ORIGINAL PAPER

THE SERRET-FRENET FRAME OF THE RATIONAL BEZIER CURVES IN THE EUCLIDEAN-3 SPACE BY ALGORITHM METHOD

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Abstract. *In this study, the Serret-Frenet frame and derivative formulas were obtained for all intermediate points of the rational Bezier curves with the algorithm method, and much more general results were computed from the previous studies. In addition, the center and radius of the osculator circle and sphere were calculated.*

Keywords: *Serret-Frenet frame; rational Bezier curve; algorithm method; osculating circle and sphere; Darboux vector.*

1. INTRODUCTION

The founding father of the Bezier curves which are one of the computer-aided geometrical design elements, Paul de Faget de Casteljau was a French automotive engineer working in the Citroen Company in 1959. Within the same years, these curves were also studied by Pierre Bezier, a French automotive engineer who was researching the cutting of the cylinder segments in the Renault automobile company. Since P. Bezier, one of the two engineers working on the Bezier curves independently, published the first article, this curve has been known as the Bezier curve. Since Bezier curves numerically have the most stable form among the bases being used, they have been used widely in geometrical design, engineering fields, industry, automobile, designing of the airframe, animation applications, and in the technics of generating fonts. The Bezier curve comes into existence in the convex polygon which is composed of combining of the given control points. Initial and ending points intersect with the curve. In order to obtain numerical values in the intermediate points of the Bezier curves, in 1959, P.F. Casteljau developed an algorithm method given his name. This algorithm Bezier curve easily provides subdivision to two parametric curves in the arbitrary parametric position. On the other hand, the rational Bezier curves compose the weight functions and the ratio of the Bezier curves. Since it is possible to obtain the properties of the polynomial Bezier curves through the rational Bezier curves, it has much more advantages than others [1-4]. Until today, a lot of studies have been done on the rational Bezier curves. In this paper, we are going to give examples of some contemporary studies. Floater (1991) studied the derivatives of the Bezier curves by utilizing the algorithm method and then gave such geometric samples as curvature and torsion of these curves [5]. Similarly, Lin (2009) analyzed the derivative formulas of the rational Bezier curves at one vertice [6]. İncesu and et al. studied some geometric properties of the Bezier curves and the surfaces [7-9]. Kuşak Samancı and et al. (2015), for the first time, computed the Serret-Frenet and Bishop frame of the initial and ending points of the Bezier curves in the Euclidean 3-space [10]. After that, Kuşak Samancı (2016) analyzed the Bezier and B-spline curves along with the properties of the textile fibers [11, 12]. The Bezier curves were analyzed in the Minkowski space for the

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first time by Georgiev (2008) [13]. In addition to that, Kusak Samancı and et al. studied the Serret-Frenet frame and some geometric properties of the Bezier and the rational Bezier curves more elaborately [14-19]. Then Erkan et al (2018) gave the results of the Serret-Frenet frame and curves in the Euclidean 3-space and the plenary results of the polynomial Bezier curves by assigning 1 to the weights in the algorithm method found by Floater. After that, the authors also analyzed these curves at 4 dimensions [20-21]. Kılıçoğlu et al (2020) gave the matrix representation of the cubic Bezier curve. Ceylan et al. (2021) investigated some different geometric properties of the rational Bezier curves [23].

In this study, the Serret-Frenet frame and derivative formulas of the rational Bezier curves have been computed through the algorithm method given in [5], and the interpretations of their curvatures and torsion have been given as well. Thus, thanks to this method, we have made the calculations that are obtained according to the initial and ending points in other studies more generally. Therefore, we have managed to compute the Serret-Frenet frame and some geometric properties of the curve for some intermediate points. Moreover, thanks to this study, we have shown that it is easy to obtain all results found in the polynomial Bezier curves and the values of the initial and ending points in the literature. For instance, the results of the polynomial Bezier curve have been obtained easily by assigning 1 to the weights in the rational Bezier curves, and again, utilizing the obtained results, values of the initial and ending points have been obtained as well. Darboux vector of the Serret-Frenet frame of the rational Bezier curves has been obtained through the algorithm method. Also, the center of the osculating circle and sphere has been computed. Finally, a numerical example has been given.

2. MATERIALS AND METHODS

2.1. THE MAIN PROPERTIES OF THE CURVES IN THE EUCLIDEAN SPACE

This section includes such basic definitions and theorems used in the differential geometry as the inner product of the Euclidean space in [24, 25], curves, Serret-Frenet frame, curvature, and torsion of the planar and space curves, Darboux vector, osculator circle, and sphere.

Definition 2.1.1. Let $\mathbb{R}$ be a real numbers field and V be a vector space. If the function $(u,v)=\langle u,v\rangle$ indicates an inner product function in V for $\forall u,v\in V$ $\langle,\rangle : V\times V\to\mathbb{R}$, vector space V is called inner product space.

Definition 2.1.2. Let the points $x=(x_1,x_2,...,x_n)$, $y=(y_1,y_2,...,y_n)$ and $z=(z_1,z_2,...,z_n)$ be taken from the n-dimensional Euclidean space. The function described as $\langle x,y\rangle=\sum_{i=1}^{n}x_iy_i$ for any two vectors is called inner product or Euclidean inner product. Then, the product $x\times y=(x_2y_3-y_2x_3, x_3y_1-y_3x_1, x_1y_2-y_1x_2)$ is called Euclidean cross-product and it satisfies the condition $(x\times y)\times z=\langle z,x\rangle y-\langle z,y\rangle x$.

Definition 2.1.3. For any two vectors $x=\left(x_1,x_2,...,x_n\right)$ and $y=\left(y_1,y_2,...,y_n\right)$ taken from n-dimensional Euclidean space, the function described as $\|x\|=\sqrt{\langle x,y\rangle}=\sqrt{\sum_{i=1}^{n}x_iy_i}$ is called the norm of the vector x.

Definition 2.1.4. Let I be an open interval of $\mathbb{R}$, a transformation which is in the form of $\alpha:I\subset\mathbb{R}\to\mathbb{R}^n$ from the set C^∞ is called as a curve in the Euclidean n-space. Since $\|\alpha'(t)\|=1$, the curve α is called a unit speed curve.

Theorem 2.1.5. Assume that $J:\mathbb{R}^2\to\mathbb{R}^2$, be a linear transformation defined by

$$J(u_1,u_2)=(-u_2,u_1)\,. \tag{2.1}$$

Geometrically J is a counterclockwise $\dfrac{\pi}{2}$ angled rotation about the origin. J is called as the complex structure of $\mathbb{R}^2$. $J^2=-1$. Since $u,v\in\mathbb{R}^2$ the transformation J has the characteristics of $\langle Ju,Jv\rangle=\langle u,v\rangle$, $\langle Ju,u\rangle=0$ and $\langle Ju,Jv\rangle=-\langle v,Ju\rangle$.

Theorem 2.1.6. Assume that $u,v\in\mathbb{R}^2$, be two non-zero vectors in the Euclidean planar. For $0\le\vartheta\le2\pi$, there is only one ϑ number as

$$\cos\vartheta=\frac{\langle u,v\rangle}{\|u\|\|v\|}\ \text{and}\ \sin\vartheta=\frac{\langle u,Jv\rangle}{\|u\|\|v\|} \tag{2.2}$$

where the angle ϑ is called a directed angle from vector u to vector v.

Theorem 2.1.7. Let $\alpha:\left(a,b\right)\to\mathbb{R}^2$ be a planar curve. The curvature of the curve α at the point $\alpha(t)$ is defined as

$$\kappa(t)=\frac{\langle\alpha''(t),J\alpha'(t)\rangle}{\|\alpha'(t)\|^3} \tag{2.3}$$

where the positive function $\rho=\dfrac{1}{\kappa(t)}$ is called the radius of curvature of the curve α.

Theorem 2.1.8. Let $\alpha:\left(a,b\right)\to\mathbb{R}^2$ be a regular planar curve. The necessary and sufficient condition for the curve α is correct is $|\kappa(t)|=0$. Also, the necessary and sufficient condition for the curve α to be a part of a radius $r>0$ circle is $|\kappa(t)|=1/r$.

Theorem 2.1.9. Assume that the curve $\alpha:\left(a,b\right)\to\mathbb{R}^2$ is a unit speed curve in the Euclidean planar. The Serret-Frenet frame is composed of the unit tangent and the normal vector field of the planar curve α is defined with the orthonormal vectors $T=\alpha'(s)$ and $N(s)=J(\alpha'(s))$.

Since $\alpha''(s)=\kappa(s)J\alpha'(s)$ the derivative formula of the Serret-Frenet frame of the curve α is computed as $T'(s)=\kappa(s)N(s)$ and $N'(s)=-\kappa(s)T(s)$.

Theorem 2.1.10. The Serret-Frenet frame which is composed of the unit tangent and normal vector field of a non-unit speed curve $\alpha:(a,b)\to\mathbb{R}^2$ in the Euclidean plane is given with

$$T=\frac{\alpha'(t)}{\|\alpha'(t)\|} \text{ and } N(t)=\frac{J(\alpha'(t))}{\|J(\alpha'(t))\|}. \tag{2.4}$$

Since $v=\|\alpha'(t)\|$ the derivative formula of the Serret-Frenet frame of the curve α is computed as $T'=v.\kappa.N$ and $N'=-v.\kappa.T$.

Definition 2.1.11. Let the curve $\alpha:I\subset\mathbb{R}\to\mathbb{R}^3$ be a unit speed curve. The tangent of the Serret-Frenet frame of the curve α is defined by $T=\alpha'(s)$, and the principal normal and binormal of which is introduced by $N=\dfrac{T'(s)}{\|T'(s)\|}$ and $B=T\times N$, respectively. If the curvature and the torsion of the curve α are $\kappa=\|T'(s)\|$ and $\tau=\dfrac{\langle\alpha',\alpha''\times\alpha'''\rangle}{\|\alpha''\|^2}$, the derivative formulas of the Serret-Frenet frame are computed by the matrix

$$\begin{pmatrix}T'\\N'\\B'\end{pmatrix}=\begin{pmatrix}0&\kappa&0\\-\kappa&0&\tau\\0&-\tau&0\end{pmatrix}\begin{pmatrix}T\\N\\B\end{pmatrix}. \tag{2.5}$$

Definition 2.1.12. Assume that the curve $\alpha:I\subset\mathbb{R}\to\mathbb{R}^3$ is a non-unit speed curve. The tangent of the Serret-Frenet frame of the curve α is defined as $T=\dfrac{\alpha'(t)}{\|\alpha'(t)\|}$ and the principal normal and binormal of which is defined as $\boldsymbol{N(t)=B(t)\times T(t)}$ and $B(t)=\dfrac{\alpha'(t)\times\alpha''(t)}{\|\alpha'(t)\times\alpha''(t)\|}$, respectively. The curvature of the curve α is defined with $\kappa(t)=\dfrac{\|\alpha'(t)\times\alpha''(t)\|}{\|\alpha'(t)\|^3}$ and the torsion of which is defined with the equation $\tau=\dfrac{\langle\alpha'(t),\alpha''(t)\times\alpha'''(t)\rangle}{\|\alpha'(t)\times\alpha''(t)\|^2}$. If the speed of the non-unit speed curve α is $v=\|\alpha'(t)\|$, then the derivative formulas of the Serret-Frenet frame are computed with the matrix

$$\begin{pmatrix}T'\\N'\\B'\end{pmatrix}=\begin{pmatrix}0&v\kappa&0\\-v\kappa&0&v\tau\\0&-v\tau&0\end{pmatrix}\begin{pmatrix}T\\N\\B\end{pmatrix}. \tag{2.6}$$

Theorem 2.1.13. Let $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$ be a curve in the Euclidean 3-space. If $\kappa = 0$, the curve α is a straight line, if $\tau = 0$, the curve α is a planar curve, if $\kappa = const. > 0$ and $\tau = 0$, the curve α is a circle segment, and finally if $\tau/\kappa = const.$, the curve α is a helix.

Theorem 2.1.14. The angular velocity vector which meets the condition $T'(t) = D \times T(t)$, $N'(t) = D \times N(t)$ and $B'(t) = D \times B(t)$ along with the Serret-Frenet frame $\{T(s), N(s), B(s)\}$ of a unit speed space curve $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$ is called as Darboux vector and is described with the equation $D = \tau T(s) + \kappa B(s)$.

Theorem 2.1.15. If $v = \|\alpha'(t)\|$, the Darboux vector of the Serret-Frenet frame $\{T(t), N(t), B(t)\}$ of a non-unit speed curve $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$ is defined by $D = v\tau T(s) + v\kappa B(s)$.

