ORIGINAL PAPER

A COMPARISON OF ESTIMATION METHODS FOR ONE PARAMETER INVERSE GOMPERTZ DISTRIBUTION

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Abstract. *In this paper, we compare the methods of estimation for one parameter lifetime distribution, which is a special case of inverse Gompertz distribution. We discuss five different estimation methods such as maximum likelihood method, least-squares method, weighted least-squares method, the method of Anderson-Darling, and the method of Crámer–von Mises. It is evaluated the performances of these estimators via Monte Carlo simulations according to the bias and mean-squared error. Furthermore, two real data applications are performed.*

Keywords: *maximum likelihood method; least-squares method; weighted least-squares method; Anderson-Darling method; Crámer–von Mises estimation method*

1. INTRODUCTION

In recent years, it is well-known that point estimation is a popular topic in the statistics literature. There are many papers about different estimation methods for various distributions. Chang and Tsai [1] provided point and interval estimation under type-II censoring for Gompertz distribution. Wu et al. [2] estimated the parameters of Gompertz distribution via least squares method. Ismail [3] tackled the problem of Bayes estimation under type-I censoring for Gompertz distribution. It is used Lindley's [4] approach of approximate Bayesian methods in [3]. Lenart [5] compared the method of moments and maximum likelihood method for the estimation of parameters of the Gompertz distribution. Torres [6] suggested nonlinear least squares procedures to estimate parameters of the shifted Gompertz distribution. It is compared the methods of estimation such as maximum likelihood, the method of moments, least squares, and weighted least squares in [6]. Dey et al. [7] discussed statistical properties and different methods of estimation for Gompertz distribution. Eliwa et al. [8] studied the estimation problem for the discrete Gompertz-G family of distribution. Karakaya and Tanış [9] discussed the estimation methods for Xgamma Weibull distribution. Tanış and Karakaya [10] focused on the problem of parameter estimation for Lindley-Geometric distribution. Karakaya and Tanış [11] studied the methods of estimation for one parameter Akash distribution. Tanış [12] compared the estimation methods for transmuted power function distribution.

The paper aims to compare the methods of estimation for one parameter inverse Gompertz distribution. Therefore, we consider five estimation methods, such as maximum likelihood estimator, least squares estimator, weighted least squares estimator, Anderson-Darling estimator, and Crámer–von Mises estimator for point estimation. This paper is

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organized as follows: Section 2 provides one parameter inverse Gompertz and two parameters inverse Gompertz distribution. The methods of estimation are described in Section 3. A comprehensive Monte Carlo simulation study is presented to compare these estimators according to bias and mean square errors (MSEs) in Section 4. Two real data illustrations are given in Section 5. Finally, concluding remarks are given in Section 6.

2. INVERSE GOMPERTZ DISTRIBUTION

The Gompertz distribution (GD) was suggested by [13]. Let the random variable Y having to the GD with α and β parameters. The cumulative distribution function (cdf) of Y is given as follows:

$$G\left(y\right)=1-\exp\left(-\frac{\alpha}{\beta}\left(\exp\left(\beta y\right)-1\right)\right), \tag{1}$$

where $\alpha > 0$ is a shape parameter, $\beta > 0$ is a scale parameter and $y > 0$. GD has an increasing failure rate. It is widely used in many areas such as medicine, engineering, agriculture, economics, etc. Inverse Gompertz distribution (IGD) is introduced by [14]. The derivation of IGD can be summarized as follows:

Let define a random variable $Z=\frac{1}{Y}$, where Y random variable with cdf given in (1). The random variable Z having to IGD with α and β parameters. The cdf and probability density function (pdf) of IGD are

$$W\left(z\right)=\exp\left(-\frac{\alpha}{\beta}\left(\exp\left(\frac{\beta}{z}-1\right)\right)\right), \tag{2}$$

and

$$w\left(z\right)=\frac{\alpha}{z^2}\exp\left(-\frac{\alpha}{\beta}\left(\exp\left(\frac{\beta}{z}-1\right)\right)+\frac{\beta}{z}\right). \tag{3}$$

respectively. Eliwa et al. [14] studied different estimation methods, such as maximum likelihood, least-squares method, weighted least-squares method, and the method of Crámer–von Mises with complete and censored data for IGD.

