ORIGINAL PAPER

A NOTE ON BIGAUSSIAN PELL AND PELL-LUCAS NUMBERS

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Abstract. *In this study, we define a new type of Pell and Pell-Lucas numbers which are called biGaussian Pell and biGaussian Pell-Lucas numbers. We also give the relationship between negabiGaussian Pell and Pell-Lucas numbers and bicomplex Pell and Pell-Lucas numbers. Moreover, we obtain the Binet's formula, generating function, d'Ocagne's identity, Catalan's identity, Cassini's identity and some sums formulas for these new type numbers. Some algebraic proporties of biGaussian Pell and Pell-Lucas numbers which are connected between biGaussian numbers and Pell and Pell-Lucas numbers are investigated. Moreover, we give the matrix representation of biGaussian Pell and Pell-Lucas numbers.*

Keywords: *biGaussian number; bicomplex number; quaternion; Pell and Pell-Lucas numbers.*

1. INTRODUCTION

Number sequences, which started their journey with a rabbit problem, found a place in many areas such as signal and image processing, time series analysis, fractal structures, queueing theory, quantum mechanics, electromagnetic waves and curved structures. Also, there are many works on squence numbers in literature. The properties, relations and results between squence numbers can be found in Dunlap, Koshy and Vajda [1-3].

Quaternions, which are a number system that extends the complex numbers were first introduced by Hamilton [4]. Segre [5] defined bicomplex numbers and gave some fundamental properties. Price [6] presented bicomplex numbers based on multi-complex spaces and functions. The bicomplex numbers are a type of Clifford algebra, one of the several possible generalizations of the ordinary complex numbers.

Recently, several remarkable studies have been conducted related with bicomplex numbers. Rochon and Tremblay, in [7], studied the bicomplex Schrödinger equations. They also mentioned that the bicomplex quantum mechanics is the generalization of both the classical and hyperbolic quantum mechanics. Kabadayi and Yayli, in [8], represented a curve by means of bicomplex numbers in a hypersurface in E^4 and then they defined the homothetic motion of this curve. Lavoie et al., in [9], determined the eigenkets and eigenvalues of the bicomplex quantum harmonic oscillator Hamiltonian. Some researchers have studied algebraic, geometric, topological and dynamic properties of bicomplex numbers [10-13].

The set of bicomplex numbers can be expressed by the basis {1, i, j, i j} as,

$$\mathbb{C}_2 = \{q = q_1 + iq_2 + jq_3 + ijq_4 \mid q_1, q_2, q_3, q_4 \in \mathbb{R}\}$$

or

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$$\mathbb{C}_2 = \{q = (q_1 + iq_2) + j(q_3 + iq_4) \mid q_1, q_2, q_3, q_4 \in \mathbb{R}\}$$

where i, j, and ij satisfy the conditions

$$i^2 = -1, j^2 = -1, ij = ji.$$

Thus, any bicomplex number q is introduced as pairs typical complex numbers with the additional structure of commutatice multiplication. The conjugations of the bicomplex numbers are defined in[14] as:

$$q = q_1 + iq_2 + jq_3 + ijq_4 = (q_1 + iq_2) + j(q_3 + iq_4), \in \mathbb{C}_2,$$

$$q_i^* = q_1 - iq_2 + jq_3 - ijq_4 = (q_1 - iq_2) + j(q_3 - iq_4),$$

$$q_j^* = q_1 + iq_2 - jq_3 - ijq_4 = (q_1 + iq_2) - j(q_3 + iq_4),$$

$$q_{ij}^* = q_1 - iq_2 - jq_3 + ijq_4 = (q_1 - iq_2) - j(q_3 - iq_4)$$

and properties of conjugation
1) $(q^*)^* = q$
2) $(q_1 q_2)^* = q_2^* q_1^*;\ q_1, q_2 \in \mathbb{C}_2$
3) $(q_1 + q_2)^* = q_1^* + q_2^*;\ q_1, q_2 \in \mathbb{C}_2$
4) $(\lambda q)^* = \lambda q^*, \lambda \in \mathbb{R}$
5) $(\lambda q_1 \pm q_2)^* = \lambda q_1^* \pm \mu q_2^*;\ q_1, q_2 \in \mathbb{C}_2;\ \lambda, \mu \in \mathbb{R}$.