Definition 2.1.16. (Osculator circle) The circle which has at least in second-order contact with the curve at a point $\alpha(s)$ of the curve α in the Euclidean 3-space is called an osculator circle at the point $\alpha(s)$ of the curve α. Assume that α is a non-unit speed curve. If $\kappa(t)$ is the curvature function of the curve and $N(t)$ is a normal vector field of the curve, the center $m(t)$ and the radius ρ of the osculator circle at the point $\alpha(t)$ is computed with

$$m(t_0) = \alpha(t_0) + \rho(t_0) N(t_0) \text{ and } \rho(t_0) = 1/\kappa. \tag{2.7}$$

The equation of the osculator circle is given with

$$\gamma(\theta) = \alpha(t_0) + \left(1 - \cos\frac{\theta}{\rho(t_0)}\right)\rho(t_0) N(t_0) + \rho(t_0)\sin\frac{\theta}{\rho(t_0)} T(t_0). \tag{2.8}$$

Definition 2.1.17. (Osculator sphere) The which has at least in third-order contact at a point $\alpha(s)$ of the curve α in the Euclidean 3-space is called an osculator sphere at the point $\alpha(s)$ of the curve α. Assume that α is a non-unit speed curve. The curvature and the torsion function of the curve are $\kappa(t)$ and $\tau(t)$, and $v = \|\alpha'(t)\|$ is the speed of it, respectively. When $N(t)$ and $B(t)$ are the normal and the binormal vector field of the curve and $\rho(t_0) = 1/\kappa$ ve $\sigma(t_0) = 1/\tau$, the center $M(t)$ and the radius Γ of the osculator sphere at the point $\alpha(t)$ are computed with

$$M(t_0) = \alpha(t_0) + \rho(t_0) N(t_0) + \frac{1}{v(t_0)} \rho'(t_0)\sigma(t_0) B(t_0) \ , \tag{2.9}$$

$$\Gamma(t) = \sqrt{\left(\frac{1}{\kappa}\right)^2 + \left(\frac{1}{v\tau}\left(\frac{1}{\kappa}\right)'\right)^2}. \tag{2.10}$$

2.2. THE POLYNOMIAL BEZIER CURVES IN THE EUCLIDEAN SPACE

In this section, the main definitions and theorems of the polynomial Bezier and the rational Bezier curves are included in the references [1-4].

Definition 2.2.1. Let $P_0, P_1, ..., P_n \in \mathbb{R}^3$ be the control points. A polynomial Bezier with n-th degree curve is defined by $P(t) = \sum_{i=0}^{n} B_{i,n}(t) P_i$ where $B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}$ are Bernstein polynomials for $t \in [0,1]$. If the degree of polynomial Bezier curve is $n = 1$, then the polynomial Bezier is called a linear Bezier curve and denoted by $P(t) = (1-t) P_0 + t P_1$. If the degree is $n = 2$, then the polynomial Bezier is called a quadratic Bezier curve and denoted by

$$P(t) = (1-t)^2 P_1 + 2(1-t) t P_2 + t^2.$$

If the degree is $n = 3$, then the polynomial Bezier is called a cubic Bezier curve and denoted by

$$P(t) = (1-t)^3 P_0 + 3(1-t)^2 t P_1 + 3(1-t) t^2 P_2 + t^3 P_3.$$

Theorem 2.2.2. Assume that $P_0, P_1, ..., P_n \in \mathbb{R}^3$ are the control points in Euclidean 3-space. The r-th order derivative of the n-th degree Bezier curve is

$$P^{(r)}(t) = \frac{n!}{(n-r)!} \sum_{i=0}^{n-r} B_{i,n-r}(t) \Delta^r P_i$$

where $\Delta^r P_i = \sum_{i=0}^{r} \binom{r}{j} (-1)^{r-1} P_{j+1}$ is the difference equation and the condition $\Delta^r P_j = \Delta^{r-1} P_{j+1} - \Delta^{r-1} P_j$ is satisfied. Especially, the first, second, and third-order derivatives of the polynomial Bezier curve are $P'(t) = n \sum_{i=0}^{n-1} B_{i,n-1}(t) \Delta P_i$, $P''(t) = n(n-1) \sum_{i=0}^{n-2} B_{i,n-2}(t) \Delta^2 P_i$, $P'''(t) = n(n-1)(n-2) \sum_{i=0}^{n-3} B_{i,n-3}(t) \Delta^3 P_i$, respectively.

Corollary 2.2.3. Let $P_0, P_1, ..., P_n \in \mathbb{R}^3$ be the control points in Euclidean 3-space. Since Bezier curves are examined with their values at the starting and ending points, the geometric values they have at these points are very important. The r-th order derivative of the n-th degree Bezier curve for the starting point $t = 0$ and ending point $t = 1$ are obtained $P^{(r)}\big|_{t=0}(t) = \frac{n!}{(n-r)!} \Delta^r P_0$, $P^{(r)}\big|_{t=1}(t) = \frac{n!}{(n-r)!} \Delta^r P_{n-r}$, respectively. In that case, the first, second, and third-order derivatives of the polynomial Bezier curve for the starting point $t = 0$ are $P'(t)\big|_{t=0} = n.\Delta P_0$, $P''(t)\big|_{t=0} = n(n-1)\Delta^2 P_0$, and $P'''(t)\big|_{t=0} = n(n-1)(n-2)\Delta^3 P_0$. Also the first, second, and third-order derivative of the polynomial Bezier curve for the ending point $t = 1$ are $P'(t)\big|_{t=1} = n.\Delta P_{n-1}$, $P''(t)\big|_{t=1} = n(n-1)\Delta^2 P_{n-1}$, and $P'''(t)\big|_{t=1} = n(n-1)(n-2)\Delta^3 P_{n-1}$.

2.3. THE RATIONAL BEZIER CURVES THROUGH ALGORITHM METHOD IN THE EUCLIDEAN 3-SPACE

In this section, the derivatives of the Bezier curves have been explained by the algorithm method given in the reference [1] and by the algorithm method of the rational Bezier curve.

Definition 2.3.1. Given the control points $P_0, P_1, ..., P_n \in \mathbb{R}^3$ in Euclidean 3-space and the associated nonnegative weights $\omega_0, \omega_1, ..., \omega_n \in \mathbb{R}$, a rational Bezier curve $P(t)$ of the n-th degree is defined by

$$P(t) = \frac{\sum_{i=0}^{n} B_{i,n}(t)\omega_i P_i}{\sum_{i=0}^{n} B_{i,n}(t)\omega_i} \tag{2.11}$$

where the coefficients $B_{0,n}, B_{1,n}, ..., B_{n,n}$ are the Bernstein polynomials. Note that if all weights are equal to 1, a rational Bezier curve reduces to a polynomial Bezier curve. Moreover, the rational Bezier curves express all conic sections. Two-degree rational Bezier curves named quadratic rational Bezier curves are conics. Since the weights are satisfied the conditions $\omega_1^2 - \omega_0\omega_2 < 0$, $\omega_1^2 - \omega_0\omega_2 = 0$, $\omega_1^2 - \omega_0\omega_2 > 0$, then the quadratic rational Bezier curve corresponds to an ellipse, a parabola, and a hyperbola, respectively.

Definition 2.3.2. (de Casteljau Algorithm) The de Casteljau algorithm was developed by Casteljau to calculate the value $P(t_0)$ of a rational Bezier curve at its point $t_0 \in [0,1]$. A cubic Bezier curve with control points P_0, P_1 and P_3 for a specified parameter value $t \in [0,1]$, the Casteljau algorithm is expressed by

$$\begin{array}{llll} P_{0,0} & P_{1,0} & P_{2,0} & P_{3,0} \\ P_{0,1} & P_{1,1} & P_{2,1} & \\ P_{0,2} & P_{1,2} & & \\ P_{0,3} & & & \end{array}$$

the recursive formula $P_{i,0} = P_i,\ P_{i,j} = (1-t)P_{i,j-1} + tP_{i+1,j-1}$ for $t \in [0,1]$, $j = 1,2,3$ and $i = 0,...,3-j$. For the cubic Bezier curve, the Casteljau algorithm generates a triangular set of values as

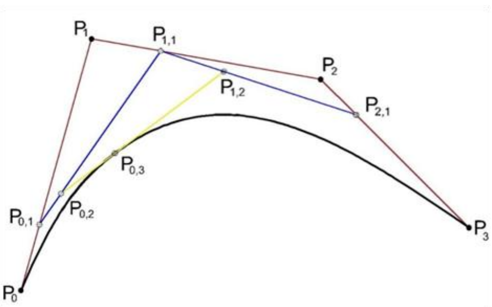

Figure 1. de Casteljau Theorem [1-6].

Farin (1983) gave a new recursive algorithm for calculating $P(t)$, analogous to the de Casteljau algorithm in the following definition:

Definition 2.3.3. Assume that $P_0, P_1, ..., P_n \in \mathbb{R}^3$ are the control points and $\omega_0, \omega_1, ..., \omega_n \in \mathbb{R}$ are the associated nonnegative weights. If the intermediate weights $\omega_{i,k}(t)$ are introduced as

$$\omega_{i,k}(t) = \sum_{j=0}^{k} B_{j,k}(t)\omega_{i+j}, \tag{2.12}$$

the algorithm gives the equation

$$\omega_{i,k} = (1-t)\,\omega_{i,k-1} + t\omega_{i+1,k-1}. \tag{2.13}$$

Then, the intermediate points $P_{i,k}(t)$ are

$$P_{i,k}(t) = \frac{\sum_{j=0}^{k} B_{j,k}(t)\omega_{i+j}P_{i+j}}{\sum_{j=0}^{k} B_{j,k}(t)\omega_{i+j}} \tag{2.14}$$

where $P_{i,0}(t) = P_i$ and $P_{0,n}(t) = P(t)$. Consequently, Farin found a different recursive algorithm for computing $P_{i,k}(t)$ as

$$\omega_{i,k}P_{i,k} = (1-t)\,\omega_{i,k-1}P_{i,k-1} + t\omega_{i+1,k-1}P_{i+1,k-1}. \tag{2.15}$$

Theorem 2.3.4. Let $P_{i,k} \in \mathbb{R}^3$ be the control points and $\omega_{i,k} \in \mathbb{R}$ be the associated nonnegative weights. Using the algorithm method, the first derivative of a rational Bezier curve $P(t)$ of the n-th degree is computed by

$$P'(t) = n\frac{\omega_{0,n-1}\omega_{1,n-1}}{\omega_{0,n}^2}\Delta P_{0,n-1} \tag{2.16}$$

where the difference between the intermediate points is denoted by $\Delta P_{i,j} = P_{i+1,j} - P_{i,j}$. Furthermore, since the weights $\omega_{i,k} \in \mathbb{R}$ are nonnegative, the norm of the first derivative of the curve $P(t)$ is obtained

$$\|P'\| = n\frac{\omega_{0,n-1}\omega_{1,n-1}}{\omega_{0,n}^2}\|\Delta P_{0,n-1}\|. \tag{2.17}$$

Theorem 2.3.5. Assume that $P_{i,k} \in \mathbb{R}^3$ be the control points and $\omega_{i,k} \in \mathbb{R}$ be the associated nonnegative weights. By the algorithm method, the second derivative of a rational Bezier curve $P(t)$ of the n-th degree is computed by

$$P'' = \frac{n}{\omega_{0,n}^3}\left\{\omega_{2,n-2}\omega_{1,n-1}.\lambda_1.\Delta P_{1,n-2} - n\omega_{0,n-2}.\lambda_2.\Delta P_{0,n-2}\right\} \tag{2.18}$$

where $\lambda_1 = 2n\omega_{0,n-1}^2 - (n-1)\omega_{0,n-2}\omega_{0,n} - 2\omega_{0,n-1}\omega_{0,n}$ and $\lambda_2 = 2n\omega_{1,n-1}^2 - (n-1)\omega_{2,n-2}\omega_{0,n} - 2\omega_{1,n-1}\omega_{0,n}$.

Theorem 2.3.6. Let $P_{i,k} \in \mathbb{R}^3$ be the control points and $\omega_{i,k} \in \mathbb{R}$ be the associated positive weights. By using the algorithm method the third derivative of a rational Bezier curve $P(t)$ of the n-th degree is

$$P''' = \gamma_1.\Delta P_{0,n-3} + \gamma_2.\Delta P_{1,n-3} + \gamma_3.\Delta P_{2,n-3} \tag{2.19}$$

where γ_i for $i = 1,2,3$ are the coefficients included the weights $\omega_{i,k}$.