Alshenawy [15] proposed one parameter IGD, which is a special case of IGD. It is shortly denoted this one parameter distribution by $A\left(\beta\right)$ in [11]. IGD is reduced $A\left(\beta\right)$ by substituting $\alpha=1$ in (2). We denote one parameter IGD as AD in this paper.

Let the random variable X having to the AD with β parameter. The cdf and pdf of X are given by

$$F\left(x\right)=\exp\left(\frac{1}{\beta}\left(1-\exp\left(\frac{\beta}{x}\right)\right)\right), \tag{4}$$

and

$$f\left(x\right)=\frac{1}{x^2}\exp\left(\frac{1}{\beta}\left(1-\exp\left(\frac{\beta}{x}\right)\right)+\frac{\beta}{x}\right), \tag{5}$$

respectively. Alshenawy [15] used the maximum likelihood method to estimate the parameter of AD based on complete and type-II censored data. Alshenawy [15] founded the maximum likelihood method is a good method to point estimation for AD. It is concluded that IGD exhibits an upside-down bathtub-shaped hazard function therefore, it is a suitable distribution for several real lifetime data sets. Further, AD provided a better fit than compared distributions such as exponential, inverse exponential, Rayleigh, inverse Rayleigh, and Lindley distributions.

3. PARAMETER ESTIMATION TECHNIQUES FOR THE ONE PARAMETER INVERSE GOMPERTZ DISTRIBUTION

In this section, estimators of the AD distribution parameter β are examined with five different estimation methods, which are maximum likelihood, least-squares, weighted least squares, Anderson-Darling, and Crámer–von Mises. Let $X_1, X_2, \ldots, X_n$ be a random sample from the $AD(\beta)$ distribution in (5). $X_1 < X_2 < \ldots < X_n$ symbolize the corresponding order statistics. Also, x_i indicate the observed value of X_i for $i = 1, 2, \ldots, n$.

3.1. MAXIMUM LIKELIHOOD ESTIMATION

The likelihood and log-likelihood function of $AD(\beta)$ distribution in (5) are given, respectively, by,

$$\begin{aligned} L(\beta) &= \prod_{i=1}^{n} f(x_i \mid \beta) \\ &= \prod_{i=1}^{n} \frac{1}{x_i^2} \exp\left[\frac{1}{\beta}\left(1-\exp\left(\frac{\beta}{x_i}\right)\right)+\frac{\beta}{x_i}\right] \end{aligned} \tag{6}$$

and

$$\ell(\beta) = -2\sum_{i=1}^{n}\log(x_i) + \beta\sum_{i=1}^{n}\frac{1}{x_i} - \frac{1}{\beta}\sum_{i=1}^{n}\left(\exp\left(\frac{\beta}{x_i}\right)-1\right). \tag{7}$$

The likelihood equation is obtained by differentiating Equation (7) for β

$$\frac{d\ell(\beta)}{d\beta} = \sum_{i=1}^{n}\frac{1}{x_i} - \frac{1}{\beta^2}\sum_{i=1}^{n}\left(\frac{\beta}{x_i}\exp\left(\frac{\beta}{x_i}\right) - \exp\left(\frac{\beta}{x_i}\right)+1\right).$$

It is noted that it is impossible to obtain an explicit solution for MLE. For this purpose, some methods are conducted. The MLE is also expressed as

$$\hat{\beta}_{MLE} = \arg\max\left(\ell(\beta)\right). \tag{8}$$

3.2. *LEAST SQUARES AND WEIGHTED LEAST SQUARES ESTIMATION*

Least square estimation (LSE) and weighted least square estimation (WLSE) methods were introduced by Swain et al. [16] for the estimation problem of Beta distribution parameters. In this study, the same methods for the $AD(\beta)$ are applied. Let us define the following two functions which are used to obtain LSE and WLSE, respectively,

$$Q_{LSE}(\beta)=\sum_{i=1}^{n}\left(\exp\left\{\frac{1}{\beta}\left(1-\exp\left(\frac{\beta}{x_{(i)}}\right)\right)\right\}-\frac{i}{n+1}\right)^2 \tag{9}$$

and

$$Q_{WLSE}(\beta)=\sum_{i=1}^{n}\frac{(n+2)(n+1)^2}{i(n-i+1)}\left(\exp\left\{\frac{1}{\beta}\left(1-\exp\left(\frac{\beta}{x_{(i)}}\right)\right)\right\}-\frac{i}{n+1}\right)^2. \tag{10}$$

