ORIGINAL PAPER

ON SOLUTIONS OF LINEAR FUNCTIONAL INTEGRAL AND INTEGRO-DIFFERENTIAL EQUATIONS VIA LAGRANGE POLYNOMIALS

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Abstract. *In this study, a matrix-collocation method is developed numerically to solve the linear Fredholm-Volterra-type functional integral and integro-differential equations. The linear functional integro-differential equations are considered under initial conditions. The mentioned type problems often appear in various branches of science and engineering such as physics, biology, mechanics, electronics. The method essentially is a collocation method based on the Lagrange polynomials and matrix operations. By using presented method, the problem is reduced to a system of linear algebraic equations. The solution of this system gives the coefficients of assumed solution. An error analysis based on the residual function is studied. Some examples are solved to demonstrate the accuracy and efficiency of the method.*

Keywords: *Interpolation and collocation points; Lagrange polynomials; Lagrange collocation method; residual error, residual function.*

1. INTRODUCTION

Functional integro-differential equations (FIDEs) have a major importance in modeling of some phenomena in many applied areas which include engineering, mechanics, physics, chemistry, astronomy, biology, economics, potential theory, electrostatics, etc. [1-11]. Since the mentioned equations are usually difficult to solve analytically, numerical methods are required.

In recent years, various techniques have been used for solving FIDEs such as He's variational iteration technique [11], the homotopy perturbation method [12], He's homotopy perturbation method [13], the modified homotopy perturbation method [14], the rationalized Haar functions method [15], the Chebyshev cardinal functions method [16], the differential transformation method [17], the Tau method with error estimation [18], He's variational iteration method [19], the collocation method [20], the Adomian decomposition method [21], the Adomian-Pade technique [22], the discontinuous Galerkin method [23], the Legendre multiwavelets [24], the trigonometric wavelets [25], the spectral methods [26, 27] the meshless method [28] and etc [35,40,41,43-50,52]. In addition, the operation matrix method, the Galerkin-like method and the matrix-collocation methods based on Taylor, Chebyshev, Bessel, Bernoulli, Laguerre, Bernstein, Legendre, Chebyshev, Morgan-Voyce, Taylor-Lucas, Dickson and Lucas, polynomials, have been studied by some authors [29-34,42,51] to solve the mentioned type equations.

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In this paper, we deal with the m-th order linear functional integro-differential equation with variable coefficients in the form

$$\sum_{j=0}^{J}\sum_{k=0}^{R} P_{jk}(x)y^{(k)}(\alpha_j x+\beta_j)+F(x)+I(x)=g(x), \quad x,t\in[a,b] \tag{1.1}$$

so that

$$F(x)=\lambda_1\int_a^b K(x,t)y^{(s)}(t)dt\,, \quad I(x)=\lambda_2\int_a^{h(x)} V(x,t)y^{(r)}(t)dt\,, \quad m=\max\{R,s,r\}.$$

under the initial conditions

$$\sum_{k=0}^{m-1} a_{jk}y^{(k)}(a)=\lambda_j\,, \quad j=0,1,2,...,m-1, \tag{1.2}$$

where $P_{jk}(x)$, $g(x)$ and $g(x)$ are analytic functions in the interval $[a,b]$; the functions $K(x,t)$ and $V(x,t)$ have the Maclaurin series expansion with N-th degree in $[a,b]$; $a_{jk},\lambda_j,\alpha_j$ and β_j are appropriate constants. Note that Eq. (1.1) corresponds to an integral equation if $m=0$ in Eq. (1.1).

Our aim in this study is to develop a collocation method based on the Lagrange polynomials and to obtain the approximate solution of the problem (1.1)-(1.2) in the truncated Lagrange series form:

$$y_N(x)=\sum_{n=0}^{N} y_n L_n(x) \; a\le x\le b \tag{1.3}$$

where $L_n(x)$, n=0,1,2,…,N are the Lagrange polynomials defined by

$$L_n(x)=\prod_{\substack{r=0\\ r\neq n}}^{N}\frac{(x-x_r)}{(x_n-x_r)}, \qquad a\le x\le b \tag{1.4}$$

and y_n, n=0,1,2,…,N are unknown coefficients. In this study, we use the evenly spaced collocation points defined by

$$t_n=a+\frac{b-a}{N}n, \quad n=0,1,...,N\,. \tag{1.5}$$

Also, the Lagrange polynomials for n=0, 1,…, N can be clearly written as follows:

$$L_0(x)=\frac{(x-x_1)(x-x_2)\ldots(x-x_N)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_N)},\; L_1(x)=\frac{(x-x_0)(x-x_2)\ldots(x-x_N)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_N)},...,$$
$$L_N(x)=\frac{(x-x_0)(x-x_1)\ldots(x-x_{N-1})}{(x_N-x_0)(x_N-x_1)\ldots(x_N-x_{N-1})}.$$

Here, x_n $(n=0,1,...,N)$ represent the interpolation points selected from $[a,b]$ of the Lagrange polynomials.