Corollary 2.3.7. Using the Equations (2.16) and (2.18), the vectorial product of P' and P'' is obtained

$$P' \times P'' = n^2 \left(n-1\right) \frac{\omega_{0,n-2}\omega_{1,n-2}\omega_{2,n-2}}{\omega_{0,n}^3} \left(\Delta P_{0,n-2} \times \Delta P_{1,n-2}\right) \tag{2.20}$$

and

$$\left\| P' \times P'' \right\| = n^2 \left(n-1\right) \frac{\omega_{0,n-2}\omega_{1,n-2}\omega_{2,n-2}}{\omega_{0,n}^3} \left\| \Delta P_{0,n-2} \times \Delta P_{1,n-2} \right\|.$$

Corollary 2.3.8. Using the Equations (2.16), (2.18), and (2.19) the triple product of P', P'' and P''' is calculated

$$\left\langle P' \times P'', P''' \right\rangle = n^3 \left(n-1\right)^2 \left(n-2\right) \frac{\omega_{0,n-3}\omega_{1,n-3}\omega_{2,n-3}\omega_{3,n-3}}{\omega_{0,n}^4} \left\langle \Delta P_{0,n-3} \times \Delta P_{1,n-3}, \Delta P_{2,n-3} \right\rangle \tag{2.21}$$

Theorem 2.3.9. Let $P_{i,k} \in \mathbb{R}^3$ be the control points of the non-unit speed rational Bezier curve and associated positive weights be $\omega_{i,k} \in \mathbb{R}^+$. The curvature of the rational Bezier curve which is at any point of it is computed by the algorithm method as

$$\kappa(t) = \frac{\left(n-1\right)}{n} \frac{\omega_{0,n}^3 \omega_{0,n-2}\omega_{1,n-2}\omega_{2,n-2}}{\omega_{0,n-1}^3 \omega_{1,n-1}^3} \frac{\left\| \Delta P_{0,n-2}(t) \times \Delta P_{1,n-2}(t) \right\|}{\left\| \Delta P_{0,n-1}(t) \right\|^3} \tag{2.22}$$

$$\tau(t) = \frac{\left(n-2\right)}{n} \frac{\omega_{0,n}^2 \omega_{0,n-3}\omega_{1,n-3}\omega_{2,n-3}\omega_{3,n-3}}{\omega_{0,n-2}^2 \omega_{1,n-2}^2 \omega_{2,n-2}^2} \frac{\left\langle \Delta P_{0,n-3}(t) \times \Delta P_{1,n-3}(t), \Delta P_{2,n-3}(t) \right\rangle}{\left\| \Delta P_{0,n-2}(t) \times \Delta P_{1,n-2}(t) \right\|^2} \tag{2.23}$$

Proof: When the curvature and torsion of the non-unit speed curves are applied for the rational Bezier curve, the result becomes $\kappa(t) = \dfrac{\left\| P'(t) \times P''(t) \right\|}{\left\| P'(t) \right\|^3}$ and $\tau(t) = \dfrac{\left\langle P'(t) \times P''(t), P'''(t) \right\rangle}{\left\| P'(t) \times P''(t) \right\|^2}$. If the equations (2.20) and (2.21) are substituted in these curvature and torsion formulas, the curvature and torsion equations become computed.

3. RESULTS AND DISCUSSION

3.1. SERRET-FRENET FRAME OF THE PLANAR RATIONAL BEZIER CURVES BY THE ALGORITHM METHOD

Theorem 3.1.1. Let $P_0, P_1, ..., P_n \in \mathbb{R}^3$ be the control points and $\omega_{i,k} \in \mathbb{R}^+$ be the positive weights. The Serret-Frenet frame of the non-unit speed planar rational Bezier curve $P(t)$ for the interval $0 \le t \le 1$ by algorithm method is computed by

$$T(t) = \frac{\Delta P_{0,n-1}(t)}{\left\|\Delta P_{0,n-1}(t)\right\|} \quad \text{and} \quad N(t) = \frac{J\left(\Delta P_{0,n-1}\right)}{\left\|\Delta P_{0,n-1}\right\|} = \frac{\left(-\Delta P^y_{0,n-1}, \Delta P^x_{0,n-1}\right)}{\left\|\Delta P_{0,n-1}\right\|}$$

where T and N are called by a tangent and a principal normal vector.

Proof: Since the equations (2.16) and (2.17) are substituted in the equation $T(t) = \frac{P'(t)}{\left\|P'(t)\right\|}$ which is the tangent vector field of the formulas of the Serret-Frenet frame for the planar curves in the equation (2.4) described in the theorem (2.1.9) and then by making some abbreviations, the tangent of the non-unit speed planar rational Bezier curve is found as $T(t) = \frac{\Delta P_{0,n-1}(t)}{\left\|\Delta P_{0,n-1}(t)\right\|}$. When the definition of the complex structure in Definition 2.5 is applied to the rational Bezier curve, the equation $JP'(t) = n\frac{\omega_{0,n-1}\omega_{1,n-1}}{\omega^2_{0,n}} J\left(\Delta P_{0,n-1}\right)$ is obtained. Assume that $J\left(\Delta P_{0,n-1}\right) = \left(-\Delta P^y_{0,n-1}, \Delta P^x_{0,n-1}\right)$, and when it is substituted in the principal normal vector $N(t) = \frac{JP'(t)}{\left\|P'(t)\right\|}$ given in the equation (2.4), the result $N(t) = \frac{J\left(\Delta P_{0,n-1}\right)}{\left\|\Delta P_{0,n-1}\right\|} = \frac{\left(-\Delta P^y_{0,n-1}, \Delta P^x_{0,n-1}\right)}{\left\|\Delta P_{0,n-1}\right\|}$ is obtained.

Theorem 3.1.2. Let $P_0, P_1, ..., P_n \in \mathbb{R}^3$ be the control points and $\omega_{i,k} \in \mathbb{R}^+$ positive weights. The complex structure and the inner product of the first and the second derivative of the planar rational Bezier curve $P(t)$ for the interval $0 \le t \le 1$ are computed as

$$\left\langle P''(t), JP'(t)\right\rangle = n^2(n-1)\frac{\omega_{0,n-2}\omega_{1,n-2}\omega_{2,n-2}}{\omega^3_{0,n}}\left\langle \Delta P_{1,n-2}, J\Delta P_{0,n-2}\right\rangle$$

Proof: Since the denotation $P''(t)$ and $P'(t)$ are substituted in the inner product with the help of the equations (2.16) and (2.18), then the inner product is calculated by

$$\begin{aligned}\left\langle P''(t), JP'(t)\right\rangle &= n^2\frac{\omega_{2,n-2}\omega_{1,n-1}\omega_{0,n-2}}{\omega_{0,n}^5}\left\{2n\omega_{0,n-1}^2-(n-1)\omega_{0,n}\omega_{2,n-2}-2\omega_{1,n-1}\omega_{0,n}\right\}(1-t)\left\langle \Delta P_{1,n-2}, J\Delta P_{0,n-2}\right\rangle\\ &-n^2\frac{\omega_{0,n-2}^2\omega_{1,n-1}}{\omega_{0,n}^5}\left\{2n\omega_{1,n-1}^2-(n-1)\omega_{0,n}\omega_{2,n-2}-2\omega_{1,n-1}\omega_{0,n}\right\}(1-t)\left\langle \Delta P_{0,n-2}, J\Delta P_{0,n-2}\right\rangle\\ &+n^2\frac{\omega_{2,n-2}^2\omega_{0,n-1}}{\omega_{0,n}^5}\left\{2n\omega_{0,n-1}^2-(n-1)\omega_{0,n-2}\omega_{0,n}-2\omega_{0,n-1}\omega_{0,n}\right\}t\left\langle \Delta P_{1,n-2}, J\Delta P_{1,n-2}\right\rangle\\ &-n^2\frac{\omega_{0,n-2}\omega_{0,n-1}\omega_{2,n-2}}{\omega_{0,n}^5}\left\{2n\omega_{0,n-1}^2-(n-1)\omega_{0,n-2}\omega_{0,n}-2\omega_{0,n-1}\omega_{0,n}\right\}t\left\langle \Delta P_{0,n-2}, J\Delta P_{1,n-2}\right\rangle.\end{aligned}$$

When it is admitted to the shared bracket by using the inner product in this equation

$\left\langle \Delta P_{0,n-2}, J\Delta P_{0,n-2}\right\rangle = 0$, $\left\langle \Delta P_{1,n-2}, J\Delta P_{1,n-2}\right\rangle = 0$ ve $\left\langle \Delta P_{0,n-2}, J\Delta P_{1,n-2}\right\rangle = -\left\langle J\Delta P_{0,n-2}, \Delta P_{1,n-2}\right\rangle$,

the result

$$n^2\frac{\omega_{0,n-2}\omega_{2,n-2}}{\omega_{0,n}^5}\begin{bmatrix}2n\omega_{0,n-1}\omega_{1,n-1}\left\{\omega_{0,n-1}(1-t)+\omega_{1,n-1}t\right\}\\ -(n-1)\omega_{0,n}\left\{\omega_{0,n-2}\omega_{1,n-1}(1-t)+\omega_{2,n-2}\omega_{0,n-1}t\right\}\\ \omega_{0,n-1}\omega_{0,n}\omega_{1,n-1}\left\{-2(1-t)-2t\right\}\end{bmatrix}\left\langle \Delta P_{1,n-2}, J\Delta P_{0,n-2}\right\rangle$$

is obtained. After using the algorithm method $\omega_{i,k}=(1-t)\omega_{i,k-1}+t\omega_{i+1,k-1}$ in the equation (2.13) and making some editing the equation

$$n^2\frac{\omega_{0,n-2}\omega_{2,n-2}}{\omega_{0,n}^5}\left[2(n-1)\omega_{0,n-1}\omega_{1,n-1}\omega_{0,n}-(n-1)\omega_{0,n}\left\{\omega_{0,n-2}\omega_{1,n-1}(1-t)+\omega_{2,n-2}\omega_{0,n-1}t\right\}\right]\left\langle \Delta P_{1,n-2}, J\Delta P_{0,n-2}\right\rangle$$

is obtained. Since it is admitted to the bracket $(n-1)\omega_{0,n}$, the equation

$$n^2(n-1)\frac{\omega_{0,n-2}\omega_{2,n-2}}{\omega_{0,n}^4}\left[\omega_{1,n-1}\left\{\omega_{0,n-1}-(1-t)\omega_{0,n-2}\right\}+\omega_{0,n-1}\left\{\omega_{1,n-1}-t\omega_{2,n-1}\right\}\right]\left\langle \Delta P_{1,n-2}, J\Delta P_{0,n-2}\right\rangle$$

is founded. Then, when the algorithm method is used again, the result becomes

$$n^2(n-1)\frac{\omega_{0,n-2}\omega_{2,n-2}}{\omega_{0,n}^4}\left[\omega_{1,n-2}\left\{t\omega_{1,n-1}+(1-t)\omega_{0,n-1}\right\}\right]\left\langle \Delta P_{1,n-2}, J\Delta P_{0,n-2}\right\rangle.$$

By the renewing of the algorithm method, the result

$$\left\langle P''(t), JP'(t)\right\rangle = n^2(n-1)\frac{\omega_{0,n-2}\omega_{1,n-2}\omega_{2,n-2}}{\omega_{0,n}^3}\left\langle \Delta P_{1,n-2}, J\Delta P_{0,n-2}\right\rangle$$

is computed.

Theorem 3.1.3. Let $P_0, P_1, ..., P_n \in \mathbb{R}^3$ be the control points and $\omega_{i,k} \in \mathbb{R}^+$ be the positive weights. The curvature of the planar rational Bezier curve $P(t)$ for the interval $0 \le t \le 1$ by algorithm method is

$$\kappa(t) = \frac{(n-1)}{n} \frac{\omega_{0,n}^3 \omega_{0,n-2} \omega_{1,n-2} \omega_{2,n-2}}{\omega_{0,n-1}^3 \omega_{1,n-1}^3} \frac{\left\langle \Delta P_{1,n-2}, J \Delta P_{0,n-2} \right\rangle}{\left\| \Delta P_{0,n-1} \right\|^3}$$

or when it is written in a more explanatory way, it is computed as

$$\kappa(t) = \frac{(n-1)}{n} \frac{\omega_{0,n}^3 \omega_{0,n-2} \omega_{1,n-2} \omega_{2,n-2}}{\omega_{0,n-1}^3 \omega_{1,n-1}^3} \frac{\Delta P_{0,n-2}^x \Delta P_{1,n-2}^y - \Delta P_{1,n-2}^x \Delta P_{0,n-2}^y}{\left\| \Delta P_{0,n-1} \right\|^3}.$$

Proof: If the inner product $\left\langle P''(t), JP'(t) \right\rangle$ and the norm $\left\| P'(t) \right\|$ which are found in the Theorem (3.1.2) and (2.17) are substituted in the equation (2.3) which is the curvature formula of the planar curves in the Euclidean space, the curvature of the planar rational Bezier curve becomes

$$\kappa(t) = \frac{n^2 (n-1) \dfrac{\omega_{0,n-2} \omega_{1,n-2} \omega_{2,n-2}}{\omega_{0,n}^3} \left\langle \Delta P_{1,n-2}, J \Delta P_{0,n-2} \right\rangle}{n^3 \dfrac{\omega_{0,n-1}^3 \omega_{1,n-1}^3}{\omega_{0,n}^6} \left\| \Delta P_{0,n-1} \right\|^3}.$$

Since the inner product is arranged by utilizing the definition of the complex structure in (2.1), it is computed as

$$\kappa(t) = \frac{(n-1)}{n} \frac{\omega_{0,n}^3 \omega_{0,n-2} \omega_{1,n-2} \omega_{2,n-2}}{\omega_{0,n-1}^3 \omega_{1,n-1}^3} \frac{\Delta P_{0,n-2}^x \Delta P_{1,n-2}^y - \Delta P_{1,n-2}^x \Delta P_{0,n-2}^y}{\left\| \Delta P_{0,n-1} \right\|^3}.$$

Theorem 3.1.4 Assume that $P_0, P_1, ..., P_n \in \mathbb{R}^3$ are control points and $\omega_{i,k} \in \mathbb{R}^+$ are weights. The derivative formulas of the Serret-Frenet frame of the planar rational Bezier curve $P(t)$ for the interval $0 \le t \le 1$ are computed as

$$T' = (n-1) \frac{\omega_{0,n} \omega_{0,n-2} \omega_{1,n-2} \omega_{2,n-2}}{\omega_{0,n-1}^2 \omega_{1,n-1}^2} \frac{\left\langle \Delta P_{1,n-2}, J \Delta P_{0,n-2} \right\rangle}{\left\| \Delta P_{0,n-1} \right\|^2} N$$

$$N' = -(n-1) \frac{\omega_{0,n} \omega_{0,n-2} \omega_{1,n-2} \omega_{2,n-2}}{\omega_{0,n-1}^2 \omega_{1,n-1}^2} \frac{\left\langle \Delta P_{1,n-2}, J \Delta P_{0,n-2} \right\rangle}{\left\| \Delta P_{0,n-1} \right\|^2} T$$

by the algorithm method.