LSE and WLSE can be obtained respectively by minimizing the function given in (9) and (10). Also, it is expressed as

$$\hat{\beta}_{LSE}=\arg\min\left(Q_{LSE}(\beta)\right) \tag{11}$$

and

$$\hat{\beta}_{WLSE}=\arg\min\left(Q_{WLSE}(\beta)\right). \tag{12}$$

LSE and WLSE can also be achieved respectively by solving the following nonlinear equations:

$$\sum_{i=1}^{n}\left(\exp\left\{\frac{1}{\beta}\left(1-\exp\left(\frac{\beta}{x_{(i)}}\right)\right)\right\}-\frac{i}{n+1}\right)\Delta\left(x_{(i)}\mid\beta\right)=0$$

and

$$\sum_{i=1}^{n}\frac{(n+2)(n+1)^2}{i(n-i+1)}\left(\exp\left\{\frac{1}{\beta}\left(1-\exp\left(\frac{\beta}{x_{(i)}}\right)\right)\right\}-\frac{i}{n+1}\right)\Delta\left(x_{(i)}\mid\beta\right)=0,$$

where

$$\Delta\left(x_{(i)}\mid\beta\right)=\frac{\exp\left\{\frac{1}{\beta}\left(1-\exp\left(\frac{\beta}{x_{(i)}}\right)\right)\right\}\left\{\left(x_{(i)}-\beta\right)\exp\left(\frac{\beta}{x_{(i)}}\right)-x_{(i)}\right\}}{\beta^2 x_{(i)}}. \tag{13}$$

3.3. *ANDERSON-DARLING ESTIMATION*

The Anderson-Darling estimation (ADE) of $AD(\beta)$ distribution is obtained minimizing function given in Equation (14)

$$Q_{ADE}(\beta)=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left(\log\left[\exp\left\{\frac{1}{\beta}\left(1-\exp\left(\frac{\beta}{x_{(i)}}\right)\right)\right\}\right]+\log\left[1-\exp\left\{\frac{1}{\beta}\left(1-\exp\left(\frac{\beta}{x_{(i)}}\right)\right)\right\}\right]\right)^2 \tag{14}$$

Also, it is expressed as

$$\hat{\beta}_{ADE} = \arg\min \left(Q_{ADE}(\beta) \right). \tag{15}$$

ADE can also be achieved by solving the following nonlinear equation:

$$\sum_{i=1}^{n} (2i-1) \left[\frac{\Delta(x_i \mid \beta)}{\exp\left\{ \frac{1}{\beta}\left(1-\exp\left(\frac{\beta}{x_i}\right)\right)\right\}} - \frac{\Delta(x_i \mid \beta)}{1-\exp\left\{ \frac{1}{\beta}\left(1-\exp\left(\frac{\beta}{x_i}\right)\right)\right\}} \right] = 0$$

where $\Delta(x_i \mid \beta)$ is in (13).

3.4. CRAMER-VON MISES ESTIMATION

The Crámer–von Mises estimation (CvME) of $AD(\beta)$ distribution is obtained minimizing function given in Equation (16)

$$Q_{CvME}(\beta) = \frac{1}{12n} + \sum_{i=1}^{n} \left(\exp\left\{ \frac{1}{\beta}\left(1-\exp\left(\frac{\beta}{x_i}\right)\right)\right\} - \frac{2i-1}{2n} \right)^2. \tag{16}$$

Also, it is expressed as

$$\hat{\beta}_{CvME} = \arg\min \left(Q_{CvME}(\beta) \right). \tag{17}$$

CvME can also be achieved by solving the following nonlinear equation:

$$\sum_{i=1}^{n} \left(\exp\left\{ \frac{1}{\beta}\left(1-\exp\left(\frac{\beta}{x_i}\right)\right)\right\} - \frac{2i-1}{2n} \right) \Delta(x_i \mid \beta) = 0$$

where $\Delta(x_i \mid \beta)$ is in (13).

Five estimates given in Equations (8), (11), (12), (15) and (17) can be obtained by optim function in R with BFGS algorithm. The BFGS algorithm is known as a quasi-Newton method and it is suggested by Fletcher [17].