Therefore, the norm of the bicomplex numbers is defined as

$$N_{qi} = \|q \times q_i^*\| = \sqrt{|q_1^2 + q_2^2 - q_3^2 - q_4^2 + 2j(q_1 q_3 + q_2 q_4)|},$$

$$N_{qj} = \left\|q \times q_j^*\right\| = \sqrt{|q_1^2 - q_2^2 + q_3^2 - q_4^2 + 2i(q_1 q_2 + q_3 q_4)|},$$

$$N_{qij} = \left\|q \times q_{ij}^*\right\| = \sqrt{|q_1^2 + q_2^2 + q_3^2 + q_4^2 + 2j(q_1 q_4 - q_2 q_3)|}.$$

Gaussian numbers are complex numbers which were investigated by Gauss. Horadam [15] introduced the concept, the complex Fibonacci numbers, called the Gaussian Fibonacci numbers. Jordan [16] considered two of the complex Fibonacci sequences and extented some relationships which are known about the common Fibonacci sequences. The author gave many identities related to them.

$$GF_n = F_n + iF_{n-1}$$

Harman [17] gave an extension of Fibonacci numbers into the complex plane and generalized the methods by Horadam. Aydın [18] defined bicomplex Pell and Pell-Lucas numbers as

$$BCP_n = \{P_n + iP_{n+1} + jP_{n+2} + ijP_{n+3} \mid P_n, nth\ Pell\ number\}$$

and

$$BCQ_n = \{Q_n + iQ_{n+1} + jQ_{n+2} + ijQ_{n+3} \mid Q_n, nth\ Pell - Lucas\ number\}$$

where i, j, and ij satisfy the conditions

$$i^2 = -1, j^2 = -1, ij = ji.$$

Some researchers have studied some properties of Gaussian numbers [19-21]. The applications of Pell and Pell-Lucas numbers in mathematics are undeniably large. They are of fundamental importance in many fields such as combinatorics, algebra and number theory.

The sequence of Pell numbers

$$1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, \dots, P_n, \dots$$

is defined by the recurrence relation

$$P_n = 2P_{n-1} + P_{n-2}, n \geq 2, \text{ with } P_0 = 0, P_1 = 1.$$

The sequence of Pell-Lucas numbers

$$2, 2, 6, 14, 34, 82, 198, 478, 1154, 2786, \dots, Q_n, \dots$$

is defined by the recurrence relation

$$Q_n = 2Q_{n-1} + Q_{n-2}, n \geq 2, \text{ with } Q_0 = 2, Q_1 = 2.$$

Also, Pell and Pell-Lucas numbers hold the following proporties [22]:

$P_{-n} = (-1)^{n+1} P_n,$

$Q_{-n} = (-1)^n Q_n.$

$P_{n+1}^2 + P_n^2 = P_{2n+1}$

$P_{2n+5} - P_{2n+1} = 2Q_{2n+3}$

$2(P_{n+1} + P_n) = Q_{n+1}$

$2(P_{n+1} - P_n) = Q_n$

$P_{n+1} + P_{n-1} = Q_n$

$P_{n+1} - P_{n-1} = 2P_n$

$P_{n+2} + P_{n-2} = 6P_n$

$P_{n+2} - P_{n-2} = 2Q_n$

$P_{2n+5} + P_{2n+1} = 6P_{2n+3}$

$P_{n+1}^2 - P_n^2 = \frac{Q_{2n+1}+2(-1)^n}{4}$

$P_{n+1}P_n = \frac{Q_{2n+1}-2(-1)^n}{8}$

$Q_{n+1} - Q_{n-1} = 2Q_n$

$Q_{n+1} + Q_n = 4P_{n+1}$

$Q_{n+1} - Q_n = 4P_n$

$Q_{n+1} + Q_{n-1} = 4P_n$

$Q_{n+2} + Q_{n-2} = 6Q_n$

$Q_{n+2} - Q_{n-2} = 8P_n$

$Q_{n+1}^2 + Q_n^2 = 8P_{2n+1}$

$Q_{n+1}^2 - Q_n^2 = 8P_{2n+1} - 4(-1)^n$

$Q_{n+1}Q_n = Q_{2n+1} + 2(-1)^n$

Above some fundamental proporties of Pell and Pell-Lucas numbers are presented. In the following sections, the biGaussian Pell and biGaussian Pell-Lucas numbers will be defined. In this work, a variety of algebraic proporties of both the bicomplex numbers and the biGaussian Pell and Pell-Lucas numbers and the negabiGaussian Pell and Pell-Lucas are presented in a unified manner. Some identities will be given for biGaussian Pell and Pell-Lucas numbers such as Binet's formula, generating function, d'Ocagne's identity, Catalan's identity, Cassini's identity and some sums formulas. The biGaussian Pell and Pell-Lucas numbers' properties will also be obtained using matrix representation.