Proof: Since the speed of the curve $v = \left\| P'(t) \right\| = n \dfrac{\omega_{0,n-1} \omega_{1,n-1}}{\omega_{0,n}^2} \left\| \Delta P_{0,n-1} \right\|$ and the curvature equations $\kappa(t)$ in Theorem (2.1.12) are substituted in the derivative formulas of the Serret-

Frenet frame for the non-unit speed planar curves in the Theorem (2.3.9), the proof of the theorem is obtained.

Corollary 3.1.5. Assume that $P_0, P_1, ..., P_n \in \mathbb{R}^3$ are the control points and $\omega_{i,k} \in \mathbb{R}^+$ are the weights. $T(t)\big|_{t=0} = \frac{\Delta P_0}{\|\Delta P_0\|}$ is the tangent at the initial point $t=0$ of the non-unit speed planar rational Bezier curve $P(t)$, $N(t)\big|_{t=0} = \frac{J\left(\Delta P_0\right)}{\left\|J\left(\Delta P_0\right)\right\|}$ is the principal normal vector field and

$$\kappa(t)\big|_{t=0} = \frac{n-1}{n}\frac{\omega_0\omega_2}{\omega_1^2}\frac{\left\langle \Delta P_1, J\Delta P_0\right\rangle}{\|\Delta P_0\|}$$

is the curvature of it.

The derivative formulas of the Serret-Frenet frame are obtained as

$$T'(t)\big|_{t=0} = (n-1)\frac{\omega_2}{\omega_1}\frac{\left\langle \Delta P_{1,n-2}, J\Delta P_{0,n-2}\right\rangle}{\left\|\Delta P_{0,n-1}\right\|^2} N\big|_{t=0}$$

$$N'(t)\big|_{t=0} = -(n-1)\frac{\omega_2}{\omega_1}\frac{\left\langle \Delta P_{1,n-2}, J\Delta P_{0,n-2}\right\rangle}{\left\|\Delta P_{0,n-1}\right\|^2} T\big|_{t=0}$$

Corollary 3.1.6. Let $P_0, P_1, ..., P_n \in \mathbb{R}^3$ be the control points and $\omega_{i,k} \in \mathbb{R}^+$ be the weights. The tangent at the ending point $t=1$ of the non-unit speed planar rational Bezier curve $P(t)$ is $T(t)\big|_{t=1} = \frac{\Delta P_{n-1}}{\|\Delta P_{n-1}\|}$, the principal normal vector field is $N(t)\big|_{t=1} = \frac{J\left(\Delta P_{n-1}\right)}{\left\|J\left(\Delta P_{n-1}\right)\right\|}$ and the curvature of it is

$$\kappa(t)\big|_{t=1} = \frac{n-1}{n}\frac{\omega_{n-1}\omega_{n-3}}{\omega_{n-1}^2}\frac{\left\langle \Delta P_{n-2}, J\Delta P_{n-1}\right\rangle}{\|\Delta P_{n-1}\|^3}.$$

Therefore, the derivative formulas of the Serret-Frenet frame are obtained as

$$T'(t)\big|_{t=1} = (n-1)\frac{\omega_{n-3}}{\omega_{n-2}}\frac{\left\langle \Delta P_{1,n-2}, J\Delta P_{0,n-2}\right\rangle}{\left\|\Delta P_{0,n-1}\right\|^2} N\big|_{t=1}$$

$$N'(t)\big|_{t=1} = -(n-1)\frac{\omega_{n-3}}{\omega_{n-2}}\frac{\left\langle \Delta P_{1,n-2}, J\Delta P_{0,n-2}\right\rangle}{\left\|\Delta P_{0,n-1}\right\|^2} T\big|_{t=1}.$$

Corollary 3.1.7. Let $P_0, P_1, ..., P_n \in \mathbb{R}^3$ be the control points and $\omega_{i,k} \in \mathbb{R}^+$ be the positive weights. Since the open writing of the non-unit speed planar rational Bezier curve $P(t)$, which is found by algorithm method for the interval $0 \le t \le 1$, is made by the Bernstein polynomial, the tangent vector field of the Serret-Frenet frame is

$$T(t)=\frac{\sum_{i=0}^{n-1}B_{i,n-1}(t)\Delta P_i(t)}{\sqrt{\sum_{i=0}^{n-1}B_{i,n-1}(t)B_{j,n-1}(t)\left\langle \Delta P_i(t),\Delta P_j(t)\right\rangle}},$$

the principal normal vector field is

$$N(t)=\frac{\sum_{i=0}^{n-1}B_{i,n-1}(t)J\left(\Delta P_i\right)(t)}{\sqrt{\sum_{i,j=0}^{n-1}B_{i,n-1}(t)B_{j,n-1}(t)\left\langle \Delta P_i,\Delta P_j\right\rangle}},$$

and the curvature is

$$\kappa(t)=\frac{n-1}{n}\frac{\omega_{0,n}^3\omega_{0,n-2}\omega_{1,n-2}\omega_{2,n-2}}{\omega_{0,n-1}^3\omega_{1,n-1}^3}\frac{\sum_{i=0}^{n-2}\sum_{j=0}^{n-1}B_{i,n-2}(t)B_{j,n-1}(t)\left\langle \Delta^2 P_i,J\Delta P_j\right\rangle}{\sqrt{\left(\sum_{i,j=0}^{n-1}B_{i,n-1}(t)B_{j,n-1}(t)\left\langle \Delta P_i,\Delta P_j\right\rangle\right)^3}}.$$

The derivative formulas of the Serret-Frenet frame are computed as

$$T'=(n-1)\frac{\omega_{0,n}\omega_{0,n-2}\omega_{1,n-2}\omega_{2,n-2}}{\omega_{0,n-1}^2\omega_{1,n-1}^2}\frac{\sum_{i=0}^{n-2}\sum_{j=0}^{n-1}B_{i,n-2}(t)B_{j,n-1}(t)\left\langle \Delta^2 P_i,J\Delta P_j\right\rangle}{\sum_{j=0}^{n-1}B_{i,n-1}(t)B_{j,n-1}(t)\left\langle \Delta P_i,\Delta P_j\right\rangle}N$$

$$N'=-(n-1)\frac{\omega_{0,n}\omega_{0,n-2}\omega_{1,n-2}\omega_{2,n-2}}{\omega_{0,n-1}^2\omega_{1,n-1}^2}\frac{\sum_{i=0}^{n-2}\sum_{j=0}^{n-1}B_{i,n-2}(t)B_{j,n-1}(t)\left\langle \Delta^2 P_i,J\Delta P_j\right\rangle}{\sum_{j=0}^{n-1}B_{i,n-1}(t)B_{j,n-1}(t)\left\langle \Delta P_i,\Delta P_j\right\rangle}T$$

Corollary 3.1.8. If the weights are taken as $\omega_{i,k}=1$ in the non-unit speed planar rational Bezier curve $P(t)$ which is composed of the control points $P_0,P_1,...,P_n\in\mathbb{R}^3$ and positive weights $\omega_{i,k}\in\mathbb{R}^+$, the planar polynomial Bezier curve occurs. Thus, the tangent of the Serret-Frenet frame of the planar polynomial Bezier curve is $T(t)=\dfrac{\Delta P_{0,n-1}(t)}{\left\|\Delta P_{0,n-1}(t)\right\|}$, the principal normal is $N(t)=\dfrac{J\left(\Delta P_{0,n-1}\right)}{\left\|\Delta P_{0,n-1}\right\|}=\dfrac{\left(-\Delta P_{0,n-1}^y,\Delta P_{0,n-1}^x\right)}{\left\|\Delta P_{0,n-1}\right\|}$ and the curvature of it becomes

$$\kappa(t)=\frac{(n-1)}{n}\frac{\left\langle \Delta P_{1,n-2},J\Delta P_{0,n-2}\right\rangle}{\left\|\Delta P_{0,n-1}\right\|^3}\ .$$

The derivative formulas of the Serret-Frenet frame are found as

$$T' = (n-1)\frac{\left\langle \Delta P_{1,n-2}, J\Delta P_{0,n-2}\right\rangle}{\left\|\Delta P_{0,n-1}\right\|^2} N \text{ and } N' = -(n-1)\frac{\left\langle \Delta P_{1,n-2}, J\Delta P_{0,n-2}\right\rangle}{\left\|\Delta P_{0,n-1}\right\|^2} T.$$

Corollary 3.1.9. Considering the Serret-Frenet frame of the non-unit speed planar rational Bezier curve $P(t)$ which is composed of the control points $P_0, P_1, ..., P_n \in \mathbb{R}^3$ and positive weights $\omega_{i,k} \in \mathbb{R}^+$ at the initial point $t=0$. The tangent and principal normal vector field are computed by $T(t)\big|_{t=0} = \frac{\Delta P_0}{\left\|\Delta P_0\right\|}$ and $N(t)\big|_{t=0} = \frac{J\left(\Delta P_0\right)}{\left\|J\left(\Delta P_0\right)\right\|}$, then the curvature is $\kappa(t)\big|_{t=0} = \frac{n-1}{n}\frac{\left\langle \Delta P_1, J\Delta P_0\right\rangle}{\left\|\Delta P_0\right\|}$, and also the derivative formulas of the Serret-Frenet frame are obtained as

$$T'(t)\big|_{t=0} = (n-1)\frac{\left\langle \Delta P_{1,n-2}, J\Delta P_{0,n-2}\right\rangle}{\left\|\Delta P_{0,n-1}\right\|^2} N\big|_{t=0}$$

and

$$N'(t)\big|_{t=0} = -(n-1)\frac{\left\langle \Delta P_{1,n-2}, J\Delta P_{0,n-2}\right\rangle}{\left\|\Delta P_{0,n-1}\right\|^2} T\big|_{t=0}$$

by taking the weights $\omega_{i,k} = 1$.

Corollary 3.1.10. The tangent at the ending point $t=0$ of the Serret-Frenet frame of the non-unit speed planar rational Bezier curve $P(t)$ which is composed of the control points $P_0, P_1, ..., P_n \in \mathbb{R}^3$ and the positive weights $\omega_{i,k} \in \mathbb{R}^+$ by taking $\omega_{i,k} = 1$ is $T(t)\big|_{t=1} = \frac{\Delta P_{n-1}}{\left\|\Delta P_{n-1}\right\|}$, the principal normal vector field is $N(t)\big|_{t=1} = \frac{J\left(\Delta P_{n-1}\right)}{\left\|J\left(\Delta P_{n-1}\right)\right\|}$, the curvature is $\kappa(t)\big|_{t=1} = \frac{n-1}{n}\frac{\left\langle \Delta P_{n-2}, J\Delta P_{n-1}\right\rangle}{\left\|\Delta P_{n-1}\right\|^3}$. The derivative formulas of the Serret-Frenet frame are obtained as

$$T'(t)\big|_{t=1} = (n-1)\frac{\left\langle \Delta P_{1,n-2}, J\Delta P_{0,n-2}\right\rangle}{\left\|\Delta P_{0,n-1}\right\|^2} N\big|_{t=1} \text{ and } N'(t)\big|_{t=1} = -(n-1)\frac{\left\langle \Delta P_{1,n-2}, J\Delta P_{0,n-2}\right\rangle}{\left\|\Delta P_{0,n-1}\right\|^2} T\big|_{t=1}.$$

Corollary 3.1.11. Since the curve is taken as $n=2$, the results are obtained similar with the quadric Bezier curve $P(t) = \left(1-t\right)^2 P_0 + 2\left(1-t\right)tP_1 + t^2 P_2$, also since it is taken as $n=3$ the results are found similar with the cubic Bezier curve $P(t) = \left(1-t\right)^3 P_0 + 3\left(1-t\right)^2 tP_1 + 3\left(1-t\right)t^2 P_2 + t^3 P_3$.