4. SIMULATION STUDY

In this section, we conducted a simulation study by the Monte Carlo method based on 5000 trials. This simulation study's primary purpose is to compare different estimators for the parameter of the AD distribution by using bias and MSE. The following algorithm was carried out in the simulation study:

Step 1: Specify the sample of size *n* and parameter of $AD\ \beta$ distribution.

Step 2: Generate data from $AD\ \beta$ distribution with size *n* using the inverse transform method.

Step 3: Using the data from $AD\ \beta$ distribution obtained in Step 2, calculate the $\hat{\beta}$ with MLE, LSE, WLSE, ADE, and CvME given in Section 2.

Step 4: Steps 2 and 3 are repeated 5000 times.

Step 5: Using $\hat{\beta}$ and $\beta,$ calculate the bias and MSE.

Nine true parameter settings are considered as $\beta = 0.25, 0.3, 0.6, 0.9, 1.2, 1.5, 1.8, 2, 3$ in Step 1. The sample size are selected as *n*=25, 50, 75, 100, 200, 300, 400, 500 in Step 1. The BFGS algorithm, which is available in R, is used to achieve five estimates given in Equations (5), (8), (9), (12), and (14). In Tables 1-3, bias and MSEs of MLE, LSE, WLSE, ADE and CvME are reported. Tables 1-3 show that the bias and MSEs of all estimators are close to zero when the sample size increases.

Table 1. Average biases and MSEs of all estimates for *β* = 0.25, 0.3 and 0.6.

β		Bias					MSE				
	n	MLE	LSE	WLSE	ADE	CvME	MLE	LSE	WLSE	ADE	CvME
0.25	25	0.0943	0.0645	0.0646	0.0271	0.1104	0.0596	0.2329	0.1878	0.0887	0.2517
	50	0.0449	0.0279	0.0292	0.0116	0.0512	0.0237	0.0942	0.0626	0.0420	0.1021
	75	0.0296	0.0168	0.0191	0.0071	0.0319	0.0132	0.0544	0.0342	0.0256	0.0562
	100	0.0197	0.0063	0.0095	0.0014	0.0175	0.0097	0.0373	0.0228	0.0191	0.0381
	200	0.0105	0.0054	0.0072	0.0024	0.0110	0.0046	0.0182	0.0106	0.0097	0.0185
	300	0.0071	0.0040	0.0053	0.0017	0.0078	0.0031	0.0123	0.0071	0.0066	0.0124
	400	0.0052	0.0020	0.0034	0.0007	0.0048	0.0022	0.0085	0.0048	0.0046	0.0086
	500	0.0038	0.0024	0.0031	0.0008	0.0046	0.0018	0.0074	0.0042	0.0040	0.0075
0.3	25	0.0976	0.0669	0.0646	0.0245	0.1129	0.0660	0.2789	0.2060	0.0940	0.2972
	50	0.0406	0.0195	0.0215	0.0050	0.0424	0.0237	0.0918	0.0598	0.0429	0.0958
	75	0.0309	0.0137	0.0167	0.0059	0.0289	0.0151	0.0551	0.0343	0.0276	0.0569
	100	0.0234	0.0175	0.0182	0.0087	0.0290	0.0109	0.0428	0.0264	0.0216	0.0439
	200	0.0092	0.0064	0.0073	0.0021	0.0121	0.0049	0.0195	0.0113	0.0101	0.0198
	300	0.0075	0.0019	0.0039	0.0007	0.0056	0.0032	0.0124	0.0072	0.0068	0.0125
	400	0.0055	0.0014	0.0029	0.0005	0.0042	0.0023	0.0091	0.0052	0.0050	0.0092
	500	0.0043	0.0018	0.0030	0.0007	0.0040	0.0018	0.0076	0.0043	0.0041	0.0077
0.6	25	0.0910	0.0525	0.0510	0.0167	0.1030	0.0809	0.2879	0.2340	0.1163	0.3103
	50	0.0490	0.0343	0.0341	0.0158	0.0591	0.0339	0.1259	0.0864	0.0584	0.1312
	75	0.0270	0.0082	0.0114	0.0022	0.0245	0.0194	0.0653	0.0421	0.0360	0.0669
	100	0.0202	0.0084	0.0104	0.0018	0.0207	0.0142	0.0507	0.0316	0.0270	0.0517
	200	0.0119	0.0068	0.0086	0.0041	0.0129	0.0070	0.0239	0.0147	0.0135	0.0242
	300	0.0071	0.0018	0.0040	0.0006	0.0059	0.0045	0.0158	0.0095	0.0090	0.0159
	400	0.0053	0.0003	0.0022	-0.0004	0.0034	0.0035	0.0121	0.0072	0.0070	0.0121
	500	0.0051	0.0021	0.0035	0.0014	0.0045	0.0026	0.0089	0.0053	0.0051	0.0090

Table 2. Average biases and MSEs of all estimates for *β* = 0.9, 1.2 and 1.5.