2. THE BIGAUSSIAN PELL AND PELL-LUCAS NUMBERS

Definition 1. BiGaussian Pell and biGaussian Pell-Lucas numbers BGP_n and BGQ_n are defined by the basis $\{1, i, j, ij\}$ as follows

$$BGP_n = \{P_n + iP_{n-1} + jP_{n-2} + ijP_{n-3} \mid P_n, nth\ Pell\ number\}$$

$$BGP_n = 2BGP_{n-1} + BGP_{n-2}$$

and

$$BGQ_n = \{Q_n + iQ_{n-1} + jQ_{n-2} + ijQ_{n-3} \mid Q_n, nth\ Pell - Lucas\ number\}$$

$$BGQ_n = 2BGQ_{n-1} + BGQ_{n-2}$$

where i, j, and ij satisfy the conditions

$$i^2 = -1, j^2 = -1, ij = ji.$$

The biGaussian Pell and biGaussian Pell-Lucas numbers starting from $n = 0$ can be written respectively as

$$BGP_0 = 0 + 1i - 2j + 5ij,\, BGP_1 = 1 + 0i + 1j - 2ij \text{ and } BGP_2 = 2 + i + 0j + 1ij,$$

$$\ldots$$

$$BGQ_0 = 2 - 2i + 6j - 14ij,\, BGQ_1 = 2 + 2i - 2j + 6ij \text{ and } BGQ_2 = 6 + 2i + 2j - 2ij, \ldots$$

Definition 2. NegabiGaussian Pell and negabiGaussian Pell-Lucas numbers BGP_{-n} and BGQ_{-n} are defined by the basis $\{1, i, j, ij\}$ as follows

$$BGP_{-n} = \{P_{-n} + iP_{-n-1} + jP_{-n-2} + ijP_{-n-3} \mid P_{-n}, -nth\ Pell\ number\}$$

and

$$BGQ_{-n} = \{Q_{-n} + iQ_{-n-1} + jQ_{-n-2} + ijQ_{-n-3} \mid Q_{-n}, -\ nth\ Pell - Lucas\ number\}$$

where i, j, and ij satisfy the conditions

$$i^2 = -1, j^2 = -1, ij = ji.$$

Let BGP_n and BGP_m be two biGaussian Pell numbers, the addition and subsraction of these numbers are defined by

$$\begin{aligned} BGP_n \pm BGP_m &= (P_n + iP_{n-1} + jP_{n-2} + ijP_{n-3}) \pm (P_m + iP_{m-1} + jP_{m-2} + ijP_{m-3}) \\ &= (P_n \pm P_m) + i(P_{n-1} \pm P_{m-1}) + j(P_{n-2} \pm P_{m-2}) + ij(P_{n-3} \pm P_{m-3}). \end{aligned}$$

The multiplication of a biGaussian Pell number by the real scalar λ is defined as

$$\lambda BGP_n = \lambda P_n + i\lambda P_{n-1} + j\lambda P_{n-2} + ij\lambda P_{n-3}\ .$$

Let BGP_n and BGP_m be two biGaussian Pell numbers, the multiplication of these numbers is defined by

$$\begin{aligned} BGP_n \times BGP_m &= (P_n + iP_{n-1} + jP_{n-2} + ijP_{n-3}) \times (P_m + iP_{m-1} + jP_{m-2} + ijP_{m-3}) \\ &= (P_nP_m - P_{n-1}P_{m-1} - P_{n-2}P_{m-2} + P_{n-3}P_{m-3}) \\ &\quad + i(P_nP_{m-1} + P_{n-1}P_m - P_{n-2}P_{m-3} - P_{n-3}P_{m-2}) \\ &\quad + j(P_nP_{m-2} + P_{n-2}P_m - P_{n-1}P_{m-3} - P_{n-3}P_{m-1}) \\ &\quad + ij(P_nP_{m-3} + P_{n-3}P_m + P_{n-1}P_{m-2} + P_{n-2}P_{m-1}) \\ &= BGP_m \times BGP_n. \end{aligned}$$

The conjugation of the biGaussian Pell numbers is defined in three different ways as follows

$$(BGP_n)_i^* = P_n - iP_{n-1} + jP_{n-2} - ijP_{n-3}, \tag{2.1}$$

$$(BGP_n)_j^* = P_n + iP_{n-1} - jP_{n-2} - ijP_{n-3}, \tag{2.2}$$

$$(BGP_n)_{ij}^* = P_n - iP_{n-1} + jP_{n-2} + ijP_{n-3}. \tag{2.3}$$

The addition and substraction, the multiplication by the real scalar, the multiplication and the conjugation for biGaussian Pell-Lucas numbers are defined similarly.

Theorem 3. For any given two biGaussian Pell numbers BGP_n and BGP_m the following relations between the conjugates of these numbers exist:

$$(BGP_n \times BGP_m)_i^* = (BGP_m)_i^* \times (BGP_n)_i^* = (BGP_n)_i^* \times (BGP_m)_i^*,$$

$$(BGP_n \times BGP_m)_j^* = (BGP_m)_j^* \times (BGP_n)_j^* = (BGP_n)_j^* \times (BGP_m)_j^*,$$

$$(BGP_n \times BGP_m)_{ij}^* = (BGP_m)_{ij}^* \times (BGP_n)_{ij}^* = (BGP_n)_{ij}^* \times (BGP_m)_{ij}^*.$$

Proof: By considering the equations (2.1), (2.2) and (2.3), the theorem can be proved easily. Also, the above three equations are valid for biGaussian Pell-Lucas numbers.

