ORIGINAL PAPER

QUARTER-SYMMETRIC METRIC CONNECTION ON TANGENT BUNDLE OF NORDEN MANIFOLD

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Abstract. *The aim of the paper is to study the tangent bundle of Norden manifolds endowed with the complete lift of quarter-symmetric metric connection. Firstly, the complete lift of quarter-symmetric metric connection is obtained on tangent bundle of almost Norden manifold. Under the condition of integrability of structure, the almost Norden manifold endowed with the quarter-symmetric metric connection is shown to be Norden manifold.*

Keywords: *Quarter-symmetric metric connection; tangent bundle; complete lift; Norden manifold.*

1. INTRODUCTION

The tangent bundle of a differentiable manifold takes a central place in the study of the differential geometry. Some studies on connnection have been built to tangent bundle of a differentiable manifold by Yano and Davies [1].

In 1975, the quarter-symmetric metric connection was defined and studied on manifold with affine connection by Golab [2]. In 1980, the quarter-symmetric metric connections and the properties of their curvature tensor were studied in Einstein, Kahler, Sasakian manifold by Mishra and Pandey [3]. In 1982, the most general form of quarter-symmetric metric connections was obtained and its applications were studied by Yano and Imai [4]. The quarter-symmetric metric connections have been studied on differentiable manifold by many authors [5-10].

In 1960, the B-metric with respect to almost complex structure on four-dimensional pseudo-Riemannian neutral space was defined by Norden [11]. Under the leadership of this study, the theory of Norden manifold has developed. The B-metric referred to as anti-Hermitian, pure or Norden metric have been studied by many authors [12-14].

In this study, we obtain the complete lift of quarter-symmetric metric connection on tangent bundle of almost Norden manifold. Under the condition of integrability of structure we show that the almost Norden manifold endowed with the quarter-symmetric metric connection becomes Norden manifold. To avoid repetation, manifold, tensor fields, linear connections are assumed to be differentiable and class C^{∞}.

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2. PRELIMINARIES

2.1. NORDEN MANIFOLD

Let (M_{2n}, φ, g) be a $2n$- dimensional almost complex manifold with Norden metric, i.e., φ is an almost complex structure and g is a Norden metric on M_{2n} such that:

$$\begin{aligned} &\varphi^2 U + U = 0, \\ &g(\varphi U, W) = g(U, \varphi W) \ (\textit{purity condition}) \end{aligned} \tag{1}$$

for any vector fields $U, W \in \Im_0^1(M_{2n})$. The triple (M_{2n}, φ, g) is called an almost Norden manifold in many studies [12, 15, 16]. Let ∇ be the Riemannian connection on M_{2n}. The tensor field F of type $(0,3)$ on M_{2n} is defined by

$$F(U,W,Z) = g((\nabla_U \varphi)W, Z). \tag{2}$$

It has the following symmetries

$$F(U,W,Z) = F(U,Z,W) = F(U,\varphi W,\varphi Z). \tag{3}$$

Then the quadruple $(M_{2n}, \varphi, g, \nabla)$ is called complex manifold with Norden metric, in another words it is called Norden manifold, if

$$F(U,W,\varphi Z) + F(W,Z,\varphi U) + F(Z,U,\varphi W) = 0 \tag{4}$$

or equivalently $N_\varphi = 0$, where

$$N_\varphi(U,W) = [\varphi U, \varphi W] - [U,W] - \varphi[\varphi U, W] - \varphi[U, \varphi W] \tag{5}$$

for any vector fields $U, W, Z \in \Im_0^1(M_{2n})$ [15, 17-19].

2.2. QUARTER-SYMMETRIC METRIC CONNECTION ON NORDEN MANIFOLD

A linear connection $\tilde{\nabla}$ on $2n$- dimensional Riemannian manifold (M_{2n}, g) is called a quarter-symmetric connection [2] if it is torsion tensor $\tilde{T}$ satisfies

$$\tilde{T}(U,W) = \omega(W)\varphi U - \omega(U)\varphi W \tag{6}$$

where ω is a differentiable 1-form and φ is a $(1,1)$ tensor fields on (M_{2n}, g). If, morever, the connection $\tilde{\nabla}$ satisfies

$$\left(\tilde{\nabla}_U \, g\right)(W, Z) = 0 \tag{7}$$

for all vector fields $U, W, Z \in \Im_0^1(M_{2n})$ on (M_{2n}, g) then it is called a quarter-symmetric metric connection.