2. FUNDAMENTAL MATRIX RELATIONS AND LAGRANGE COLLOCATION METHOD

In this section, we consider the functional integro-differential equation (1.1) and find the matrix forms of each term in the equation. Firstly, we can convert the solution form (1.3), the term $y_N(\alpha_j x+\beta_j)$ and its k-th order derivative $y_N^{(k)}(\alpha_j x+\beta_j)$ to matrix forms as follows:

$$y_N\left(x\right)=\boldsymbol{L}(x)\boldsymbol{Y}\,,\ y_N^{(k)}\left(x\right)=\boldsymbol{L}^{(k)}(x)\boldsymbol{Y} \tag{2.1}$$

and

$$y_N(\alpha_j x+\beta_j)=\boldsymbol{L}(\alpha_j x+\beta_j)\boldsymbol{Y}\,,\ y_N^{(k)}(\alpha_j x+\beta_j)=\boldsymbol{L}^{(k)}(\alpha_j x+\beta_j)\boldsymbol{Y} \tag{2.2}$$

where, for j=0,1,…,J and k=0,1,…,R,

$$\boldsymbol{L}^{(k)}(x)=\left[L_0^{(k)}(x)\quad L_1^{(k)}(x)\quad ...\quad L_N^{(k)}(x)\right],\ \boldsymbol{Y}=\left[y_0\quad y_1\quad ...\quad y_N\right]^T,$$

$$\boldsymbol{L}^{(k)}(\alpha_j x+\beta_j)=\left[L_0^{(k)}(\alpha_j x+\beta_j)\quad L_1^{(k)}(\alpha_j x+\beta_j)\quad ...\quad L_N^{(k)}(\alpha_j x+\beta_j)\right].$$

Let's denote the term with the sum symbol on the left-side of the Eq. (1.1) by $\boldsymbol{D(x)}$. By substituting the matrix relation (2.2) into the part $\boldsymbol{D(x)}$, we obtain the matrix form

$$\boldsymbol{D(x)}=\sum_{j=0}^{J}\sum_{k=0}^{R}\boldsymbol{P_{jk}}(x)\mathbf{L}^{(k)}(\alpha_j x+\beta_j)\boldsymbol{Y}\ . \tag{2.3}$$

Then, we put the collocation points $x=t_i$ (i=0,1,2,…,N), which are selected in the interval [a,b] such that $t_i\neq x_n;\ i,n=0,1,...,N,$ in the (2.3) as

$$\boldsymbol{D(t_i)}=\sum_{j=0}^{J}\sum_{k=0}^{R}\boldsymbol{P}_{jk}(t_i)\boldsymbol{L}^{(k)}(\alpha_j t_i+\beta_j)\boldsymbol{Y},\ \text{i=0,1,2,…,N.}$$

Here, we note that x_n $(n=0,1,...,N)$ show the interpolation points of the Lagrange polynomials in Eq. (1.4).

The compact form of the above system can be written as

$$\boldsymbol{D}=\sum_{j=0}^{J}\sum_{k=0}^{R}\boldsymbol{P}_{jk}\boldsymbol{L}^{(k)}(\alpha_j,\beta_j)\boldsymbol{Y} \tag{2.4}$$

where

$$\boldsymbol{Y}=\left[y_0\quad y_1\quad ...\quad y_N\right]^T,$$

$$\boldsymbol{P}_{jk}=\begin{bmatrix} P_{jk}(t_0) & 0 & 0 & \cdots & 0 \\ 0 & P_{jk}(t_1) & 0 & \cdots & 0 \\ 0 & 0 & P_{jk}(t_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & P_{jk}(t_N) \end{bmatrix},$$

$$\boldsymbol{L}^{(k)}(\alpha_j,\beta_j)=\underbrace{\begin{bmatrix} L_0^{(k)}(\alpha_j t_0+\beta_j) & L_1^{(k)}(\alpha_j t_0+\beta_j) & \dots & L_N^{(k)}(\alpha_j t_0+\beta_j) \\ L_0^{(k)}(\alpha_j t_1+\beta_j) & L_1^{(k)}(\alpha_j t_1+\beta_j) & \cdots & L_N^{(k)}(\alpha_j t_1+\beta_j) \\ \vdots & \vdots & \ddots & \vdots \\ L_0^{(k)}(\alpha_j t_N+\beta_j) & L_1^{(k)}(\alpha_j t_N+\beta_j) & \cdots & L_N^{(k)}(\alpha_j t_N+\beta_j) \end{bmatrix}}_{L^{(k)}}.$$

In Eq. (2.4), the full dimensions of the matrices $\boldsymbol{P}_{jk}$, $\boldsymbol{L}^{(k)}(\alpha_j,\beta_j)$ and are respectively $(N+1)x(N+1)$, $(N+1)x(N+1)$ and $(N+1)x1$.