Theorem 4. Let $(BGP_n)_i^*$, $(BGP_n)_j^*$ and $(BGP_n)_{ij}^*$ be three kinds of conjugations of the biGaussian Pell numbers. The following relations hold

$$BGP_n \times (BGP_n)_i^* = 2Q_{2n-3} + 2jP_{2n-3},$$
$$BGP_n \times (BGP_n)_j^* = \frac{Q_{2n-1}+Q_{2n-5}-4(-1)^n}{4} + i\,\frac{Q_{2n-1}+Q_{2n-5}+4(-1)^n}{4},$$
$$BGP_n \times (BGP_n)_{ij}^* = 6P_{2n-3} - 4ij(-1)^n.$$

Proof:

$$\begin{aligned} BGP_n \times (BGP_n)_i^* &= P_n^2 + P_{n-1}^2 - P_{n-2}^2 - P_{n-3}^2 + 2j(P_nP_{n-2} + P_{n-1}P_{n-3}) \\ &= 2Q_{2n-3} + 2jP_{2n-3}, \end{aligned}$$

$$\begin{aligned} BGP_n \times (BGP_n)_j^* &= P_n^2 - P_{n-1}^2 + P_{n-2}^2 - P_{n-3}^2 + 2i(P_nP_{n-1} + P_{n-2}P_{n-3}) \\ &= \frac{Q_{2n-1}+Q_{2n-5}-4(-1)^n}{4} + i\,\frac{Q_{2n-1}+Q_{2n-5}+4(-1)^n}{4}, \end{aligned}$$

$$\begin{aligned} BGP_n \times (BGP_n)_{ij}^* &= P_n^2 + P_{n-1}^2 + P_{n-2}^2 + P_{n-3}^2 + 2ij(P_nP_{n-3} - P_{n-1}P_{n-2}) \\ &= 6P_{2n-3} - 4ij(-1)^n. \end{aligned}$$

These three kinds of conjugations are valid for biGaussian Pell-Lucas numbers.

$$\begin{aligned} BGQ_n \times (BGQ_n)_i^* &= Q_n^2 + Q_{n-1}^2 - Q_{n-2}^2 - Q_{n-3}^2 + 2j(Q_nQ_{n-2} + Q_{n-1}Q_{n-3}) \\ &= 8(P_{2n-1} - P_{2n-5}) + 2j(Q_{2n-2} + Q_{2n-4}), \end{aligned}$$

$$\begin{aligned} BGQ_n \times (BGQ_n)_j^* &= Q_n^2 - Q_{n-1}^2 + Q_{n-2}^2 - Q_{n-3}^2 + 2i(Q_nQ_{n-1} + Q_{n-2}Q_{n-3}) \\ &= 8(P_{2n-1} + P_{2n-5} - (-1)^{n-1}) + 2i(Q_{2n-1} + Q_{2n-5} - 4(-1)^n), \end{aligned}$$

$$\begin{aligned} BGQ_n \times (BGQ_n)_{ij}^* &= Q_n^2 + Q_{n-1}^2 + Q_{n-2}^2 + Q_{n-3}^2 + 2ij(Q_nQ_{n-3} - Q_{n-1}Q_{n-2}) \\ &= 8(P_{2n-1} + P_{2n-5}) + 4ij(Q_{2n-3} - 6(-1)^n). \end{aligned}$$

Also, the norm of the biGaussian Pell and Pell-Lucas numbers BGP_n and BGQ_n is defined in three different ways as follows

$$N_{BGP_{n_i}} = \|GBP_n \times GBP_{n\,i}^*\| = \sqrt{2|Q_{2n-3} + jP_{2n-3}|},$$

$$N_{BGP_{n_j}} = \left\| GBP_n \times GBP^*_{n\,j} \right\| = \sqrt{\tfrac{1}{4}|Q_{2n-1} + Q_{2n-5} - 4(-1)^n + iQ_{2n-1} + Q_{2n-5} + 4(-1)^n|},$$

$$N_{BGP_{n_{ij}}} = \left\| GBP_n \times GBP^*_{n\,ij} \right\| = \sqrt{2|3P_{2n-3} - 2ij(-1)^n|},$$

$$N_{BGQ_{n_i}} = \left\| GBQ_n \times GBQ^*_{n_i} \right\| = \sqrt{2|4(P_{2n-1} - P_{2n-5}) + j(Q_{2n-2} + Q_{2n-4})|},$$

$$N_{BGQ_{n_j}} = \left\| GBQ_n \times GBQ^*_{n\,j} \right\| = \sqrt{|8(P_{2n-1} + P_{2n-5} - (-1)^{n-1}) + 2i(Q_{2n-1} + Q_{2n-5} - 4(-1)^n)|},$$

$$N_{BGQ_{n_{ij}}} = \left\| GBQ_n \times GBQ^*_{n_{ij}} \right\| = \sqrt{4|2(P_{2n-1} + P_{2n-5}) + ij(Q_{2n-3} - 6(-1)^n)|}.$$