Now we give a relation between the Levi-Civita connection ∇ and the quarter-symmetric metric connection ∇ on (M_{2n}, φ, g) Norden manifold. Let us write

$$\tilde{\nabla}_U \, W = \nabla_U \, W + H(U, W) \tag{8}$$

where $H(U,W)$ is a tensor af type $(1,2)$. From $\tilde{\nabla} g = 0$ we get

$$\begin{aligned}
Ug(W,Z) - g\left(\tilde{\nabla}_U \, W, Z\right) - g\left(W, \tilde{\nabla}_U \, \mathrm{Z}\right) = 0 \\
Ug(W,Z) - g\left(\nabla_U \, W + H(U,W), Z\right) - g\left(W, \nabla_U \, \mathrm{Z} + \mathrm{H}(U,Z)\right) = 0 \\
Ug(W,Z) - g\left(\nabla_U \, W + H(U,W), Z\right) - g\left(W, \nabla_U \, \mathrm{Z} + \mathrm{H}(U,Z)\right) = 0 \\
Ug(W,Z) - g(\nabla_U \, W, Z) - g(W, \nabla_U \, \mathrm{Z}) - g\left(H(U,W), Z\right) - g\left(W, H(U,Z)\right) = 0 \\
(\nabla_U \, g)(W,Z) - g\left(H(U,W), Z\right) - g\left(H(U,Z), W\right) = 0 \\
g\left(H(U,W), Z\right) = -g\left(H(U,Z), W\right) \\
g\left(H(U,W), Z\right) = -g\left(H(W,U), Z\right).
\end{aligned} \tag{9}$$

Hence we obtain the $H(U,W)$ that is an antisymmetric tensor of type $(1,2)$.

From (8) it follows that

$$\begin{aligned}
H(U,W) - H(W,U) &= \tilde{\nabla}_U W - \nabla_U W - \tilde{\nabla}_W U + \nabla_W U \\
&= \tilde{\nabla}_U W - \tilde{\nabla}_W U - [U, W] \\
&= \tilde{T}(U, W).
\end{aligned}$$

Using relation (6) we obtain

$$H(U,W) - H(W,U) = \omega(W)\varphi U - \omega(U)\varphi W \tag{10}$$

Using relation (10) and purity condition we obtain

i. $g\left(H(U,W), Z\right) - g\left(H(W,U), Z\right) = \omega(W) g(\varphi U, Z) - \omega(U) g(\varphi W, Z)$

ii. $g\left(H(U,Z), W\right) - g\left(H(Z,U), W\right) = \omega(Z) g(\varphi U, W) - \omega(U) g(\varphi Z, W)$

iii. $g\left(H\left(W,Z\right),U\right)-g\left(H\left(Z,W\right),U\right)=\omega\left(Z\right)g\left(\varphi W,U\right)-\omega\left(W\right)g\left(\varphi Z,U\right)$

Using the purity condition and the relation (9) we add (i) and (ii), then we subtract (iii) from the result. After considering the purity condition and the relation (9), we obtain

$$\begin{aligned} 2g\left(H\left(U,W\right),Z\right) &= 2\omega\left(W\right)g\left(\varphi Z,U\right)-2\omega\left(Z\right)g\left(\varphi W,U\right) \\ &= 2\omega\left(W\right)g\left(\varphi U,Z\right)-2\omega\left(Z\right)g\left(\varphi W,U\right) \\ &= 2g\left(\omega\left(W\right)\varphi U,Z\right)-2g\left(P,Z\right)g\left(W,\varphi U\right) \\ &= 2g\left(\omega\left(W\right)\varphi U,Z\right)-2g\left(g\left(W,\varphi U\right)P,Z\right) \\ &= 2g\left(\omega\left(W\right)\varphi U-g\left(W,\varphi U\right)P,Z\right) \end{aligned}$$

Then we get

$$H\left(U,W\right)=\omega\left(W\right)\varphi U-g\left(\varphi U,W\right)P$$

Hence a quarter-symmetric metric connection $\tilde{\nabla}$ on Norden Manifold $\left(M_{2n},\varphi,g\right)$ is given by

$$\tilde{\nabla}_U W=\nabla_U W+\omega\left(W\right)\varphi U-g\left(\varphi U,W\right)P. \tag{11}$$

2.3. TANGENT BUNDLE

Let M_{2n} be a differentiable manifold and $T_K\left(M_{2n}\right)$ the tangent space at any point $K\in M_{2n}$. Then the set $TM_{2n}=\bigcup_{K\in M_{2n}} T_K\left(M_{2n}\right)$ is called the total space of the tangent bundle of M_{2n}. The local coordinate system $\left(x^i,y^i\right)$ on TM_{2n} is induced from the base coordinate $\left(x^i\right)$ on M_{2n}. For any point $\overline{K}\in TM_{2n}$, the compatibility $\overline{K}\to K$ determines the bundle projection $\pi:TM_{2n}\to M_{2n}$, i.e., $\pi\left(\overline{K}\right)=K$. The set $\pi^{-1}\left(K\right)$ is called the fibre over $K\in M_{2n}$ and M_{2n} is called the base space.