3.2. THE SERRET-FRENET FRAME OF THE RATIONAL BEZIER CURVES IN THE EUCLIDEAN 3-SPACE BY THE ALGORITHM METHOD

Theorem 3.2.1. Let $P_0, P_1, ..., P_n \in \mathbb{R}^3$ be the control points and $\omega_{i,k} \in \mathbb{R}^+$ be the weights. Also, assume that the speed of the rational Bezier curve $P(t)$ for the interval $0 \le t \le 1$ is $v = \|P'(t)\| = n \frac{\omega_{0,n-1}\omega_{1,n-1}}{\omega_{0,n}^2} \|\Delta P_{0,n-1}(t)\|$ and the length of the arc parameter is $s = n \frac{\omega_{0,n-1}\omega_{1,n-1}}{\omega_{0,n}^2} \int_{t_0}^{t_1} \|\Delta P_{0,n-1}(t)\| dt$. The tangent, principal normal, and the binormal vector field of the Serret-Frenet frame of the rational Bezier curve $P(t)$ for the interval $0 \le t \le 1$ through the algorithm method is computed as

$$T(t) = \frac{\Delta P_{0,n-1}(t)}{\|\Delta P_{0,n-1}(t)\|},$$

$$\begin{aligned} N(t) &= \frac{\left(\Delta P_{0,n-2}(t) \times \Delta P_{1,n-2}(t)\right) \times \Delta P_{0,n-1}(t)}{\|\Delta P_{0,n-1}(t)\| \|\Delta P_{0,n-2}(t) \times \Delta P_{1,n-2}(t)\|} \\ &= \frac{\cos\theta_{0,n-1}^{0,n-2}(t)}{\sin\theta_{0,n-2}^{1,n-2}(t)} \frac{\Delta P_{1,n-2}(t)}{\|\Delta P_{1,n-2}(t)\|} - \frac{\cos\theta_{0,n-1}^{1,n-2}(t)}{\sin\theta_{0,n-2}^{1,n-2}(t)} \frac{\Delta P_{0,n-2}(t)}{\|\Delta P_{0,n-2}(t)\|} \end{aligned}$$

$$B(t) = \frac{\Delta P_{0,n-2}(t) \times \Delta P_{1,n-2}(t)}{\|\Delta P_{0,n-2}(t) \times \Delta P_{1,n-2}(t)\|}$$

where the angle $\theta_{i,k}^{j,m} = \sphericalangle\left(\Delta P_{i,k}, \Delta P_{j,m}\right)$ is an angle between the vectors $\Delta P_{i,k}$ and $\Delta P_{j,m}$.

Proof: Since $\omega_{0,n}, \omega_{0,n-1}, \omega_{1,n-1} \in \mathbb{R}^+$ are the positive weights, if the values $P'(t)$ and $\|P'(t)\|$ in the equations (2.16) and (2.17) are substituted in the tangent formula $T(t) = \frac{P'(t)}{\|P'(t)\|}$ of the Serret-Frenet frame of the non-unit speed rational Bezier curve and the abbreviations are made

$$T(t) = \frac{n \frac{\omega_{0,n-1}\omega_{1,n-1}}{\omega_{0,n}^2} \left(P_{1,n-1}(t) - P_{0,n-1}(t)\right)}{\left\| n \frac{\omega_{0,n-1}\omega_{1,n-1}}{\omega_{0,n}^2} \left(P_{1,n-1}(t) - P_{0,n-1}(t)\right) \right\|} = \frac{\Delta P_{0,n-1}(t)}{\|\Delta P_{0,n-1}(t)\|}$$

is obtained. Since the binormal vector field of the rational Bezier curve is substituted for the equations Corollary 2.3.7 in the binormal vector field formula in Definition (2.1.12), it is found as

$$B(t)=\frac{P'(t)\times P''(t)}{\left\|P'(t)\times P''(t)\right\|}$$

$$=\frac{n^2(n-1)\frac{\omega_{0,n-2}\omega_{1,n-2}\omega_{2,n-2}}{\omega_{0,n-2}^3}\Delta P_{0,n-2}(t)\times\Delta P_{1,n-2}(t)}{n^2(n-1)\frac{\omega_{0,n-2}\omega_{1,n-2}\omega_{2,n-2}}{\omega_{0,n-2}^3}\left\|\Delta P_{0,n-2}(t)\times\Delta P_{1,n-2}(t)\right\|}$$

$$=\frac{\Delta P_{0,n-2}(t)\times\Delta P_{1,n-2}(t)}{\left\|\Delta P_{0,n-2}(t)\times\Delta P_{1,n-2}(t)\right\|}.$$

The normal vector field is computed by vectorial product $N(t)=B(t)\times T(t)$ as

$$N(t)=\frac{\Delta P_{0,n-2}(t)\times\Delta P_{1,n-2}(t)}{\left\|\Delta P_{0,n-2}(t)\times\Delta P_{1,n-2}(t)\right\|}\times\frac{\Delta P_{0,n-1}(t)}{\left\|\Delta P_{0,n-1}(t)\right\|}$$

$$=\frac{\left(\Delta P_{0,n-2}(t)\times\Delta P_{1,n-2}(t)\right)\times\Delta P_{0,n-1}(t)}{\left\|\Delta P_{0,n-1}(t)\right\|\left\|\Delta P_{0,n-2}(t)\times\Delta P_{1,n-2}(t)\right\|}$$

where the angle between the vectors $\Delta P_{i,k}$ and $\Delta P_{j,m}$ is $\theta_{i,k}^{j,m}=\sphericalangle\left(\Delta P_{i,k},\Delta P_{j,m}\right)$. When the vector products are arranged in the principal normal vector of the rational Bezier curve, they can also be shown as

$$N(t)=\frac{\left\langle\Delta P_{0,n-1}(t),\Delta P_{0,n-2}(t)\right\rangle\Delta P_{1,n-2}(t)-\left\langle\Delta P_{0,n-1}(t),\Delta P_{1,n-2}(t)\right\rangle\Delta P_{0,n-2}(t)}{\left\|\Delta P_{0,n-1}(t)\right\|\left\|\Delta P_{0,n-2}(t)\right\|\left\|\Delta P_{1,n-2}(t)\right\|\sin\theta_{0,n-2}^{1,n-2}}$$

$$=\frac{\cos\theta_{0,n-1}^{0,n-2}(t)}{\sin\theta_{0,n-2}^{1,n-2}(t)}\frac{\Delta P_{1,n-2}(t)}{\left\|\Delta P_{1,n-2}(t)\right\|}-\frac{\cos\theta_{0,n-1}^{1,n-2}(t)}{\sin\theta_{0,n-2}^{1,n-2}(t)}\frac{\Delta P_{0,n-2}(t)}{\left\|\Delta P_{0,n-2}(t)\right\|}.$$

Theorem 3.2.2. Assume that $P_0,P_1,...,P_n\in\mathbb{R}^3$ are the control points and $\omega_{i,k}\in\mathbb{R}^+$ are the weights. The tangent formula of the Serret-Frenet frame of the rational Bezier curve $P(t)$ for the interval $0\le t\le 1$ through the algorithm method is

$$T'(t)=(n-1)\frac{\omega_{0,n}\omega_{0,n-2}\omega_{1,n-2}\omega_{2,n-2}}{\omega_{0,n-1}^2\omega_{1,n-1}^2}\frac{\left\|\Delta P_{0,n-2}(t)\times\Delta P_{1,n-2}(t)\right\|}{\left\|\Delta P_{0,n-1}(t)\right\|^2}N$$

$$N'(t)=-(n-1)\frac{\omega_{0,n}\omega_{0,n-2}\omega_{1,n-2}\omega_{2,n-2}}{\omega_{0,n-1}^2\omega_{1,n-1}^2}\frac{\left\|\Delta P_{0,n-2}(t)\times\Delta P_{1,n-2}(t)\right\|}{\left\|\Delta P_{0,n-1}(t)\right\|^2}T(t)$$

$$+(n-2)\frac{\omega_{0,n-1}\omega_{0,n-3}\omega_{1,n-1}\omega_{1,n-3}\omega_{2,n-3}\omega_{3,n-3}}{\omega_{0,n}}\frac{\left\langle\Delta P_{0,n-3}(t)\times\Delta P_{1,n-3}(t),\Delta P_{2,n-3}(t)\right\rangle\left\|\Delta P_{0,n-1}(t)\right\|}{\left\|\Delta P_{0,n-2}(t)\times\Delta P_{1,n-2}(t)\right\|^2}B(t)$$

$$B'(t)=-(n-2)\frac{\omega_{0,n-3}\omega_{1,n-3}\omega_{2,n-3}\omega_{3,n-3}\omega_{0,n-1}\omega_{1,n-1}}{\omega_{0,n-2}^2\omega_{1,n-2}^2\omega_{2,n-2}^2}\frac{\left\langle\Delta P_{0,n-3}(t)\times\Delta P_{1,n-3}(t),\Delta P_{2,n-3}(t)\right\rangle\left\|\Delta P_{0,n-1}(t)\right\|}{\left\|\Delta P_{0,n-2}(t)\times\Delta P_{1,n-2}(t)\right\|^2}N(t).$$

Proof: Since the speed of the non-unit speed rational Bezier curve in the Euclidean 3-space is

$v=\left\|P'(t)\right\|=n\frac{\omega_{0,n-1}\omega_{1,n-1}}{\omega_{0,n}^{2}}\left\|\Delta P_{0,n-1}(t)\right\|$, the tangent formulas of the Serret-Frenet frame of the rational Bezier curve is obtained as

$$T'(t)=v\kappa(t)N(t)=(n-1)\frac{\omega_{0,n}\omega_{0,n-2}\omega_{1,n-2}\omega_{2,n-2}}{\omega_{0,n-1}^{2}\omega_{1,n-1}^{2}}\frac{\left\|\Delta P_{0,n-2}(t)\times\Delta P_{1,n-2}(t)\right\|}{\left\|\Delta P_{0,n-1}(t)\right\|^{2}}N,$$

$$\begin{aligned}N'(t)&=-v\kappa(t)T(t)+v\tau(t)B(t)\\&=-(n-1)\frac{\omega_{0,n}\omega_{0,n-2}\omega_{1,n-2}\omega_{2,n-2}}{\omega_{0,n-1}^{2}\omega_{1,n-1}^{2}}\frac{\left\|\Delta P_{0,n-2}(t)\times\Delta P_{1,n-2}(t)\right\|}{\left\|\Delta P_{0,n-1}(t)\right\|^{2}}T\\&\quad+(n-2)\frac{\omega_{0,n-1}\omega_{0,n-3}\omega_{1,n-1}\omega_{1,n-3}\omega_{2,n-3}\omega_{3,n-3}}{\omega_{0,n}}\frac{\left\langle\Delta P_{0,n-3}(t)\times\Delta P_{1,n-3}(t),\Delta P_{2,n-3}(t)\right\rangle\left\|\Delta P_{0,n-1}(t)\right\|}{\left\|\Delta P_{0,n-2}(t)\times\Delta P_{1,n-2}(t)\right\|^{2}}B,\end{aligned}$$

$$\begin{aligned}B'(t)&=-v\tau(t)N(t)\\&=-(n-2)\frac{\omega_{0,n-3}\omega_{1,n-3}\omega_{2,n-3}\omega_{3,n-3}\omega_{0,n-1}\omega_{1,n-1}}{\omega_{0,n-2}^{2}\omega_{1,n-2}^{2}\omega_{2,n-2}^{2}}\frac{\left\langle\Delta P_{0,n-3}(t)\times\Delta P_{1,n-3}(t),\Delta P_{2,n-3}(t)\right\rangle\left\|\Delta P_{0,n-1}(t)\right\|}{\left\|\Delta P_{0,n-2}(t)\times\Delta P_{1,n-2}(t)\right\|^{2}}N(t).\end{aligned}$$

Corollary 3.2.3. Let $P_0,P_1,...,P_n\in\mathbb{R}^3$ be the control points and $\omega_{i,k}\in\mathbb{R}^+$ be the positive weights. The tangent at the initial point $t=0$ of the non-unit speed rational Bezier curve $P(t)$ in the Euclidean 3-space is $T(t)\big|_{t=0}=\frac{\Delta P_0(t)}{\left\|\Delta P_0(t)\right\|}$, the principal normal vector field is $N(t)\big|_{t=0}=\frac{(\Delta P_0\times\Delta P_1)\times\Delta P_0}{\left\|\Delta P_0\right\|\left\|\Delta P_0\times\Delta P_1\right\|}$ or $N(t)\big|_{t=0}=\operatorname{cosec}\theta_0^1\frac{\Delta P_1}{\left\|\Delta P_1\right\|}-\cot\theta_0^1\frac{\Delta P_0(t)}{\left\|\Delta P_0(t)\right\|}$, and the binormal vector field becomes $B(t)\big|_{t=0}=\frac{\Delta P_0\times\Delta P_1}{\left\|\Delta P_0\times\Delta P_1\right\|}$. The curvature is obtained $\kappa(t)\big|_{t=0}=\frac{(n-1)}{n}\frac{\omega_0\omega_2}{\omega_1^2}\frac{\left\|\Delta P_0\times\Delta P_1\right\|}{\left\|\Delta P_0\right\|^3}$ or $\kappa(t)\big|_{t=0}=\frac{(n-1)}{n}\frac{\omega_0\omega_2}{\omega_1^2}\frac{\left\|\Delta P_1\right\|}{\left\|\Delta P_0\right\|^2}\sin\theta_0^1$, and the torsion is obtained as

$$\tau(t)\big|_{t=0}=\frac{(n-2)}{n}\frac{\omega_0\omega_3}{\omega_1\omega_2}\frac{\left\langle\Delta P_0\times\Delta P_1,\Delta P_2\right\rangle}{\left\|\Delta P_0\times\Delta P_1\right\|^2}\text{ or }\tau(t)\big|_{t=0}=\frac{(n-2)}{n}\frac{\omega_0\omega_3}{\omega_1\omega_2}\frac{\det(\Delta P_0,\Delta P_1,\Delta P_2)}{\left\|\Delta P_0\times\Delta P_1\right\|^2}.$$