Theorem 5. BCP_n, BCQ_n, BGP_{-n} and BGQ_{-n} are bicomplex Pell, bicomplex Pell-Lucas, negabiGaussian Pell and negabiGaussian Pell-Lucas numbers, respectively. There are relations between these numbers as follows

$$BGP_{-n} = (-1)^n BCP_n + 2(-1)^{n+1} P_n$$

and

$$BGQ_{-n} = 3(-1)^n BCQ_n - 2(-1)^n Q_n.$$

Proof:

$$\begin{aligned} BGP_{-n} &= P_{-n} + iP_{-n-1} + jP_{-n-2} + ijP_{-n-3} \\ &= P_{-n} + iP_{-(n+1)} + jP_{-(n+2)} + ijP_{-(n+3)} \\ &= (-1)^{n+1}P_n + i(-1)^{n+2}P_{n+1} + j(-1)^{n+3}P_{n+2} + ij(-1)^{n+4}P_{n+3} \\ &= (-1)^{n+1}[P_n + iP_{n+1} + jP_{n+2} + ijP_{n+3}] - i(-1)^{n+1}P_{n+1} - j(-1)^{n+1}P_{n+2} - \\ &\quad ij(-1)^{n+1}P_{n+3} + i(-1)^{n+2}P_{n+1} + j(-1)^{n+3}P_{n+2} + ij(-1)^{n+4}P_{n+3} \\ &= (-1)^{n+1}BCP_n - i(-1)^{n+1}[P_{n+1} + P_{n+1}] - j(-1)^{n+1}[P_{n+2} + P_{n+2}] - \\ &\quad ij(-1)^{n+1}[P_{n+3} + P_{n+3}] \\ &= (-1)^{n+1}BCP_n - 2(-1)^{n+1}[P_n + iP_{n+1} + jP_{n+2} + ijP_{n+3} - P_n] \\ &= (-1)^n BCP_n + 2(-1)^{n+1}P_n. \end{aligned}$$

$$\begin{aligned} BGQ_{-n} &= Q_{-n} + iQ_{-n-1} + jQ_{-n-2} + ijQ_{-n-3} \\ &= Q_{-n} + iQ_{-(n+1)} + jQ_{-(n+2)} + ijQ_{-(n+3)} \\ &= (-1)^n Q_n + i(-1)^{n+1}Q_{n+1} + j(-1)^{n+2}Q_{n+2} + ij(-1)^{n+3}Q_{n+3} \\ &= (-1)^n[Q_n + iQ_{n+1} + jQ_{n+2} + ijQ_{n+3}] - i(-1)^n Q_{n+1} - j(-1)^n Q_{n+2} - \\ &\quad ij(-1)^n Q_{n+3} + i(-1)^{n+1}Q_{n+1} + j(-1)^{n+2}Q_{n+2} + ij(-1)^{n+3}Q_{n+3} \\ &= (-1)^n BCQ_n - i(-1)^n[Q_{n+1} + Q_{n+1}] - j(-1)^n[Q_{n+2} + Q_{n+2}] - ij(-1)^n[Q_{n+3} + \\ &\quad Qn{+}3 \\ &= (-1)^n BCQ_n - 2(-1)^{n+1}[Q_n + iQ_{n+1} + jQ_{n+2} + ijQ_{n+3} - Q_n] \\ &= 3(-1)^n BCQ_n - 2(-1)^n Q_n. \end{aligned}$$

Theorem 6. (Binet's Formula) Let BGP_n and BGQ_n be the biGaussian Pell and the biGaussian Pell-Lucas numbers, respectively. Binet's Formula for these numbers is as follows

$$BGP_n = \frac{\hat{\alpha}\alpha^{n-3} - \hat{\beta}\beta^{n-3}}{\alpha - \beta}$$

and

$$BGQ_n = \hat{\alpha}\alpha^{n-3} + \hat{\beta}\beta^{n-3}$$

where $\hat{\alpha} = \alpha^3 + i\alpha^2 + j\alpha + ij$, $\alpha = 1 + \sqrt{2}$ and $\hat{\beta} = \beta^3 + i\beta^2 + j\beta + ij$, $\alpha = 1 - \sqrt{2}$.