The composition of two maps $\pi:TM_{2n}\to M_{2n}$ and $f:M_{2n}\to\mathbb{R}$ defined by ${}^V f=f\circ\pi$ is called the vertical lift of $f\in\Im_0^0\left(M_{2n}\right)$. For any point $\overline{K}$ according to induced coordinates $\left(x^i,y^i\right)$, the equation

$${}^V f\left(\overline{K}\right)={}^V f\left(x,y\right)=f\circ\pi\left(\overline{K}\right)=f\left(K\right)=f\left(x\right) \tag{12}$$

is obtained. The relation (12) show that the value of ${}^V f\left(\overline{K}\right)$ is constant along each fibre $T_K\left(M_{2n}\right)$ and equal to the value $f\left(K\right)$ of f.

For $f \in \Im_0^0(M_{2n})$, the complete lift of f is defined by

$$^{C}f = \iota(df) = y^i \partial_i f \tag{13}$$

with respect to the induced coordinates (x^i, y^i) on TM_{2n}.

Let $f, t \in \Im_0^0(M_{2n})$, $U \in \Im_0^1(M_{2n})$, $\omega \in \Im_1^0(M_{2n})$, $\varphi \in \Im_1^1(M_{2n})$, $g \in \Im_2^0(M_{2n})$, $T \in \Im_2^1(M_{2n})$ be a function, a vector field, a 1- form, type- $(1,1)$, type- $(0,2)$, type- $(1,2)$ tensor field, respectively. We denote, respectively, by ^{V}f, ^{V}U, $^{V}\omega$, $^{V}\varphi$, ^{V}g, ^{V}T their vertical lifts and by ^{C}f, ^{C}U, $^{C}\omega$, $^{C}\varphi$, ^{C}g, ^{C}T their complete lifts. This lifts have the properties:

$$\begin{aligned}
&\left[{}^{C}U, {}^{C}W\right] = {}^{C}[U,W]\ ; && {}^{C}\varphi\left({}^{C}U\right) = {}^{C}\left(\varphi(U)\right) \\
&{}^{V}\omega\left({}^{C}U\right) = {}^{V}\left(\omega(U)\right)\ ; && {}^{C}\omega\left({}^{C}U\right) = {}^{C}\left(\omega(U)\right) \\
&{}^{C}g\left({}^{V}U, {}^{C}W\right) = {}^{C}g\left({}^{C}U, {}^{V}W\right) = {}^{V}\left(g(U,W)\right)\ ; && {}^{C}g\left({}^{C}U, {}^{C}W\right) = {}^{C}\left(g(U,W)\right) \\
&{}^{C}\nabla_{{}^{C}U}{}^{C}W = {}^{C}\left(\nabla_U W\right); && {}^{C}\nabla_{{}^{C}U}{}^{V}W = {}^{V}\left(\nabla_U W\right) \\
&{}^{C}T\left({}^{C}U, {}^{C}W\right) = {}^{C}\left(T(U,W)\right); && {}^{C}f\ {}^{V}t + {}^{V}f\ {}^{C}t = {}^{C}\left(f\ t\right)
\end{aligned} \tag{14}$$

where $W \in \Im_0^1(M_{2n})$ [20].

3. COMPLETE LIFT OF QUARTER-SYMMETRİC METRIC CONNECTION ON A TANGENT BUNDLE OF NORDEN MANIFOLD

Let the triple (M_{2n}, φ, g) be an almost Norden manifold and the triple $(T\,M_{2n}, {}^{C}\varphi, {}^{C}g)$ be its tangent bundle where the complete lift of φ and g are $^{C}\varphi$ and ^{C}g, respectively. And the almost Norden manifold satisfies the following relations

$$\begin{aligned}
&\left({}^{C}\varphi\right)^2\ {}^{C}U + {}^{C}U = 0, \\
&{}^{C}g\left({}^{C}(\varphi U), {}^{C}W\right) = {}^{C}g\left({}^{C}U, {}^{C}(\varphi W)\right).
\end{aligned} \tag{15}$$

Let $^{C}\nabla$ be complete lift of the Riemannian connection on TM_{2n}. Then the quadruple $(TM_{2n}, {}^{C}\varphi, {}^{C}g, {}^{C}\nabla)$ is called Norden manifold endowed with complete lift of the Riemannian connection if

$${}^{C}N_{\varphi}\left({}^{C}U, {}^{C}W\right) = 0 \tag{16}$$

for all $U,W \in \Im_0^1(M_{2n})$, where we denoted by ${}^C N_\varphi$ the Nijenhuis tensor field of $\left(TM_{2n},\ {}^C\varphi,\ {}^C g,\ {}^C\nabla\right)$.