β		Bias					MSE				
	n	MLE	LSE	WLSE	ADE	CvME	MLE	LSE	WLSE	ADE	CvME
0.9	25	0.1025	0.0514	0.0521	0.0186	0.1036	0.1067	0.3558	0.2714	0.1576	0.3768
	50	0.0482	0.0297	0.0298	0.0143	0.0559	0.0433	0.1389	0.0948	0.0725	0.1443
	75	0.0319	0.0154	0.0182	0.0078	0.0327	0.0262	0.0835	0.0556	0.0466	0.0856
	100	0.0240	0.0089	0.0122	0.0036	0.0219	0.0188	0.0614	0.0397	0.0345	0.0625
	200	0.0112	0.0080	0.0092	0.0040	0.0145	0.0091	0.0309	0.0194	0.0181	0.0313
	300	0.0079	0.0020	0.0041	0.0007	0.0063	0.0059	0.0194	0.0120	0.0115	0.0195
	400	0.0070	0.0028	0.0043	0.0020	0.0060	0.0045	0.0137	0.0086	0.0083	0.0137
	500	0.0047	0.0014	0.0028	0.0005	0.0040	0.0037	0.0117	0.0072	0.0070	0.0118
1.2	25	0.1129	0.0727	0.0686	0.0352	0.1285	0.1301	0.4159	0.3146	0.1936	0.4444
	50	0.0513	0.0202	0.0231	0.0100	0.0477	0.0528	0.1559	0.1091	0.0872	0.1611
	75	0.0358	0.0123	0.0164	0.0071	0.0306	0.0331	0.0941	0.0634	0.0554	0.0963
	100	0.0249	0.0144	0.0155	0.0070	0.0281	0.0240	0.0729	0.0483	0.0427	0.0743
	200	0.0116	0.0055	0.0075	0.0030	0.0123	0.0112	0.0332	0.0214	0.0201	0.0335
	300	0.0083	0.0017	0.0040	0.0006	0.0063	0.0076	0.0226	0.0144	0.0140	0.0227
	400	0.0068	0.0012	0.0034	0.0008	0.0046	0.0055	0.0167	0.0105	0.0103	0.0167
	500	0.0062	0.0036	0.0047	0.0026	0.0063	0.0043	0.0133	0.0083	0.0081	0.0133
1.5	25	0.1117	0.0656	0.0630	0.0271	0.1246	0.1451	0.4513	0.3619	0.2108	0.4886
	50	0.0589	0.0227	0.0267	0.0119	0.0514	0.0644	0.1870	0.1318	0.1055	0.1927
	75	0.0359	0.0102	0.0137	0.0055	0.0293	0.0393	0.1057	0.0721	0.0642	0.1079
	100	0.0279	0.0121	0.0147	0.0070	0.0265	0.0286	0.0815	0.0551	0.0498	0.0829
	200	0.0189	0.0166	0.0172	0.0125	0.0238	0.0139	0.0397	0.0263	0.0248	0.0402
	300	0.0081	0.0021	0.0042	0.0008	0.0069	0.0092	0.0258	0.0170	0.0163	0.0260
	400	0.0063	0.0036	0.0046	0.0019	0.0072	0.0066	0.0197	0.0127	0.0124	0.0198
	500	0.0052	0.0010	0.0026	0.0004	0.0039	0.0055	0.0151	0.0098	0.0096	0.0152

Table 3. Average biases and MSEs of all estimates for *β* = 1.8, 2.0 and 3.0.