Proof:

$$\begin{aligned} BGP_n &= P_n + iP_{n-1} + jP_{n-2} + ijP_{n-3} \\ &= \frac{\alpha^n - \beta^n}{\alpha - \beta} + i\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} + j\frac{\alpha^{n-2} - \beta^{n-2}}{\alpha - \beta} + ij\frac{\alpha^{n-3} - \beta^{n-3}}{\alpha - \beta} \\ &= \frac{\alpha^{n-3}(\alpha^3 + i\alpha^2 + j\alpha + ij) - \beta^{n-3}(\beta^3 + i\beta^2 + j\beta + ij)}{\alpha - \beta} \\ &= \frac{\hat{\alpha}\alpha^{n-3} - \hat{\beta}\beta^{n-3}}{\alpha - \beta} \end{aligned}$$

Similarly, the Binet's Formula for biGaussian Pell-Lucas numbers is obtained.

Theorem 7. (Generating Function Formula) Let BGP_n and BGQ_n be the biGaussian Pell and the biGaussian Pell-Lucas numbers, respectively. Generating function for these numbers is as follows

$$h(t) = \frac{1 - 2j + (1 - 2i + 5j - 12ij)t}{1 - 2t - t^2}$$

and

$$m(t) = \frac{2 - 2i + 6j - 14ij + (-2 + 6i - 14j + 34ij)t}{1 - 2t - t^2}.$$

Proof: Let $h(t)$ be the generating function for biGaussian Pell numbers as

$$h(t) = \sum_{n=0}^{\propto} BGP_n t^n.$$

Using $h(t)$, $2th(t)$ and $t^2h(t)$, we get the following equations

$$th(t) = \sum_{n=0}^{\propto} BGP_n t^{n+1},\ t^2h(t) = \sum_{n=0}^{\propto} BGP_n t^{n+2}.$$

After the needed calculations, the generating function for biGaussian Pell numbers is obtained as

$$h(t) = \frac{BGP_0 + BGP_1 t - 2BGP_0 t}{1 - 2t - t^2}$$

$$h(t) = \frac{1 - 2j + (1 - 2i + 5j - 12ij)t}{1 - 2t - t^2}.$$

Similarly, the generating function for biGaussian Pell-Lucas numbers is obtained.

Theorem 8. (d'Ocagne's Identity) Let BGP_n and BGQ_n be the biGaussian Pell and the biGaussian Pell-Lucas numbers, respectively. d'Ocagne's identity for these numbers is as follows

$$\begin{aligned} BGP_m BGP_{n+1} - BGP_{m+1} BGP_n = 12(-1)^{n-1} iP_{m-n+2} \\ +6jP_{m-n}[(-1)^{2m-n} + (-1)^n] + 12(-1)^{n-1} ijP_{m-n+2} \end{aligned}$$

$$BGQ_mBGQ_{n+1} - BGQ_{m+1}BGQ_n = 16[(-1)^n + (-1)^{2m-n}]iP_{m-n} \\ -48[(-1)^{2m-n} + (-1)^n]jP_{m-n} + 96(-1)^n ijP_{m-n}$$

Proof:

$$\begin{aligned} &BGP_mBGP_{n+1} - BGP_{m+1}BGP_n \\ &= (P_m + iP_{m-1} + jP_{m-2} + ijP_{m-3})(P_{n+1} + iP_n + jP_{n-1} + ijP_{n-2}) \\ &\quad -(P_{m+1} + iP_m + jP_{m-1} + ijP_{m-2})(P_n + iP_{n-1} + jP_{n-2} + ijP_{n-3}) \\ &= 2[(-1)^{n-1} + (-1)^{2m-n}]iP_{m-n} + 6[(-1)^{2m-n} + (-1)^n]jP_{m-n} + 12(-1)^{n-1}ijP_{m-n+2} \end{aligned}$$

$$\begin{aligned} &BGQ_mBGQ_{n+1} - BGQ_{m+1}BGQ_n = (Q_m + iQ_{m-1} + jQ_{m-2} + ijQ_{m-3})(Q_{n+1} + iQ_n + jQ_{n-1} + ijQ_{n-2}) \\ &\quad -(Q_{m+1} + iQ_m + jQ_{m-1} + ijQ_{m-2})(Q_n + iQ_{n-1} + jQ_{n-2} + ijQ_{n-3}) \\ &= 16[(-1)^n + (-1)^{2m-n}]iP_{m-n} - 48[(-1)^{2m-n} + (-1)^n]jP_{m-n} + 96(-1)^n ijP_{m-n} \end{aligned}$$