From the derivative of the Serret-Frenet frame, the derivative of the tangent formula is computed by

$$T'(t)\big|_{t=0}=(n-1)\frac{\omega_2}{\omega_1}\frac{\left\|\Delta P_0\times\Delta P_1\right\|}{\left\|\Delta P_0\right\|^2}N\big|_{t=0}\text{ or }T'(t)\big|_{t=0}=(n-1)\frac{\omega_2}{\omega_1}\frac{\left\|\Delta P_1\right\|}{\left\|\Delta P_0\right\|}\sin\theta_0^1\,N\big|_{t=0}.$$

The principal normal vector becomes

$$N'(t)\big|_{t=0} = -(n-1)\frac{\omega_2}{\omega_1}\frac{\left\|\Delta P_0 \times \Delta P_1\right\|}{\left\|\Delta P_0\right\|^2}T\big|_{t=0} + (n-2)\frac{\omega_3}{\omega_2}\frac{\left\langle \Delta P_0 \times \Delta P_1, \Delta P_2\right\rangle \left\|\Delta P_0\right\|}{\left\|\Delta P_0 \times \Delta P_1\right\|^2}B\big|_{t=0}$$

or

$$N'(t) = -(n-1)\frac{\omega_2}{\omega_1}\frac{\left\|\Delta P_1\right\|}{\left\|\Delta P_0\right\|}\sin\theta_0^1\, T\big|_{t=0} + (n-2)\frac{\omega_3}{\omega_2}\frac{\det\left(\Delta P_0, \Delta P_1, \Delta P_2\right)}{\left\|\Delta P_0\right\|\left\|\Delta P_1\right\|^2 \sin\theta_0^1}B\big|_{t=0}.$$

Moreover, the binormal vector is obtained as

$$B'(t)\big|_{t=0} = -(n-2)\frac{\omega_3}{\omega_2}\frac{\left\langle \Delta P_0 \times \Delta P_1, \Delta P_2\right\rangle \left\|\Delta P_0\right\|}{\left\|\Delta P_0 \times \Delta P_1\right\|}N\big|_{t=0}$$

or

$$B'(t)\big|_{t=0} = -(n-2)\frac{\omega_3}{\omega_2}\frac{\det\left(\Delta P_0, \Delta P_1, \Delta P_2\right)}{\left\|\Delta P_0\right\|\left\|\Delta P_1\right\|^2 \sin^2\theta_0^1}N\big|_{t=0}.$$

Corollary 3.2.4. Let $P_0, P_1, \dots, P_n \in \mathbb{R}^3$ be the control points and $\omega_{i,k} \in \mathbb{R}^+$ be the positive weights. The tangent at the ending point $t=1$ of the non-unit speed rational Bezier curve $P(t)$ in the Euclidean 3-space is $T(t)\big|_{t=1} = \dfrac{\Delta P_{n-1}(t)}{\left\|\Delta P_{n-1}(t)\right\|}$**,** the principal normal vector field is $N(t)\big|_{t=1} = \dfrac{\left(\Delta P_{n-1} \times \Delta P_{n-2}\right) \times \Delta P_{n-1}}{\left\|\Delta P_{n-1}\right\|\left\|\Delta P_{n-1} \times \Delta P_{n-2}\right\|}$ or $N(t)\big|_{t=1} = \cos ec\theta_{n-1}^{n-2}\dfrac{\Delta P_{n-2}}{\left\|\Delta P_{n-2}\right\|} - \cot\theta_{n-1}^{n-2}\dfrac{\Delta P_{n-1}(t)}{\left\|\Delta P_{n-1}(t)\right\|}$, and the binormal vector field is $B(t)\big|_{t=1} = \dfrac{\Delta P_{n-1} \times \Delta P_{n-2}}{\left\|\Delta P_{n-1} \times \Delta P_{n-2}\right\|}$. The curvature is obtained as $\kappa(t)\big|_{t=1} = \dfrac{(n-1)}{n}\dfrac{\omega_{n-1}\omega_{n-3}}{\omega_{n-2}^2}\dfrac{\left\|\Delta P_{n-1} \times \Delta P_{n-2}\right\|}{\left\|\Delta P_{n-1}\right\|^3}$ or $\kappa(t)\big|_{t=1} = \dfrac{(n-1)}{n}\dfrac{\omega_{n-1}\omega_{n-3}}{\omega_{n-2}^2}\dfrac{\left\|\Delta P_{n-2}\right\|}{\left\|\Delta P_{n-1}\right\|^2}\sin\theta_{n-1}^{n-2}$, and the torsion is obtained as

$$\tau(t)\big|_{t=1} = \frac{(n-2)}{n}\frac{\omega_{n-1}\omega_{n-4}}{\omega_{n-2}\omega_{n-3}}\frac{\left\langle \Delta P_{n-1} \times \Delta P_{n-2}, \Delta P_{n-3}\right\rangle}{\left\|\Delta P_{n-1} \times \Delta P_{n-2}\right\|^2}$$

or

$$\tau(t)\big|_{t=1} = \frac{(n-2)}{n}\frac{\omega_{n-1}\omega_{n-4}}{\omega_{n-2}\omega_{n-3}}\frac{\det\left(\Delta P_0, \Delta P_1, \Delta P_2\right)}{\left\|\Delta P_0 \times \Delta P_1\right\|^2}$$

From the tangent formula of the Serret-Frenet frame

$$T'(t)\big|_{t=1} = (n-1)\frac{\omega_{n-3}}{\omega_{n-2}}\frac{\left\|\Delta P_{n-1} \times \Delta P_{n-2}\right\|}{\left\|\Delta P_{n-1}\right\|^2}N\big|_{t=1} \text{ or } T'(t)\big|_{t=1} = (n-1)\frac{\omega_{n-3}}{\omega_{n-2}}\frac{\left\|\Delta P_{n-2}\right\|}{\left\|\Delta P_{n-1}\right\|}\sin\theta_{n-1}^{n-2}\, N\big|_{t=1},$$

the principal normal vector becomes

$$N'(t)\big|_{t=1} = -(n-1)\frac{\omega_{n-3}}{\omega_{n-2}}\frac{\left\|\Delta P_{n-1} \times \Delta P_{n-2}\right\|}{\left\|\Delta P_{n-1}\right\|^2}T\big|_{t=1} + (n-2)\frac{\omega_{n-4}}{\omega_{n-3}}\frac{\left\langle \Delta P_{n-1} \times \Delta P_{n-2}, \Delta P_{n-3}\right\rangle \left\|\Delta P_{n-1}\right\|}{\left\|\Delta P_{n-1} \times \Delta P_{n-2}\right\|^2}B\big|_{t=1}$$

or

$$N'(t)\big|_{t=1}=-\left(n-1\right)\frac{\omega_{n-3}}{\omega_{n-2}}\frac{\left\|\Delta P_{n-2}\right\|}{\left\|\Delta P_{n-1}\right\|}\sin\theta_{n-1}^{n-2}\,T\big|_{t=1}+\left(n-2\right)\frac{\omega_{n-4}}{\omega_{n-3}}\frac{\det\left(\Delta P_{n-1},\Delta P_{n-2},\Delta P_{n-3}\right)}{\left\|\Delta P_{n-1}\right\|\left\|\Delta P_{n-2}\right\|^{2}\sin^{2}\theta_{n-1}^{n-2}}B\big|_{t=1}.$$

The binormal vector is obtained as

$$B'(t)\big|_{t=1}=-\left(n-2\right)\frac{\omega_{n-4}}{\omega_{n-3}}\frac{\left\langle\Delta P_{n-1}\times\Delta P_{n-2},\Delta P_{n-3}\right\rangle\left\|\Delta P_{n-1}\right\|}{\left\|\Delta P_{n-1}\times\Delta P_{n-2}\right\|}N\big|_{t=1}$$

or

$$B'(t)\big|_{t=1}=-\left(n-2\right)\frac{\omega_{n-4}}{\omega_{n-3}}\frac{\det\left(\Delta P_{n-1},\Delta P_{n-2},\Delta P_{n-3}\right)}{\left\|\Delta P_{n-1}\right\|\left\|\Delta P_{n-2}\right\|^{2}\sin^{2}\theta_{n-1}^{n-2}}N\big|_{t=1}.$$

Corollary 3.2.5. Assume that $P_0,P_1,...,P_n\in\mathbb{R}^3$ are control points and $\omega_{i,k}\in\mathbb{R}^+$ are positive weights. When the open writing of the non-unit speed rational Bezier curve $P(t)$ is made by the Bernstein polynomial in the Euclidean 3-space found by the algorithm method for the interval $0\le t\le 1$, the tangent vector field of the Serret-Frenet frame is

$$T(t)=\frac{\sum_{i=0}^{n-1}B_{i,n-1}(t)\Delta P_i(t)}{\sqrt{\sum_{i=0}^{n-1}B_{i,n-1}(t)B_{j,n-1}(t)\left\langle\Delta P_i(t),\Delta P_j(t)\right\rangle}},$$

the principal normal vector field is

$$N(t)=\frac{\sum_{i=0}^{n-1}\sum_{j=0}^{n-2}\sum_{k=0}^{n-1}B_{i,n-1}(t)B_{j,n-2}(t)B_{k,n-1}(t)\left(\Delta P_i\times\Delta^2P_j\right)\times\Delta P_k}{\left\|\sum_{i=0}^{n-1}B_{i,n-1}(t)\Delta P_i(t)\right\|\left\|\sum_{j=0}^{n-2}\sum_{i=0}^{n-2}B_{i,n-2}(t)B_{j,n-2}(t)\left(\Delta P_i\times\Delta^2P_j\right)\right\|},$$

the binormal vector field is

$$B(t)=\frac{\sum_{j=0}^{n-2}\sum_{i=0}^{n-2}B_{i,n-2}(t)B_{j,n-2}(t)\left(\Delta P_i\times\Delta^2P_j\right)}{\left\|\sum_{j=0}^{n-2}\sum_{i=0}^{n-2}B_{i,n-2}(t)B_{j,n-2}(t)\left(\Delta P_i\times\Delta^2P_j\right)\right\|}$$

and the curvature is

$$\kappa(t)=\frac{\left(n-1\right)}{n}\frac{\omega_{0,n}^{3}\omega_{0,n-2}\omega_{1,n-2}\omega_{2,n-2}}{\omega_{0,n-1}^{3}\omega_{1,n-1}^{3}}\frac{\left\|\sum_{j=0}^{n-2}\sum_{i=0}^{n-2}B_{i,n-2}(t)B_{j,n-2}(t)\left(\Delta P_i\times\Delta^2P_j\right)\right\|}{\left\|\sum_{i=0}^{n-1}B_{i,n-1}(t)\Delta P_i(t)\right\|^{3}}$$

$$\tau(t)=\frac{(n-2)}{n}\frac{\omega_{0,n}^2\omega_{0,n-3}\omega_{1,n-3}\omega_{2,n-3}\omega_{3,n-3}}{\omega_{0,n-2}^2\omega_{1,n-2}^2\omega_{2,n-2}^2}\frac{\sum_{i=0}^{n-1}\sum_{j=0}^{n-2}\sum_{k=0}^{n-3}B_{i,n-1}(t)B_{j,n-2}(t)B_{k,n-3}(t)\left\langle\Delta P_i(t)\times\Delta^2 P_j(t),\Delta^3 P_k(t)\right\rangle}{\left\|\sum_{j=0}^{n-2}\sum_{k=0}^{n-2}B_{i,n-1}(t)B_{j,n-2}(t)B_{k,n-3}(t)\left(\Delta P_i(t)\times\Delta^2 P_j(t)\right)\right\|^2}$$

The derivative formulas of the Serret-Frenet frame could be shown similarly.