Now, let's consider the Fredholm integral part

$$F(x)=\lambda_1\int_a^b K(x,t)y^{(s)}(t)dt\,. \tag{2.5}$$

Since it is assumed that $K(x,t)$ function can be expanded to the maclaurin series in the problem (1.1), the function $K(x,t)$ can be written as

$$K(x,t)=\sum_{i=0}^{N}\sum_{j=0}^{N}k_{ij}x^i t^j\,,$$

$$k_{ij}=\frac{1}{i!\,j!}\frac{\partial^{i+j}K(0,0)}{\partial x^i t^j}\,,\ \boldsymbol{K}=\left[k_{ij}\right],\ \ \boldsymbol{K}=\begin{bmatrix} k_{00} & k_{01} & \cdots & k_{0N} \\ k_{10} & k_{11} & \cdots & k_{1N} \\ \vdots & \vdots & .... & \vdots \\ k_{N0} & k_{N1} & \cdots & k_{NN} \end{bmatrix}.$$

From here, the kernel function can be written as $K(x,t)=\boldsymbol{X}(x)\boldsymbol{K}\boldsymbol{X}^T(t)$ and so by using the matrix relation (2.2), we can write

$$\lambda_1\int_a^b K(x,t)y^{(s)}(t)dt=\lambda_1\int_a^b \boldsymbol{X}(x)\boldsymbol{K}\boldsymbol{X}^T(t)\boldsymbol{L}^{(s)}(\alpha_j,\beta_j)(t)dt\boldsymbol{Y}\,. \tag{2.6}$$

By arranging the expression (2.6), we get

$$F(x)=\lambda_1\boldsymbol{X}(x)\boldsymbol{K}\underbrace{\int_a^b \boldsymbol{X}^T(t)\boldsymbol{L}^{(s)}(\alpha_j,\beta_j)(t)dt}_{\boldsymbol{M}}\boldsymbol{Y}$$

and hereby we have

$$F(x)=\lambda_1\boldsymbol{X}(x)\boldsymbol{KMY}. \tag{2.7}$$

By using the collocation points in the expression (1.5) in here, we obtain the system

$$F(t_i)=\lambda_1\boldsymbol{X}(t_i)\boldsymbol{KMY} \tag{2.8}$$

where

$$\boldsymbol{X}(x)=\begin{bmatrix}1 & x & x^2 & \cdots & x^N\end{bmatrix},\ \boldsymbol{M}=\int_a^b\begin{bmatrix} t^0L_0(t) & t^0L_1(t) & \cdots & t^0L_N(t)\\ tL_0(t) & tL_1(t) & \cdots & tL_N(t)\\ \vdots & \vdots & \cdots & \vdots\\ t^NL_0(t) & t^NL_1(t) & \cdots & t^NL_N(t)\end{bmatrix}dt.$$

The system (2.8) can be written in compact form, which is called as the matrix form of the Fredholm integral part,

$$\boldsymbol{F}=\lambda_1\boldsymbol{XKMY}, \tag{2.9}$$

where

$$\boldsymbol{X}=\begin{bmatrix}\boldsymbol{X}(t_0)\\ \boldsymbol{X}(t_1)\\ \vdots\\ \boldsymbol{X}(t_N)\end{bmatrix},$$

Similarly, the kernel function $V(x,t)$ like $K(x,t)$ can be expressed to Maclaurin series. Its Maclaurin series becomes $V(x,t)=\boldsymbol{X}(x)\boldsymbol{VX}^T(t)$. Here,

$$V(x,t)=\sum_{i=0}^{N}\sum_{j=0}^{N}v_{ij}x^it^j,\ v_{ij}=\frac{1}{i!\,j!}\frac{\partial^{i+j}V(0,0)}{\partial x^it^j},\ \boldsymbol{V}=\begin{bmatrix}v_{ij}\end{bmatrix},\ \boldsymbol{V}=\begin{bmatrix} v_{00} & v_{01} & \cdots & v_{0N}\\ v_{10} & v_{11} & \cdots & v_{1N}\\ \vdots & \vdots & .... & \vdots\\ v_{N0} & v_{N1} & \cdots & v_{NN}\end{bmatrix}.$$

Hence, the Volterra integral part can be written as

$$\lambda_2\int_a^{h(x)}V(x,t)y^{(r)}(t)dt=\lambda_2\int_a^{h(x)}\boldsymbol{X}(x)\boldsymbol{VX}^T(t)\boldsymbol{L}^{(r)}(\alpha_j,\beta_j)(t)dt\boldsymbol{Y}. \tag{2.10}$$