Theorem 9. (Catalan's Identity) Let BGP_n and BGQ_n be the biGaussian Pell and the biGaussian Pell-Lucas numbers, respectively. Catalan's identity for these numbers is as follows

$$(BGP_n)^2 - BGP_{n+r}BGP_{n-r} = 2(-1)^{n-r}[P_{r+2} + P_{r-2}]jP_r \\ +(-1)^{n-r}[P_{r-3} - P_{r+3} + P_{r+1} - P_{r-1}]ijP_r$$

$$(BGQ_n)^2 - BGQ_{n+r}BGQ_{n-r} = 16(-1)^{n-r+1}[P_{r+2} + P_{r-2}]jP_r \\ +8(-1)^{n-r}[P_{r+3} - P_{r-3} + P_{r-1} - P_{r+1}]ijP_r$$

Proof:

$$\begin{aligned} &(BGP_n)^2 - BGP_{n+r}BGP_{n-r} = (P_n + iP_{n-1} + jP_{n-2} + ijP_{n-3})(P_n + iP_{n-1} + jP_{n-2} + ijP_{n-3}) \\ &\quad -(P_{n+r} + iP_{n+r-1} + jP_{n+r-2} + ijP_{n+r-3})(P_{n-r} + iP_{n-r-1} + jP_{n-r-2} + ijP_{n-r-3}) \\ &= 2(-1)^{n-r}[P_{r+2} + P_{r-2}]jP_r + (-1)^{n-r}[P_{r-3} - P_{r+3} + P_{r+1} - P_{r-1}]ijP_r \end{aligned}$$

$$\begin{aligned} &(BGQ_n)^2 - BGQ_{n+r}BGQ_{n-r} = \left(Q_n + iQ_{n-1} + jQ_{n-2} + ijQ_{n-3}\right)\left(Q_n + iQ_{n-1} + jQ_{n-2} + ijQ_{n-3}\right) \\ &\quad -(Q_{n+r} + iQ_{n+r-1} + jQ_{n+r-2} + ijQ_{n+r-3})(Q_{n-r} + iQ_{n-r-1} + jQ_{n-r-2} + ijQ_{n-r-3}) \\ &= 16(-1)^{n-r+1}[P_{r+2} + P_{r-2}]jP_r + 8(-1)^{n-r}[P_{r+3} - P_{r-3} + P_{r-1} - P_{r+1}]ijP_r \end{aligned}$$

Theorem 10. (Cassini's Identity) Let BGP_n and BGQ_n be the biGaussian Pell and the biGaussian Pell-Lucas numbers, respectively. Cassini's identity for these numbers is as follows

$$BGP_{n+1}BGP_{n-1} - (BGP_n)^2 = 12(-1)^n j - 12(-1)^n ij$$

$$BGQ_{n+1}BGQ_{n-1} - (BGQ_n)^2 = 96(-1)^{n+1} j - 96(-1)^{n+1} ij$$

Proof: If $r = 1$ is taken in the Catalan's identity, Cassini's identity is obtained.

Theorem 11. Let BGP_n and BGQ_n be the biGaussian Pell and the biGaussian Pell-Lucas numbers, respectively. The following relations are satisfied

- $2(BGP_{n+1} + BGP_n) = BGQ_{n+1}$
- $2(BGP_{n+1} - BGP_n) = BGQ_n$
- $BGP_{n+1} + BGP_{n-1} = BGQ_n$

- $BGP_{n+1} - BGP_{n-1} = 2BGP_n$
- $BGP_{n+2} + BGP_{n-2} = 6BGP_n$
- $BGP_{n+2} - BGP_{n-2} = 2BGQ_n$
- $BGQ_{n+1} + BGQ_n = 4BGP_{n+1}$
- $BGQ_{n+1} - BGQ_n = 4BGP_n$
- $BGQ_{n+1} + BGQ_{n-1} = 4BGP_n$
- $BGQ_{n+1} - BGQ_{n-1} = 2BGQ_n$
- $BGQ_{n+2} + BGQ_{n-2} = 6BGP_n$
- $BGQ_{n+2} - BGQ_{n-2} = 8BGP_n$

Proof: By considering the definition 1 and definition 2, the theorem can be proved easily.

Lemma 12. Let P_n and Q_n be the Pell and the Pell-Lucas numbers, respectively. The following relations are satisfied

- $\sum_{i=1}^{n} P_i = \frac{Q_{n+1}-2}{4}$
- $\sum_{i=1}^{n} Q_i = 2P_{n+1} - 2$
- $\sum_{i=1}^{n} P_{2i-1} = \frac{P_{2n}}{2}$
- $\sum_{i=1}^{n} Q_{2i-1} = \frac{Q_{2n}-1}{2}$
- $\sum_{i=1}^{n} P_{2i} = \frac{P_{2n+1}-1}{2}$
- $\sum_{i=1}^{n} Q_{2i} = \frac{Q_{2n+1}-1}{2}$

Proof: The proofs are seen by induction on *n*.