Corollary 3.2.6. Since the weights are taken as $\omega_{i,k}=1$ in the equation of the non-unit speed rational Bezier curve $P(t)$, which is composed of the control points $P_0,P_1,...,P_n\in\mathbb{R}^3$ and the positive weights $\omega_{i,k}\in\mathbb{R}^+$, in the Euclidean 3-space, the polynomial Bezier curve occurs. Therefore, the Serret-Frenet frame of the polynomial Bezier curve

$$T(t)=\frac{\Delta P_{0,n-1}(t)}{\left\|\Delta P_{0,n-1}(t)\right\|},$$

$$N(t)=\frac{\left(\Delta P_{0,n-2}(t)\times\Delta P_{1,n-2}(t)\right)\times\Delta P_{0,n-1}(t)}{\left\|\Delta P_{0,n-1}(t)\right\|\left\|\Delta P_{0,n-2}(t)\times\Delta P_{1,n-2}(t)\right\|}$$

$$B(t)=\frac{\Delta P_{0,n-2}(t)\times\Delta P_{1,n-2}(t)}{\left\|\Delta P_{0,n-2}(t)\times\Delta P_{1,n-2}(t)\right\|}$$

the curvature, and the torsion of the polynomial Bezier curve

$$\kappa(t)=\frac{(n-1)}{n}\frac{\left\|\Delta P_{0,n-2}(t)\times\Delta P_{1,n-2}(t)\right\|}{\left\|\Delta P_{0,n-1}(t)\right\|^3}$$

and

$$\tau(t)=\frac{(n-2)}{n}\frac{\left\langle\Delta P_{0,n-3}(t)\times\Delta P_{1,n-3}(t),\Delta P_{2,n-3}(t)\right\rangle}{\left\|\Delta P_{0,n-2}(t)\times\Delta P_{1,n-2}(t)\right\|^2}$$

are obtained similarly with the polynomial Bezier curves. Also, the derivative formulas of the Serret-Frenet frame are

$$T'(t)=(n-1)\frac{\left\|\Delta P_{0,n-2}(t)\times\Delta P_{1,n-2}(t)\right\|}{\left\|\Delta P_{0,n-1}(t)\right\|^2}N$$

$$N'(t)=-(n-1)\frac{\left\|\Delta P_{0,n-2}(t)\times\Delta P_{1,n-2}(t)\right\|}{\left\|\Delta P_{0,n-1}(t)\right\|^2}T(t)$$
$$+(n-2)\frac{\left\langle\Delta P_{0,n-3}(t)\times\Delta P_{1,n-3}(t),\Delta P_{2,n-3}(t)\right\rangle\left\|\Delta P_{0,n-1}(t)\right\|}{\left\|\Delta P_{0,n-2}(t)\times\Delta P_{1,n-2}(t)\right\|^2}B(t)$$

$$B'(t) = -(n-2)\frac{\left\langle \Delta P_{0,n-3}(t)\times \Delta P_{1,n-3}(t), \Delta P_{2,n-3}(t)\right\rangle \left\|\Delta P_{0,n-1}(t)\right\|}{\left\|\Delta P_{0,n-2}(t)\times \Delta P_{1,n-2}(t)\right\|^2} N(t).$$

Corollary 3.2.7. The tangent at the initial point $t=0$ of the Serret-Frenet frame of the non-unit speed polynomial Bezier curve $P(t)$, which is composed of the control points $P_0, P_1, ..., P_n \in \mathbb{R}^3$ and the positive weights $\omega_{i,k} \in \mathbb{R}^+$ and $\omega_{i,k} = 1$ taken in the Euclidean 3-space is $T(t)\big|_{t=0} = \frac{\Delta P_0(t)}{\left\|\Delta P_0(t)\right\|}$, the principal normal vector field is $N(t)\big|_{t=0} = \frac{\left(\Delta P_0 \times \Delta P_1\right)\times \Delta P_0}{\left\|\Delta P_0\right\|\left\|\Delta P_0 \times \Delta P_1\right\|}$ or $N(t)\big|_{t=0} = \operatorname{cosec}\theta_0^1 \frac{\Delta P_1}{\left\|\Delta P_1\right\|} - \cot\theta_0^1 \frac{\Delta P_0(t)}{\left\|\Delta P_0(t)\right\|}$, and the binormal vector field becomes $B(t)\big|_{t=0} = \frac{\Delta P_0 \times \Delta P_1}{\left\|\Delta P_0 \times \Delta P_1\right\|}$. The curvature is obtained as $\kappa(t)\big|_{t=0} = \frac{(n-1)}{n}\frac{\left\|\Delta P_0 \times \Delta P_1\right\|}{\left\|\Delta P_0\right\|^3}$ or $\kappa(t)\big|_{t=0} = \frac{(n-1)}{n}\frac{\left\|\Delta P_1\right\|}{\left\|\Delta P_0\right\|^2}\sin\theta_0^1$, and the torsion is obtained as $\tau(t)\big|_{t=0} = \frac{(n-2)}{n}\frac{\left\langle \Delta P_0 \times \Delta P_1, \Delta P_2\right\rangle}{\left\|\Delta P_0 \times \Delta P_1\right\|^2}$ or $\tau(t)\big|_{t=0} = \frac{(n-2)}{n}\frac{\det\left(\Delta P_0, \Delta P_1, \Delta P_2\right)}{\left\|\Delta P_0 \times \Delta P_1\right\|^2}$. From the derivative formulas of the Serret-Frenet frame $T'(t)\big|_{t=0} = (n-1)\frac{\left\|\Delta P_0 \times \Delta P_1\right\|}{\left\|\Delta P_0\right\|^2} N\big|_{t=0}$ or $T'(t)\big|_{t=0} = (n-1)\frac{\left\|\Delta P_1\right\|}{\left\|\Delta P_0\right\|}\sin\theta_0^1 N\big|_{t=0}$, the principal normal vector becomes

$$N'(t)\big|_{t=0} = -(n-1)\frac{\left\|\Delta P_0 \times \Delta P_1\right\|}{\left\|\Delta P_0\right\|^2} T\big|_{t=0} + (n-2)\frac{\left\langle \Delta P_0 \times \Delta P_1, \Delta P_2\right\rangle \left\|\Delta P_0\right\|}{\left\|\Delta P_0 \times \Delta P_1\right\|^2} B\big|_{t=0}$$

or

$$N'(t) = -(n-1)\frac{\left\|\Delta P_1\right\|}{\left\|\Delta P_0\right\|}\sin\theta_0^1 T\big|_{t=0} + (n-2)\frac{\det\left(\Delta P_0, \Delta P_1, \Delta P_2\right)}{\left\|\Delta P_0\right\|\left\|\Delta P_1\right\|^2 \sin\theta_0^1} B\big|_{t=0}.$$

The binormal vector is obtained by

$$B'(t)\big|_{t=0} = -(n-2)\frac{\left\langle \Delta P_0 \times \Delta P_1, \Delta P_2\right\rangle \left\|\Delta P_0\right\|}{\left\|\Delta P_0 \times \Delta P_1\right\|} N\big|_{t=0}$$

or

$$B'(t)\big|_{t=0} = -(n-2)\frac{\det\left(\Delta P_0, \Delta P_1, \Delta P_2\right)}{\left\|\Delta P_0\right\|\left\|\Delta P_1\right\|^2 \sin^2\theta_0^1} N\big|_{t=0}.$$

It seems that the results here are the same as those in the study [10].

Corollary 3.2.8. The tangent at the ending point $t=1$ of the Serret-Frenet frame of the non-unit speed polynomial Bezier curve $P(t)$, which is composed of the control points $P_0, P_1, ..., P_n \in \mathbb{R}^3$ and the positive weights $\omega_{i,k} \in \mathbb{R}^+$ and taking the weights as $\omega_{i,k} = 1$ in the

Euclidean 3-space is $T(t)\big|_{t=1}=\dfrac{\Delta P_{n-1}(t)}{\left\|\Delta P_{n-1}(t)\right\|}$, the principal normal vector field is $N(t)\big|_{t=1}=\dfrac{\left(\Delta P_{n-1}\times\Delta P_{n-2}\right)\times\Delta P_{n-1}}{\left\|\Delta P_{n-1}\right\|\left\|\Delta P_{n-1}\times\Delta P_{n-2}\right\|}$ or $N(t)\big|_{t=1}=\operatorname{cosec}\theta_{n-1}^{n-2}\dfrac{\Delta P_{n-2}}{\left\|\Delta P_{n-2}\right\|}-\cot\theta_{n-1}^{n-2}\dfrac{\Delta P_{n-1}(t)}{\left\|\Delta P_{n-1}(t)\right\|}$, the binormal vector field is $B(t)\big|_{t=1}=\dfrac{\Delta P_{n-1}\times\Delta P_{n-2}}{\left\|\Delta P_{n-1}\times\Delta P_{n-2}\right\|}$. The curvature is

$$\kappa(t)\big|_{t=1}=\frac{(n-1)}{n}\frac{\left\|\Delta P_{n-1}\times\Delta P_{n-2}\right\|}{\left\|\Delta P_{n-1}\right\|^3}\ \text{or}\ \kappa(t)\big|_{t=1}=\frac{(n-1)}{n}\frac{\left\|\Delta P_{n-2}\right\|}{\left\|\Delta P_{n-1}\right\|^2}\sin\theta_{n-1}^{n-2},$$

and the torsion is obtained as

$$\tau(t)\big|_{t=1}=\frac{(n-2)}{n}\frac{\left\langle\Delta P_{n-1}\times\Delta P_{n-2},\Delta P_{n-3}\right\rangle}{\left\|\Delta P_{n-1}\times\Delta P_{n-2}\right\|^2}\ \text{or}\ \tau(t)\big|_{t=1}=\frac{(n-2)}{n}\frac{\det\left(\Delta P_0,\Delta P_1,\Delta P_2\right)}{\left\|\Delta P_0\times\Delta P_1\right\|^2}.$$

From the derivative formula of the Serret-Frenet frame $T'(t)\big|_{t=1}=(n-1)\dfrac{\left\|\Delta P_{n-1}\times\Delta P_{n-2}\right\|}{\left\|\Delta P_{n-1}\right\|^2}N\big|_{t=1}$ or $T'(t)\big|_{t=1}=(n-1)\dfrac{\left\|\Delta P_{n-2}\right\|}{\left\|\Delta P_{n-1}\right\|}\sin\theta_{n-1}^{n-2}N\big|_{t=1}$, the principal normal vector becomes

$$N'(t)\big|_{t=1}=-(n-1)\frac{\left\|\Delta P_{n-1}\times\Delta P_{n-2}\right\|}{\left\|\Delta P_{n-1}\right\|^2}T\big|_{t=1}+(n-2)\frac{\left\langle\Delta P_{n-1}\times\Delta P_{n-2},\Delta P_{n-3}\right\rangle\left\|\Delta P_{n-1}\right\|}{\left\|\Delta P_{n-1}\times\Delta P_{n-2}\right\|^2}B\big|_{t=1}$$

or

$$N'(t)\big|_{t=1}=-(n-1)\frac{\left\|\Delta P_{n-2}\right\|}{\left\|\Delta P_{n-1}\right\|}\sin\theta_{n-1}^{n-2}T\big|_{t=1}+(n-2)\frac{\det\left(\Delta P_{n-1},\Delta P_{n-2},\Delta P_{n-3}\right)}{\left\|\Delta P_{n-1}\right\|\left\|\Delta P_{n-2}\right\|^2\sin^2\theta_{n-1}^{n-2}}B\big|_{t=1}.$$

The binormal vector is obtained by

$$B'(t)\big|_{t=1}=-(n-2)\frac{\left\langle\Delta P_{n-1}\times\Delta P_{n-2},\Delta P_{n-3}\right\rangle\left\|\Delta P_{n-1}\right\|}{\left\|\Delta P_{n-1}\times\Delta P_{n-2}\right\|}N\big|_{t=1}$$

or

$$B'(t)\big|_{t=1}=-(n-2)\frac{\det\left(\Delta P_{n-1},\Delta P_{n-2},\Delta P_{n-3}\right)}{\left\|\Delta P_{n-1}\right\|\left\|\Delta P_{n-2}\right\|^2\sin^2\theta_{n-1}^{n-2}}N\big|_{t=1}.$$

It seems that the results obtained here are the same as those in the study [10]. As it seems, since taking the weights as $\omega_{i,k}=1$, the rational Bezier curve transforms into polynomial Bezier curves. Therefore, the Serret-Frenet frame of the rational Bezier curves involves the results of the Serret-Frenet frame of the polynomial Bezier curve as well. Thus, since the Serret-Frenet frame of the rational Bezier curves is a general demonstration that involves all situations, it is a more preferred curve.

Corollary 3.2.9. Since the curve is taken as $n=2$ in the above results, the results of the quadratic rational Bezier curve $P(t)$ are obtained. One advantage of the quadratic rational

Bezier curves is that the cone classification as an ellipse, a parabola, and a hyperbola can be determined easily by which the weights $\omega_1, \omega_2, \omega_3 \in \mathbb{R}^+$ satisfy the conditions $\omega_1^2 - \omega_0\omega_2 < 0$. $\omega_1^2 - \omega_0\omega_2 = 0$, $\omega_1^2 - \omega_0\omega_2 > 0$. Also, if $n = 3$, the results for the cubic rational Bezier curve can be obtained easily.

Corollary 3.2.11. Assume that $P_0, P_1, ..., P_n \in \mathbb{R}^3$ are the control points and $\omega_{i,k} \in \mathbb{R}^+$ are the weights of the rational Bezier curve $\boldsymbol{P(t)}$ in the Euclidean 3-space. The geometric interpretation of the curvature and the torsion of the rational Bezier curves found by the algorithm method in Theorem 2.3.9 can be given. If the curvature of the rational Bezier curve $\boldsymbol{P(t)}$ is $\kappa = 0$, then the curve $\boldsymbol{P(t)}$ indicates a straight line. If the torsion is $\tau = 0$, $\boldsymbol{P(t)}$ is a planar curve. If $\kappa = const. > 0$ $\tau = 0$, $\boldsymbol{P(t)}$ is a circle segment. And if $\tau/\kappa = const.$, $\boldsymbol{P(t)}$ is a helix.