We arrange the expression (2.10) and thus it becomes

$$\mathrm{I}(x)=\lambda_2\boldsymbol{X}(x)\boldsymbol{V}\underbrace{\int_a^{h(x)}\boldsymbol{X}^T(t)\boldsymbol{L}^{(r)}(\alpha_j,\beta_j)(t)dt}_{M(x)}\boldsymbol{Y}$$

where

$$\boldsymbol{M}(x)=\int_a^{h(x)}\begin{bmatrix} t^0L_0^{(r)}(\alpha_j,\beta_j)(t) & t^0L_1^{(r)}(\alpha_j,\beta_j)(t) & \cdots & t^0L_N^{(r)}(\alpha_j,\beta_j)(t) \\ tL_0^{(r)}(\alpha_j,\beta_j)(t) & tL_1^{(r)}(\alpha_j,\beta_j)(t) & \cdots & tL_N^{(r)}(\alpha_j,\beta_j)(t) \\ \vdots & \vdots & \cdots & \vdots \\ t^NL_0^{(r)}(\alpha_j,\beta_j)(t) & t^NL_1^{(r)}(\alpha_j,\beta_j)(t) & \cdots & t^NL_N^{(r)}(\alpha_j,\beta_j)(t) \end{bmatrix}dt$$

and thus, we have

$$I(x)=\lambda_2\boldsymbol{X}(x)\boldsymbol{V}\boldsymbol{M}(x)\boldsymbol{Y}. \tag{2.11}$$

By substituting the collocation points (1.5) into the matrix form (2.11), we get the system

$$I(t_i)=\lambda_2\boldsymbol{X}(t_i)\boldsymbol{V}\boldsymbol{M}(t_i)\boldsymbol{Y}\,. \tag{2.12}$$

We write this system in compact form and so we have the matrix form of the Volterra integral part

$$\boldsymbol{I}=\lambda_2\overline{\boldsymbol{X}}\,\overline{\boldsymbol{V}}\,\overline{\boldsymbol{M}}\boldsymbol{Y} \tag{2.13}$$

Where

$$\overline{\boldsymbol{X}}=\begin{bmatrix} \boldsymbol{X}(t_0) & 0 & \dots & 0 \\ 0 & \boldsymbol{X}(t_1) & \dots & 0 \\ 0 & 0 & \boldsymbol{X}(t_2) & 0 \\ 0 & 0 & \dots & \boldsymbol{X}(t_N) \end{bmatrix}_{(N+1)x(N+1)^2},\quad \overline{\boldsymbol{V}}=\begin{bmatrix} \boldsymbol{V} & 0 & \dots & 0 \\ 0 & \boldsymbol{V} & \dots & 0 \\ 0 & 0 & \boldsymbol{V} & 0 \\ 0 & 0 & \dots & \boldsymbol{V} \end{bmatrix}_{(N+1)^2x(N+1)^2},\quad \overline{\boldsymbol{M}}=\begin{bmatrix} \boldsymbol{M}(t_0) \\ \boldsymbol{M}(t_1) \\ \vdots \\ \boldsymbol{M}(t_N) \end{bmatrix}_{N+1)^2x(N+1)}.$$

By means of the matrix forms (2.4), (2.9) and (2.13), the fundamental matrix equation of the Eq. (1.1) is formed as

$$\underbrace{\left\{\sum_{j=0}^{J}\sum_{k=0}^{R}\boldsymbol{P}_{jk}\boldsymbol{L}^{(k)}(\alpha_j,\beta_j)+\lambda_1\boldsymbol{X}\boldsymbol{K}\boldsymbol{M}+\lambda_2\overline{\boldsymbol{X}}\,\overline{\boldsymbol{V}}\,\overline{\boldsymbol{M}}\right\}}_{\boldsymbol{W}}\boldsymbol{Y}=\boldsymbol{G}$$

or it can be briefly written as

$$\boldsymbol{W}\boldsymbol{Y}=\boldsymbol{G}\quad\text{or }[\boldsymbol{W};\boldsymbol{G}] \tag{2.14}$$

where for $p,q=0,1,2,...,N$

$$\boldsymbol{W}=[w_{pq}]=\sum_{j=0}^{J}\sum_{k=0}^{R}\boldsymbol{P}_{jk}\boldsymbol{L}^{(k)}(\alpha_j,\beta_j)+\lambda_1\boldsymbol{X}\boldsymbol{K}\boldsymbol{M}+\lambda_2\overline{\boldsymbol{X}}\,\overline{\boldsymbol{V}}\,\overline{\boldsymbol{M}},\ \boldsymbol{G}=\begin{bmatrix} g(t_0) & g(t_1) & \dots & g(t_N)\end{bmatrix}^{\boldsymbol{T}}.$$

On the other hand, by means of the relation (2.1), the corresponding matrix forms of the initial conditions (1.2) are obtained as

$$\boldsymbol{U}_j\boldsymbol{Y}=\lambda_j\ \text{ or }\left[\boldsymbol{U}_j;\boldsymbol{\lambda}_j\right];\ j=0,1,2,...,m-1, \tag{2.15}$$

such that

$$\boldsymbol{U}_j=\sum_{k=0}^{m-1}a_{jk}\boldsymbol{L}^{(k)}(a)=\begin{bmatrix} u_{j0} & u_{j1} & u_{j2} & \cdots & u_{jN}\end{bmatrix}.$$