Theorem 13: Let BGP_n and BGQ_n be the biGaussian Pell and the biGaussian Pell-Lucas numbers, respectively. In this case

- $\sum_{i=1}^{n} GBP_i = \left(\frac{Q_{n+1}-2}{4}\right) + i\left(\frac{Q_n-2}{4}\right) + j\left(\frac{Q_{n-1}+2}{4}\right) + ij\left(\frac{Q_{n-2}-6}{4}\right)$
- $\sum_{i=1}^{n} GBQ_i = (2P_{n+1} - 2) + i(2P_n) + j(2P_{n-1} - 2) + ij(2P_{n-2} + 4)$

- $\sum_{i=1}^{n} GBP_{2i-1} = \left(\frac{P_{2n}}{2}\right) + i\left(\frac{P_{2n-1}-1}{2}\right) + j\left(\frac{P_{2n-2}+1}{2}\right) + ij\left(\frac{P_{2n-3}-5}{2}\right)$
- $\sum_{i=1}^{n} GBQ_{2i-1} = \left(\frac{Q_{2n}-1}{2}\right) + i\left(\frac{Q_{2n-1}+3}{2}\right) + j\left(\frac{Q_{2n-2}-5}{2}\right) + ij\left(\frac{Q_{2n-3}+15}{2}\right)$
- $\sum_{i=1}^{n} GBP_{2i} = \left(\frac{P_{2n+1}-1}{2}\right) + i\left(\frac{P_{2n}}{2}\right) + j\left(\frac{P_{2n-1}-1}{2}\right) + ij\left(\frac{P_{2n-2}+2}{2}\right)$
- $\sum_{i=1}^{n} GBQ_{2i} = \left(\frac{Q_{2n+1}-1}{2}\right) + i\left(\frac{Q_{2n}-1}{2}\right) + j\left(\frac{Q_{2n-1}+3}{2}\right) + ij\left(\frac{Q_{2n-2}-5}{2}\right)$

Proof:

$$\begin{aligned}\sum_{i=1}^{n} GBP_i &= \sum_{i=1}^{n}(P_n + iP_{n-1} + jP_{n-2} + ijP_{n-3})\\ &= \sum_{i=1}^{n} P_i + i\sum_{i=0}^{n-1} P_i + j\sum_{i=-1}^{n-2} P_i + ij\sum_{i=-2}^{n-3} P_i\\ &= \left(\frac{Q_{n+1}-2}{4}\right) + i\left(\frac{Q_n-2}{4}\right) + j\left(\frac{Q_{n-1}+2}{4}\right) + ij\left(\frac{Q_{n-2}-6}{4}\right).\end{aligned}$$

Other sums are proven through the same method.

Theorem 14. Let BGP_n and BGQ_n be the biGaussian Pell and the biGaussian Pell-Lucas numbers, respectively. For $n \geq 1$ be integer. Then

$$\begin{bmatrix}2 & 1\\ 1 & 0\end{bmatrix}^n \begin{bmatrix}BGP_2 & BGP_1\\ BGP_1 & BGP_0\end{bmatrix} = \begin{bmatrix}BGP_{n+2} & BGP_{n+1}\\ BGP_{n+1} & BGP_n\end{bmatrix}$$

and

$$\begin{bmatrix}2 & 1\\ 1 & 0\end{bmatrix}^n \begin{bmatrix}BGQ_2 & BGQ_1\\ BGQ_1 & BGQ_0\end{bmatrix} = \begin{bmatrix}BGQ_{n+2} & BGQ_{n+1}\\ BGQ_{n+1} & BGQ_n\end{bmatrix}.$$

Proof: The proof is seen by induction on *n*.

Theorem 15: Let BGP_n and BGQ_n be the biGaussian Pell and the biGaussian Pell-Lucas numbers, respectively. For $n \geq 1$ be integer. Then

$$\begin{bmatrix}0 & 1\\ 1 & 2\end{bmatrix}^n \begin{bmatrix}BGP_0\\ BGP_1\end{bmatrix} = \begin{bmatrix}BGP_n\\ BGP_{n+1}\end{bmatrix}$$

and

$$\begin{bmatrix}0 & 1\\ 1 & 2\end{bmatrix}^n \begin{bmatrix}BGQ_0\\ BGQ_1\end{bmatrix} = \begin{bmatrix}BGQ_n\\ BGQ_{n+1}\end{bmatrix}.$$

Proof: The proof is seen by induction on *n*.

3. CONCLUSION

This study presents the biGaussian Pell and Pell-Lucas quaternions. We obtain these new quaternions not defined in the literature before. We generate Binet's formula, generating function, matrix representation and the summation formulas for these quaternions. Also we give d'Ocagne's identity, Catalan's identity and Cassini's identity. Since this study includes some new results, it contributes to literature by providing essential information concerning the bicomplex quaternions. For further studies, we plan to find some proporties for these new numbers.

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