Corollary 3.2.12 Assume that $P_0, P_1, ..., P_n \in \mathbb{R}^3$ are the control points and $\omega_{i,k} \in \mathbb{R}^+$ are the weights of the rational Bezier curve $\boldsymbol{P(t)}$ in the Euclidean 3-space. The Darboux vector which shows the rotation axis, which is around a center, of the Serret-Frenet frame of the rational Bezier curves by the algorithm method is given by

$$D = (n-2)\mu_1 \frac{\left\langle \Delta P_{0,n-3} \times \Delta P_{1,n-3}, \Delta P_{2,n-3} \right\rangle \Delta P_{0,n-1}}{\left\| \Delta P_{0,n-2} \times \Delta P_{1,n-2} \right\|^2} + (n-1)\mu_2 \frac{\Delta P_{0,n-2} \times \Delta P_{1,n-2}}{\left\| \Delta P_{0,n-1} \right\|^2}$$

where the coefficients are given by

$$\mu_1 = \frac{\omega_{0,n-1}\omega_{1,n-1}\omega_{0,n-3}\omega_{1,n-3}\omega_{2,n-3}\omega_{3,n-3}}{\omega_{0,n-2}^2\omega_{1,n-2}^2\omega_{2,n-2}^2} \text{ and } \mu_2 = \frac{\omega_{0,n-2}\omega_{1,n-2}\omega_{2,n-2}\omega_{0,n}}{\omega_{0,n-1}^2\omega_{1,n-1}^2}.$$

Corollary 3.2.13 Let $P_0, P_1, ..., P_n \in \mathbb{R}^3$ be the control points and $\omega_{i,k} \in \mathbb{R}^+$ be the weights. The curve $\boldsymbol{P(t)}$ is a rational Bezier curve. The center of the osculating circle of the non-unit speed rational Bezier curve $\boldsymbol{P(t)}$ is computed by

$$\boldsymbol{M} = P_{0,n}(t_0) + \frac{n}{(n-1)} \frac{\omega_{0,n-1}^3\omega_{1,n-1}^3}{\omega_{0,n-2}\omega_{1,n-2}\omega_{2,n-2}\omega_{0,n}^3} \frac{\left\| \Delta P_{0,n-1} \right\|^2 \left(\Delta P_{0,n-2} \times \Delta P_{1,n-2} \right) \times \Delta P_{0,n-1}}{\left\| \Delta P_{0,n-2} \times \Delta P_{1,n-2} \right\|}$$

and the radius of the osculating circle is

$$\rho(t_0) = \frac{n}{(n-1)} \frac{\omega_{0,n-1}^3\omega_{1,n-1}^3}{\omega_{0,n}^3\omega_{0,n-2}\omega_{1,n-2}\omega_{2,n-2}} \frac{\left\| \Delta P_{0,n-1}(t) \right\|^3}{\left\| \Delta P_{0,n-2}(t) \times \Delta P_{1,n-2}(t) \right\|}$$

by utilizing (2.7). Thus, the osculating circle equation of the rational Bezier curve $\boldsymbol{P(t)}$ found by the algorithm method is obtained as

$$\gamma(\theta)=P_{0,n}(t_0)+\rho(t_0)\left(1-\cos\frac{\theta}{\rho(t_0)}\right)\frac{\left(\Delta P_{0,n-2}\times\Delta P_{1,n-2}\right)\times\Delta P_{0,n-1}}{\left\|\Delta P_{0,n-2}\times\Delta P_{1,n-2}\right\|\left\|\Delta P_{0,n-1}\right\|}+\rho(t_0)\sin\frac{\theta}{\rho(t_0)}\frac{\Delta P_{0,n-1}}{\left\|\Delta P_{0,n-1}\right\|}.$$

Corollary 3.2.14 Let $P_0,P_1,...,P_n\in\mathbb{R}^3$ and $\omega_{i,k}\in\mathbb{R}^+$ be the control points and the weights of a rational Bezier curve $P(t)$. And assume that $v=\left\|P'(t)\right\|$, $\rho(t)=\dfrac{1}{\kappa(t)}$ and $\sigma(t)=\dfrac{1}{\tau(t)}$. Thus the center of the osculating sphere of the rational Bezier curve $P(t)$ is

$$\begin{aligned}M=P(t_0)&+\frac{n}{n-1}\frac{\omega_{0,n-1}^3\omega_{1,n-1}^3}{\omega_{0,n-2}\omega_{1,n-2}\omega_{2,n-2}\omega_{0,n}^3}\frac{\left\|\Delta P_{0,n-1}\right\|^2\left(\Delta P_{0,n-2}\times\Delta P_{1,n-2}\right)\times\Delta P_{1,n-2}}{\left\|\Delta P_{0,n-2}\times\Delta P_{1,n-2}\right\|^2}\\&+\frac{\rho'(t)\omega_{0,n-2}^2\omega_{1,n-2}^2\omega_{2,n-2}^2}{(n-2)\omega_{0,n-1}\omega_{1,n-1}\omega_{0,n-3}\omega_{1,n-3}\omega_{2,n-3}\omega_{3,n-3}}\frac{\left\|\Delta P_{0,n-2}\times\Delta P_{1,n-2}\right\|}{\left\|\Delta P_{0,n-1}\right\|}\frac{\Delta P_{0,n-2}\times\Delta P_{1,n-2}}{\left\langle\Delta P_{0,n-3}\times\Delta P_{1,n-3},\Delta P_{2,n-3}\right\rangle}\end{aligned}$$

by utilizing the equation (2.9). And with the help of the equation (2.10), the radius of the osculating sphere can be computed similarly.

Corollary 3.2.15 The results of the Bezier curve with the Osculator circle and sphere equations at the points $t=0$, $t=1$ and their applications in textile yarn can be analyzed from the references in [11-12] produced from the BEBAP 2014.08 from the Bitlis Eren University Scientific Research Projects.

3. A NUMERIC EXAMPLE

Assume that $P_0(1,0,0)$, $P_1(1,0,1)$, $P_2(0,2,2)$ and $P_3(1,1,1)$ are the control points and $\omega_0=1$, $\omega_1=2$, $\omega_2=1$ and $\omega_3=2$ are the weights of the rational Bezier curve. Now we will compute the Serret-Frenet frame, the curvature, and the torsion of the rational Bezier curve at the intermediate point $t=0.25$ of the algorithm method. The rational Bezier curve given by the four control points is a third-order curve and it is called a cubic rational Bezier curve from the Definition (2.2.1). Since the Bernstein polynomials $B_{0,3}(t)=(1-t)^3$, $B_{1,3}(t)=3t(1-t)^2$, $B_{2,3}(t)=3t^2(1-t)$ and $B_{3,3}(t)=t^3$ are substituted in the equation

$$P(t)=\frac{B_{0,3}(t)\omega_0P_0+B_{1,3}(t)\omega_1P_1+B_{2,3}(t)\omega_2P_2+B_{3,3}(t)\omega_3P_3}{B_{0,3}(t)\omega_0+B_{1,3}(t)\omega_1+B_{2,3}(t)\omega_2+B_{3,3}(t)\omega_3}$$

from (2.1) for $n=3$, the equation of the cubic rational Bezier curve is obtained by

$$P(t)=\frac{\left(1+3t-9t^2+7t^3,6t^2-4t^3,6t-6t^2+2t^3\right)}{1+3t-6t^2+4t^3}.$$

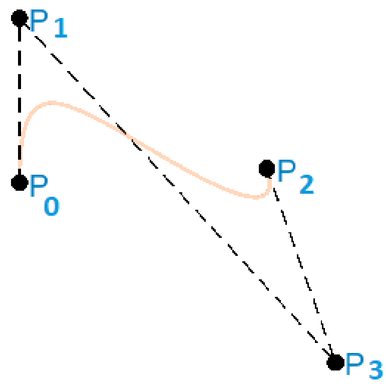

Figure 2. Rational Cubic Bezier Curve.

By using the recursion formula $P_{i,j} = (1-t)P_{i,j-1} + tP_{i+1,j-1}$ of the algorithm method in Definition 2.3.2, the control points form a triangular structure as

$$
\begin{array}{llll}
P_{0,0} = (1,0,0) & P_{1,0} = (1,0,1) & P_{2,0} = (0,2,2) & P_{3,0} = (1,1,1) \\
P_{0,1} = \left(1,0,\frac{1}{4}\right) & P_{1,1} = \left(\frac{3}{4},\frac{2}{4},\frac{5}{4}\right) & P_{2,1} = \left(\frac{1}{4},\frac{7}{4},\frac{7}{4}\right) & \\
P_{0,2} = \left(\frac{15}{16},\frac{2}{16},\frac{8}{16}\right) & P_{1,2} = \left(\frac{10}{16},\frac{13}{16},\frac{22}{16}\right) & & \\
P_{0,3} = \left(\frac{55}{64},\frac{19}{64},\frac{23}{32}\right) & & &
\end{array}
$$

for the point $t = 0.25$. Similarly, the weights from (2.13) are obtained as

$$
\begin{array}{llll}
\omega_{0,0} = 1 & \omega_{1,0} = 2 & \omega_{2,0} = 1 & \omega_{3,0} = 2 \\
\omega_{0,1} = \frac{5}{4} & \omega_{1,1} = \frac{7}{4} & \omega_{2,1} = \frac{5}{4} & \\
\omega_{0,2} = \frac{11}{8} & \omega_{1,2} = \frac{13}{8} & & \\
\omega_{0,3} = \frac{23}{16} & & &
\end{array}
$$

for the point $t = 0.25$. Now we will investigate the Serret-Frenet frame and the torsion of the rational Bezier curve for the intermediate point $t = 0.25$. Since $P'(0.25) = \left(-\frac{5}{4},\frac{11}{4},\frac{14}{4}\right)$, the tangent at the intermediate point $t = 0.25$ of the cubic rational Bezier curve $P(t)$ from the Theorem 3.2.1 is

$$T(0.25) = \frac{P_{1,2} - P_{0,2}}{\left\| P_{1,2} - P_{0,2} \right\|} = \frac{1}{\sqrt{342}}(-5,11,14) \cong \frac{1}{18}(-5,11,14),$$

the principle normal is

$$N(0.25)=\frac{\left[\left(P_{1,1}-P_{0,1}\right)\times\left(P_{2,1}-P_{1,1}\right)\right]\times\left(P_{1,2}-P_{0,2}\right)}{\left\|P_{1,2}-P_{0,2}\right\|.\left\|\left(P_{1,1}-P_{0,1}\right)\times\left(P_{2,1}-P_{1,1}\right)\right\|},$$
$$=\frac{\left(-73,229,-206\right)}{\sqrt{342}\sqrt{293}}\cong\frac{1}{317}\left(-73,229,-206\right)$$

and the binormal is $B(0.25)=\dfrac{\left(P_{1,1}-P_{0,1}\right)\times\left(P_{2,1}-P_{1,1}\right)}{\left\|\left(P_{1,1}-P_{0,1}\right)\times\left(P_{2,1}-P_{1,1}\right)\right\|}=\dfrac{-1}{204}\left(16,6,1\right).$

Moreover, the curvature is

$$\kappa(0.25)=\frac{2}{3}\frac{\omega_{0,3}^3\omega_{0,1}\omega_{1,1}\omega_{2,1}}{\omega_{0,2}^3\omega_{1,2}^3}\frac{\left\|\left(P_{1,1}-P_{0,1}\right)-\left(P_{2,1}-P_{1,1}\right)\right\|}{\left\|\left(P_{1,1}-P_{0,1}\right)\right\|^3}$$
$$=\frac{23^3.5^2.32.\sqrt{14}}{11^3.13^3.9.\sqrt{21}}\cong 0.302$$

and the torsion is

$$\tau(t)=\frac{1}{3}\frac{\omega_{0,3}^2\omega_{0,0}\omega_{1,0}\omega_{2,0}\omega_{3,0}}{\omega_{0,1}^2\omega_{1,1}^2\omega_{2,1}^2}\frac{\left\langle\left(P_{1,0}-P_{0,0}\right)\times\left(P_{2,0}-P_{1,0}\right),\left(P_{3,0}-P_{2,0}\right)\right\rangle}{\left\|\left(P_{1,1}-P_{0,1}\right)\times\left(P_{2,1}-P_{1,1}\right)\right\|^2}=-\frac{2^{10}.23^2}{3.5^4 7^2.293}\cong -0.02012$$

at the point $t=0.25$.

4. CONCLUSIONS

In this study, the Serret-Frenet frame of the rational Bezier curves is computed by the algorithm method defined in the reference [5]. Though the Serret-Frenet frame and the curvature at the initial and ending points of the polynomial and the rational Bezier curves have been studied before, the Serret-Frenet frame of the intermediate points hasn’t been studied. However, with this study, a more general approach has been obtained, and acquiring the geometric properties at the intermediate points has been accomplished. Moreover, in this study when the weights are chosen as $\omega_{i,k}=1$, the polynomial Bezier curves can be obtained. Thus, both the Serret-Frenet frame and the geometric properties of the rational Bezier curves and the polynomial Bezier curves can be obtained by a single study.

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