Hence, the augmented matrices of the expressions (2.14) and (2.15) can be clearly written in the forms

$$[\boldsymbol{W};\boldsymbol{G}]=\begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & g(t_0) \\ w_{10} & w_{11} & \cdots & w_{1N} & ; & g(t_1) \\ \vdots & \vdots & \vdots & \vdots & ; & \vdots \\ w_{(N-1)0} & w_{(N-1)1} & \cdots & w_{(N-1)N} & ; & g(t_{N-1}) \\ w_{N0} & w_{N1} & \cdots & w_{NN} & ; & g(t_N) \end{bmatrix}$$

and

$$[\boldsymbol{U}_j;\lambda_j]=\begin{bmatrix} u_{00} & u_{01} & \cdots & u_{0N} & ; & \lambda_0 \\ u_{10} & u_{11} & \cdots & u_{1N} & ; & \lambda_1 \\ \vdots & \vdots & \cdots & \vdots & ; & \vdots \\ u_{(m-1)0} & u_{(m-1)1} & \cdots & u_{(m-1)N} & ; & \lambda_{m-1} \end{bmatrix}.$$

Consequently, in order to obtain the Lagrange polynomial solution of the Eq. (1.1) under the condition (1.2), we replace any m rows of the augmented matrix (2.14) by the m row matrices of the Eq. (2.15). Thereby we obtain the new augmented matrix

$$[\tilde{\boldsymbol{W}};\tilde{\boldsymbol{G}}] \quad \text{or} \quad \tilde{\boldsymbol{W}}\boldsymbol{Y}=\tilde{\boldsymbol{G}}. \tag{2.16}$$

If last m rows of the augmented matrix (2.14) are replaced, then the new augmented matrix becomes as follows

$$[\tilde{\boldsymbol{W}};\tilde{\boldsymbol{G}}]=\begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0N} & ; & g(t_0) \\ w_{10} & w_{11} & \cdots & w_{1N} & ; & g(t_1) \\ \vdots & \vdots & \vdots & \vdots & ; & \vdots \\ w_{(N-1-m)0} & w_{(N-1-m)1} & \cdots & w_{(N-1-m)N} & ; & g(t_{N-1-m}) \\ w_{(N-m)0} & w_{(N-m)1} & \cdots & w_{(N-m)N} & ; & g(t_{N-m}) \\ u_{00} & u_{01} & \cdots & u_{0N} & ; & \lambda_0 \\ u_{10} & u_{11} & \cdots & u_{1N} & ; & \lambda_1 \\ \vdots & \vdots & \vdots & \vdots & ; & \vdots \\ u_{(m-1)0} & u_{(m-1)1} & \vdots & u_{(m-1)N} & ; & \lambda_{m-1} \end{bmatrix}.$$

If $rank\,\widetilde{\boldsymbol{W}} = rank\,[\widetilde{\boldsymbol{W}};\widetilde{\boldsymbol{G}}] = N+1$ in Eq. (2.16), then we can write $\boldsymbol{Y}=(\tilde{\boldsymbol{W}})^{-1}\tilde{\boldsymbol{G}}$. Thus, the matrix $\boldsymbol{Y}$ (the coefficients of the Lagrange solution) is uniquely determined. Hence, the Eq. (1.1) under the conditions (1.2) has a unique solution. This solution is given by the truncated Lagrange series (1.3).

3. ACCURACY OF SOLUTIONS AND RESIDUAL ERROR ESTIMATION

In this section, we can easily check the accuracy of the obtained solutions and the error estimation [29-32, 36-38]. Since the truncated Lagrange series (1.3) is approximate solution of the Eq. (1.1), when the function $y_N(x)$ and its derivatives are substituted in the Eq. (1.1), the resulting equation must be satisfied approximately. That is, it becomes

$$R_N(x) = \sum_{j=0}^{J}\sum_{k=0}^{R} P_{jk}(x) y_N^{(k)}(\alpha_j x + \beta_j) + F(x) + I(x) - g(x) \cong 0, \quad x \in [\mathrm{a,b}],$$

where

$$F(x) = \lambda_1 \int_a^b K(x,t) y_N^{(s)}(t)dt \quad \text{and} \quad I(x) = \lambda_2 \int_a^{h(x)} V(x,t) y_N^{(r)}(t)dt .$$

For $x = x_l \in [\mathrm{a,b}], l = 0,1,2,...;$ $R_N(x_l) \le 10^{-k_l}$, $(k_l$ is any positive integer$)$.

If $\max 10^{-k_l} = 10^{-h}$ (h is an positive integer) is prescribed, then the truncation *limit N* is increased until the difference $R_N(x_l)$ at each of the points becomes smaller than the prescribed 10^{-h}.

