aligned}&\|(u,v)\|_X \\ &\le \frac{2\left(\|u_a\|+\|v_a\|\right)}{1-\left(\frac{2M^{\alpha_1+\alpha_2}\|\psi_1\|_\infty}{\Gamma(\alpha_1+\alpha_2+1)}+\frac{2M^{\alpha_3+\alpha_4}\|\psi_3\|_\infty}{\Gamma(\alpha_3+\alpha_4+1)}+\frac{2\mu_1 M^{\alpha_2}}{\Gamma(\alpha_2+1)}+\frac{2\mu_2 M^{\alpha_4}}{\Gamma(\alpha_4+1)}+n(A+B)+\|\psi_2\|_\infty+\|\psi_4\|_\infty\right)} \\ &<+\infty.\end{aligned}$$

Then the set E is thus bounded. By Steps 1 to 4, we can conclude that T has at least one fixed point in U_r which is a solution of the system (1.1)-(1.2).

Example: Consider the following nonlinear Langevin equation of fractional orders

$$\begin{cases} {}^{c}\mathbf{D}_{1^+}^{0,75;t^2}\left({}^{c}\mathbf{D}_{1^+}^{0,5;t^2}+\pi\right)u(t)=f(t,u(t),v(t)),t\in[1,2], \\ {}^{c}\mathbf{D}_{1^+}^{0,45;t^2}\left({}^{c}\mathbf{D}_{1^+}^{0,8;t^2}+2\pi\right)u(t)=g(t,u(t),v(t)) \\ u(1)=v(1)=2, \\ f(t,u(t),v(t))=\frac{\sin t}{49t^{0,2}(1+t)}\frac{u(t)-v(t)}{(u(t)+v(t))^2} \\ g(t,u(t),v(t))=\frac{\cos t}{100(1+t^2)}\frac{u(t)-2v(t)}{(2u(t)+v(t))^2}; \end{cases} \tag{3.5}$$

Observe that the function *f*, *g* is also continuous.

Thus, the assumptions (H_1)-(H_3) are satisfied Theorem and Theorem implies that the system (3.5) has a solution.

5. CONCLUSION

In this paper, we discuss the existence and uniqueness of solutions to the conjugated system of partial differential equations by using the fixed point theorem in the metric space of matrices, we also supported this with an illustrative example to demonstrate the applicability of our results.

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