β		Bias					MSE				
	n	MLE	LSE	WLSE	ADE	CvME	MLE	LSE	WLSE	ADE	CvME
1.8	25	0.1256	0.0501	0.0517	0.0243	0.1100	0.1704	0.4769	0.3666	0.2451	0.5024
	50	0.0533	0.0200	0.0223	0.0075	0.0500	0.0712	0.1990	0.1426	0.1148	0.2048
	75	0.0357	0.0134	0.0170	0.0066	0.0334	0.0439	0.1267	0.0878	0.0750	0.1293
	100	0.0275	0.0124	0.0150	0.0060	0.0274	0.0332	0.0961	0.0648	0.0582	0.0976
	200	0.0144	0.0069	0.0090	0.0043	0.0143	0.0157	0.0446	0.0296	0.0282	0.0449
	300	0.0097	0.0039	0.0059	0.0022	0.0089	0.0109	0.0304	0.0202	0.0194	0.0306
	400	0.0067	-0.0007	0.0020	-0.0005	0.0030	0.0078	0.0211	0.0140	0.0137	0.0212
	500	0.0059	-0.0006	0.0017	-0.0004	0.0024	0.0062	0.0171	0.0111	0.0110	0.0171
2	25	0.1146	0.0529	0.0495	0.0191	0.1148	0.1829	0.5248	0.4062	0.2689	0.5526
	50	0.0626	0.0280	0.0313	0.0168	0.0588	0.0836	0.2248	0.1641	0.1325	0.2314
	75	0.0362	0.0088	0.0130	0.0032	0.0293	0.0514	0.1347	0.0944	0.0838	0.1372
	100	0.0318	0.0049	0.0116	0.0036	0.0202	0.0369	0.0990	0.0679	0.0623	0.1003
	200	0.0145	0.0068	0.0094	0.0042	0.0145	0.0170	0.0476	0.0316	0.0300	0.0480
	300	0.0105	0.0035	0.0059	0.0025	0.0086	0.0113	0.0313	0.0207	0.0201	0.0314
	400	0.0078	0.0039	0.0056	0.0030	0.0077	0.0086	0.0240	0.0160	0.0157	0.0241
	500	0.0060	0.0021	0.0035	0.0012	0.0052	0.0068	0.0185	0.0122	0.0120	0.0186
3	25	0.1396	0.0617	0.0606	0.0351	0.1303	0.2629	0.6732	0.5260	0.3843	0.7065
	50	0.0647	0.0264	0.0289	0.0127	0.0608	0.1147	0.2961	0.2142	0.1816	0.3039
	75	0.0469	0.0233	0.0265	0.0149	0.0462	0.0732	0.1936	0.1379	0.1216	0.1974
	100	0.0362	0.0184	0.0214	0.0126	0.0356	0.0555	0.1386	0.0988	0.0908	0.1407
	200	0.0201	0.0138	0.0157	0.0103	0.0224	0.0269	0.0679	0.0475	0.0454	0.0685
	300	0.0095	0.0005	0.0034	-0.0003	0.0062	0.0169	0.0439	0.0303	0.0294	0.0441
	400	0.0051	0.0006	0.0019	-0.0009	0.0049	0.0126	0.0324	0.0222	0.0217	0.0325
	500	0.0072	0.0019	0.0043	0.0018	0.0053	0.0100	0.0256	0.0174	0.0172	0.0257

5. REAL DATA APPLICATIONS

In this section, two real data applications for the AD distribution are conducted. AD distribution is fitted to two real data sets and we estimate the parameter of AD distribution using five methods of estimation given in Section 2. The MLEs, LSEs, WLSEs, ADEs and CVMEs of the AD distribution parameter is obtained by BFGS algorithm and given in Table 4.

Table 4. Parameter estimation, KS and corresponding p value for two data sets based on all estimators.

Methods	Data set 1			Data set 2		
	$\hat{\beta}$	KS	p value	$\hat{\beta}$	KS	p value
MLE	0.8360	0.1912	0.0516	6.9277	0.3248	0.1559
LSE	1.3841	0.1847	0.0659	6.0676	0.2996	0.2263
WLSE	1.3577	0.1843	0.0669	5.8709	**0.2941**	**0.2444**
ADE	1.3161	**0.1836**	**0.0685**	6.1884	0.3030	0.2155
CvME	1.4035	0.1850	0.0652	6.1797	0.3028	0.2163