In addition, the error function $e_N(x)$ can be defined as

$$e_N(x) = y(x) - y_N(x) ,$$

where $y(x)$ is the exact solution of the problem (1.1)-(1.2). Thus, we obtain the error differential equation in the form

$$\sum_{j=0}^{n}\sum_{k=0}^{R} P_{jk}(x) e_N^{(k)}(\alpha_{jk} x + \beta_{jk}) + \int_a^b K(x,t) e_N^{(s)}(t)dt + \int_a^{h(x)} V(x,t) e_N^{(r)}(t)dt = -R_N(x), \qquad (3.1)$$

with the homogeneous conditions

$$\sum_{k=0}^{m-1} a_{jk} e_N^{(k)}(a) = 0, \ j = 0,1,...,m-1 .$$

The error problem (3.1) is solved by using the method given in Section 2 and thus we obtain the approximation $e_{N,M}(x)$ to $e_N(x)$. Also, we can improve the approximate solution $y_N(x)$ by $y_{N,M}(x) = y_N(x) + e_{N,M}(x)$.

4. NUMERICAL EXAMPLES

In this section, we give applications of the method on some numerical examples. In the first two examples, we solve the integral equation which corresponds to case $m = 0$ of Eq. (1.1). In the last two examples, two applications of the integro-differential equations are given.

Example 1. We consider the linear Fredholm-Volterra integral equation

$$e^x y(2x) + x^2 y(x) = g(x) + \int_0^{2x} e^{x+t} y(t)dt - \int_0^1 e^{x-2t} y(2t)dt, \quad 0 \le x,t \le 1 \qquad (4.1)$$

with the exact solution $y(x) = \sin x$. Here, $P_{00}(x) = x^2$, $P_{10}(x) = e^x$, $\mathrm{V}(x,t) = -e^{x+t}$, $\mathrm{K}(x,t) = e^{x-2t}$, $g(x) = -\frac{e^x}{4} - \frac{1}{4}e^{-2+x}\cos 2 + \frac{1}{2}e^{3x}\cos 2x - \frac{1}{4}e^{-2+x}\sin 2 + x^2\sin x + e^x\cos 2x - \frac{1}{2}e^{3x}\sin 2x$, $\lambda_1 = 1$, $\lambda_2 = -1$, $h(x) = 2x$, $\alpha_0 = 1$, $\beta_0 = 0$, $\alpha_1 = 2$, $\beta_1 = 0$.

According to the method in Section 3, the fundamental matrix form of the Eq. (4.1) is

$$\underbrace{\left\{\boldsymbol{P}_{00}\boldsymbol{L}^{(0)}(\alpha_0,\beta_0) + \boldsymbol{P}_{10}\boldsymbol{L}^{(0)}(\alpha_1,\beta_1) + \boldsymbol{XKM} - \overline{\boldsymbol{X}}\overline{\boldsymbol{V}}\overline{\boldsymbol{M}}\right\}}_{\boldsymbol{W}}\boldsymbol{Y} = \boldsymbol{G}.$$

We note that the problem is applied without initial condition since the problem is unconditional that is it is an integral equation problem.

In Table 1 and Fig. 1, the exact solution of equation (4.1) and the obtained numerical results are compared for $N = 5$, $N = 6$ and $N = 10$. According to this information, it can be said that the errors decrease as the value of N increases.

Table 1. Comparison of the values of the exact solution and the approximate solutions of the Example 1 for values *x*.

x_i	Exact solution	Lagrange collocation method			
	$y(x) = \sin x$	$y(x_i), N=5$	$\mathrm{E}(x_i), N=5$	$y(x_i), N=6$	$\mathrm{E}(x_i), N=6$
0	0	-0.00495619	4.9561e-03	-0.00065869869	6.5869e-04
0.2	0.198669330	0.2344778152	3.5808e-02	0.200638310438	1.9689e-03
0.4	0.389418342	0.3952434767	5.8251e-03	0.388579599448	8.3874e-04
0.6	0.564642473	0.5541787256	1.0463e-02	0.563716333962	9.2613e-04
0.8	0.717356090	0.7196883210	2.332e-03	0.717363861002	7.7701e-06
1	0.841470984	0.8646744913	2.3203e-02	0.840939293397	5.3169e-04

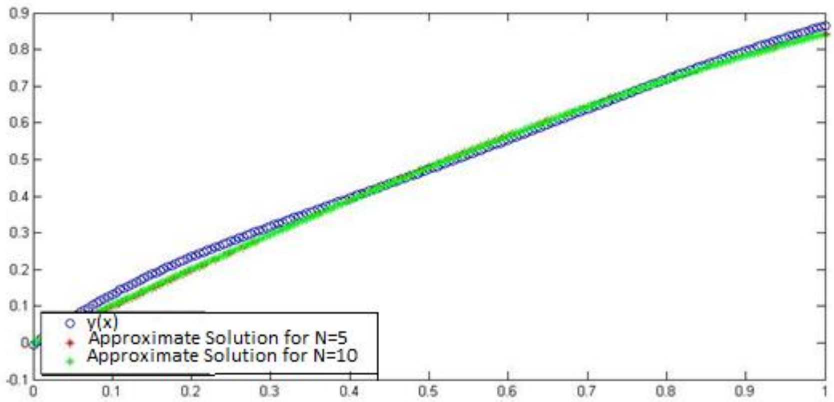

Figure 1. Comparison of the exact solution and the approximate solutions for the Example 1.

Example 2. Secondly, let's take the functional Volterra integral equation

$$e^x y(0.8x) + y(x) + \int_0^x e^{x+t} y(t)\,dt = g(x), 0 \le x \le 1.1 \tag{4.2}$$

with the exact solution $y(x)=e^{-x}$. Here, $g(x)=e^{-x}-e^{x-0.8x}+xe^{x}, a=0, b=1.1$, $P_{00}(x)=1$, $P_{10}(x)=e^{x}$, $V(x,t)=e^{x+t}$, $h(x)=x$, $\lambda_1=0$, $\lambda_2=1$, $\alpha_0=1$, $\beta_0=0$, $\alpha_1=0.8$, $\beta_1=0$ and thus the fundamental matrix form becomes

$$\underbrace{\left\{\boldsymbol{P}_{00}\boldsymbol{L}^{(0)}(\alpha_0,\beta_0)+\boldsymbol{P}_{10}\boldsymbol{L}^{(0)}(\alpha_1,\beta_1)+\overline{\boldsymbol{X}}\overline{\boldsymbol{V}}\overline{\boldsymbol{M}}\right\}}_{\boldsymbol{W}}\boldsymbol{Y}=\boldsymbol{G}.$$

In Table 2, values of the exact solution and the approximate solution for $N=5$ are compared. Fig. 2 show a comparison of the exact solution and the approximate solution for $N=5$. From Fig. 2, it can be said that the approximate solution is quite close to the exact solution.

Table 2. Comparison of the values of the exact solution and the approximate solutions of the Example 2 for values x.

	Exact solution	**Approximate solution**	**Absolute error**
x_i	$y(x_i)=e^{-x_i}$	$N=5, y_5(x_i)$	$\lvert e_5(x_i)\rvert$
0.1	0.904837418035960	0.904837430610626	1.2574e-08
0.4	0.670320046035639	0.670321948648831	1.9026e-06
0.6	0.548811636094027	0.548841279982742	2.9643e-05
0.8	0.449328964117222	0.449516281156010	1.8731e-04
1	0.367879441171442	0.368577080743251	6.9763e-04
1.1	0.332871083698080	0.334059536022421	1.1884e-03

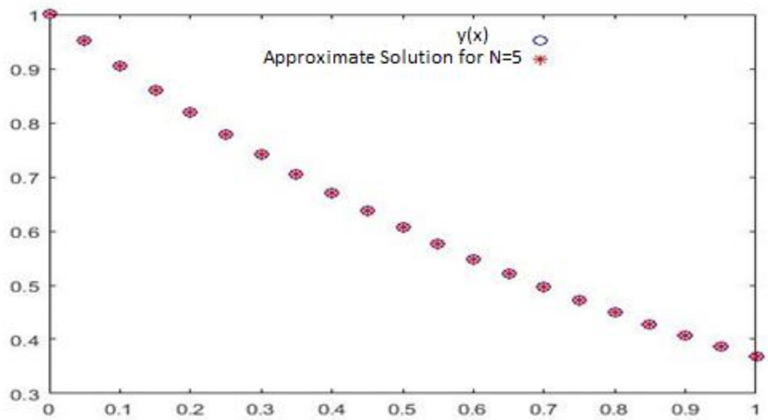

Figure 2. Comparison of the exact solution and the approximate solutions for the Example 2.

Example 3. [34,39] Now we consider the third-order linear delay Fredholm integro-differential equation

$$y'''(x)-xy'(x)+y''(x-1)-xy(x-1)=-(x+1)(\sin(x-1)+\cos(x))-\cos(2)+1+\int_{-1}^{1}y(t-1)dt \quad (4.3)$$

with conditions $y(0)=0$, $y'(0)=1$, $y''(0)=0$. The exact solution of problem is $y(x)=\sin(x)$.