Kolmogorov-Smirnov statistics (KS) and corresponding p-values given in Table 4 for all estimators. Therefore, if one will use the AD distribution in real data modeling, one of the five estimation methods can be used. The first data consists of 50 observations about breaking stress of carbon fibers (GPa) from Nichols and Padgett [18]. The first data given as: 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03,1.80,1.57,1.08,2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65. The second data consists of 11 observations about times to breakdown of an electrical insulating fluid subjected to 30 kilovolts from Lawless [19]. The second data set is given as follows: 2.836, 3.120, 3.045, 5.169, 4.934, 4.970, 3.018, 3.770, 5.272, 3.856, 2.046. Total time on test (TTT) plot due to [20] is a very considerable graphical approach to confirm the data can be applied to a particular distribution or not. TTT plot is provided for two data set in Figs. 1-2. Figs. 1-2 illustrate that both data with an increasing failure rate. Therefore, AD distribution can be used for modelling these data.

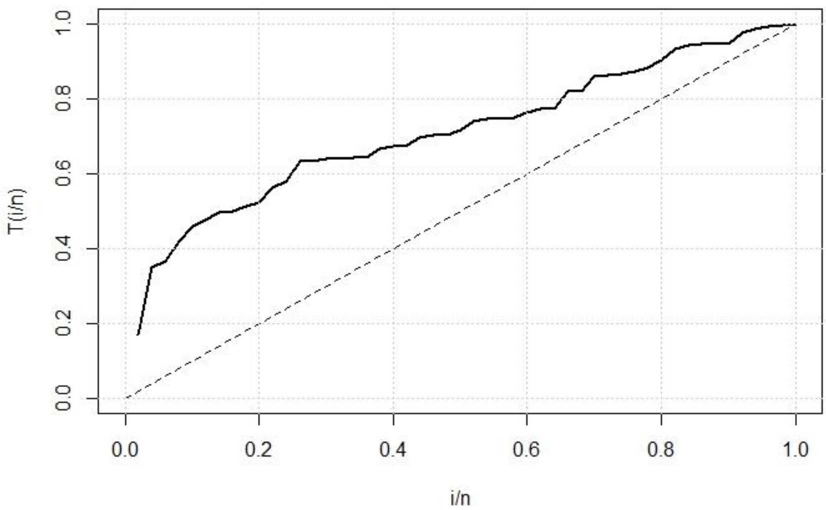

Figure 1. TTT plot for the first data set.

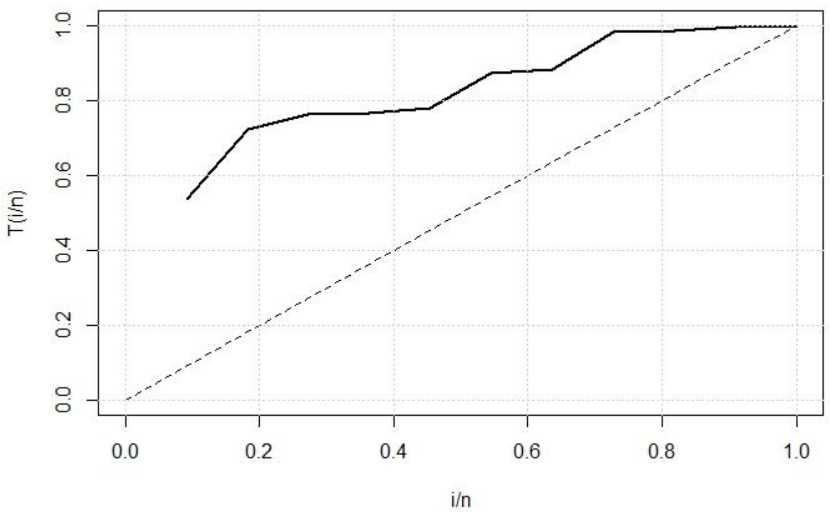

Figure 2. TTT plot for the second data set.

6. CONCLUSIONS

In this study, AD distribution introduced by [15] is examined concerning some point estimations. Five estimators are analyzed to estimate the parameter of AD distribution. A new extension has been provided for the estimation of the parameter for the AD distribution. Extensively Monte Carlo simulations are accomplished for nine different parameter values and different sample sizes. It has been determined that when the sample size increases, the MSE and bias decrease and approach to zero as expected. The parameter estimates and KS of AD distribution are obtained using five different estimation methods for two real data sets. We recommend one of the examined estimators to estimate the parameter of AD distribution according to simulation results. Thus, we provide a new extension for the point estimation of AD distribution. In future studies, other estimation methods can be examined for AD distribution.

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