Here, $g(x)=-(x+1)(\sin(x-1)+\cos(x))-\cos(2)+1,\ P_{00}(x)=-x,\ P_{13}(x)=1,$ $P_{11}(x)=-x,\ \ P_{02}(x)=1,\ \mathrm{K}(x,t)=1,\ \lambda_1=-1,\ \ \lambda_2=0,\ \ \alpha_0=1,\ \ \beta_0=-1,\ \ \alpha_1=1,\ \ \beta_1=0$. Thus, fundamental matrix form becomes as follows

$$\underbrace{\left\{\boldsymbol{P}_{13}\boldsymbol{L}^{(3)}(\alpha_1,\beta_1)+\boldsymbol{P}_{11}\boldsymbol{L}^{(1)}(\alpha_1,\beta_1)+\boldsymbol{P}_{02}\boldsymbol{L}^{(2)}(\alpha_0,\beta_0)+\boldsymbol{P}_{00}\boldsymbol{L}^{(0)}(\alpha_0,\beta_0)-\boldsymbol{XKM}\right\}}_{\boldsymbol{W}}\boldsymbol{Y}=\boldsymbol{G}.$$

In Table 4, the obtained results from the present method are compared with the results obtained from the Taylor and Legendre methods. From this, it can be said that although it gives similar results with the Legendre method, it gives better results at more points compared to the Taylor method.

Table 4. Comparison of the values of the approximate solutions of the Example 3 for x values with other methods.

	Taylor series [39]	**Legendre series [34]**	**Present method**
$\mathbf{x_i}$	m=7	m=7	N=5
-1	6.03 e-02	5.05 e-03	5.6815e-03
-0.8	2.28 e-02	2.38 e-03	2.6828e-03
-0.6	6.63 e-03	9.14 e-04	1.0277e-03
-0.4	1.20 e-03	2.42 e-04	2.7199e-04
-0.2	6.90 e-05	2.65 e-05	2.9829e-05
0	0	0	0
0.2	5.30 e-05	1.91 e-05	2.1534e-05
0.4	8.09 e-04	1.25 e-04	1.4039e-04
0.6	3.82 e-03	3.30 e-04	3.7091e-04
0.8	1.14 e-02	5.78 e-04	6.4853e-04
1	2.73 e-02	7.53 e-04	8.4147e-04

Example 4. [34,39] Finally, let's take the second-order linear functional Fredholm integro-differential equation

$$y''(x)+xy'(x)+xy(x)+y'(x-1)+y(x-1)=e^{-x}+e+\int_{-1}^{0}ty(t-1)dt,\ -1\le x,t\le 0, \quad (4.4)$$

with the conditions $y(0)=1$, $y'(0)=-1$. The exact solution of the problem is $y(x)=e^{-x}$.

In Table 5, the exact solution of equation (4.4) and the obtained numerical results according to the method in Section 2 are compared for N = 5 and N = 10. According to this information, as in the previous tables and figures, it can be said that the better the result can be obtained for the higher the N value,. Additionally in Table 5, the obtained results from the present method are compared with the results obtained from the Taylor method. From here, it can be observed that the present method gives better results than the Taylor method.

Table 5. Comparison of the values of the exact solution and the approximate solutions of the Example 4 for x values with Taylor matrix method.

	Exact solution	Lagrange Collocation method		Taylor series [39]
x_i	$y(x_i)=e^{-x_i}$	$N=5, y_5(x_i)$	$\lvert e_5(x_i)\rvert$	N=6
0	1	1	0	0
-0.1	1.105170902716915	1.105264084493846	9.3181e-05	0.100e-3
-0.2	1.221402758160170	1.221671552582331	2.6879e-04	0.200e-3
-0.3	1.349858807576003	1.350215521556035	3.5671e-04	0.100e-3
-0.4	1.491824697641270	1.491993887696942	1.6919e-04	0.100e-3
-0.5	1.648721270700128	1.648223141050780	4.9812e-04	0.200e-3
-0.6	1.822118800390509	1.820252180199343	1.8666e-03	0.100e-2
-0.7	2.013752707470477	2.009576127032827	4.1765e-03	0.240e-2
-0.8	2.225540928492468	2.217850141522157	7.6907e-03	0.450e-2
-0.9	2.459603111156950	2.446903236491322	1.2699e-02	0.740e-2

Lagrange Collocation method

x_i	$N=10, y_{10}(x_i)$	$\lvert e_{10}(x_i)\rvert$
0	1	0
-0.1	1.105169340463458	1.562253456821594e-06
-0.2	1.221397221328834	5.536831336083736e-06
-0.3	1.349851459450560	7.348125442785403e-06
-0.4	1.491821211271796	3.486369474448026e-06
-0.5	1.648731526704309	1.025600418103068e-05
-0.6	1.822157235003633	3.843461312391661e-05
-0.7	2.013838699950223	8.599247974627744e-05
-0.8	2.225699263531193	1.583350387255322e-04
-0.9	2.459864554785385	2.614436284353872e-04

5. CONCLUSIONS

In this paper, we have presented the Lagrange collocation method to solve m-th order linear functional integro-differential equations with variable coefficients. In numerical examples, we have calculated the numerical values of the approximate solutions obtained by the method. Also, the results have been compared with exact solution and the results of other methods in tables and figures. These comparisons show that the suggested method is quite effective. Moreover, the method can be developed for nonlinear integro-differential equations and partial integro-differential equations.

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