Theorem 1. Let $\tilde{\nabla}$ be a quarter-symmetric metric connection with respect to the Riemannian connection on almost Norden manifold $\left(M_{2n}\ ,\varphi,g\right)$. Then ${}^C\tilde{\nabla}$ is also a quarter-symmetric metric connection with respect to the complete lift of the Riemannian ${}^C\nabla$ on tangent bundle of almost Norden manifold $\left(TM_{2n}\ ,{}^C\varphi,{}^C g\right)$.

Proof: Applying the complete lift to both sides of the relation (11), we get

$$ {}^C\left(\tilde{\nabla}_U\ W\right)= {}^C(\nabla_U\ \mathrm{W})+ {}^C\left(\omega(W)\varphi U\right)- {}^C\left(g\left(\varphi U,W\right)P\right) $$

$$ \begin{aligned} {}^C\tilde{\nabla}_{{}^C U}\ {}^C W &= {}^C\nabla_{{}^C U}\ {}^C W+\left({}^C\omega\ {}^C W\right)\left({}^C\varphi\ {}^V U\right)+\left({}^C\omega\ {}^V W\right)\left({}^C\varphi\ {}^C U\right) \\ &\quad -\left({}^C g\left({}^C\varphi\ {}^C U,\ {}^C W\right)\right){}^V P-\left({}^C g\left({}^V\varphi\ {}^C U,\ {}^C W\right)\right){}^C P \end{aligned} \tag{17} $$

$$ \begin{aligned} {}^C\nabla_{{}^C W}\ {}^C U &= {}^C\nabla_{{}^C W}\ {}^C U+\left({}^C\omega\ {}^C U\right)\left({}^C\varphi\ {}^V W\right)+\left({}^C\omega\ {}^V U\right)\left({}^C\varphi\ {}^C W\right) \\ &\quad -\left({}^C g\left({}^C\varphi\ {}^C W,\ {}^C U\right)\right){}^V P-\left({}^C g\left({}^V\varphi\ {}^C W,\ {}^C U\right)\right){}^C P. \end{aligned} \tag{18} $$

Using the relations (17) and (18) we obtain the complete lift of torsion tensor of the quarter-symmetric metric connection

$$ {}^C\tilde{T}\left({}^C\mathrm{U},\ {}^C\mathrm{W}\right)={}^C\tilde{\nabla}_{{}^C U}\ {}^C W-{}^C\tilde{\nabla}_{{}^C W}\ {}^C U-\left[{}^C U,{}^C W\right] $$

$$ \begin{aligned} &= {}^C\nabla_{{}^C U}\ {}^C\mathrm{W}-{}^C\nabla_{{}^C W}\ {}^C U-\left[{}^C U,\ {}^C W\right] \\ &\quad +\left({}^C\omega\ {}^C W\right)\left({}^C\varphi\ {}^V U\right)+\left({}^C\omega\ {}^V W\right)\left({}^C\varphi\ {}^C U\right) \\ &\quad -\left({}^C\omega\ {}^C U\right)\left({}^C\varphi\ {}^V W\right)-\left({}^C\omega\ {}^V U\right)\left({}^C\varphi\ {}^C W\right) \\ &\quad -\left({}^C g\left({}^C U,\ {}^C\varphi\ {}^C W\right)\right){}^V P+\left({}^C g\left({}^C\varphi\ {}^C W,\ {}^C U\right)\right){}^V P \\ &\quad -\left({}^C g\left({}^V\varphi\ {}^C U,\ {}^C W\right)\right){}^C P+\left({}^C g\left({}^C W,\ {}^V\varphi\ {}^C U\right)\right){}^C P \end{aligned} \tag{19} $$

$$ = {}^C\left(\left(\omega W\right)\left(\varphi U\right)\right)-{}^C\left(\left(\omega U\right)\left(\varphi W\right)\right). $$

Then we obtain

$$
\begin{aligned}
&{}^{C}g\left({}^{C}\tilde{\nabla}_{{}^{C}U}\,{}^{C}W,\ {}^{C}Z\right)+{}^{C}g\left({}^{C}W,\ {}^{C}\overline{\nabla}_{{}^{C}U}\,{}^{C}Z\right)\\
&= {}^{C}g({}^{C}\nabla_{{}^{C}U}\,{}^{C}W+\left({}^{C}\omega\,{}^{C}W\right)\left({}^{C}\varphi\,{}^{V}U\right)+\left({}^{C}\omega\,{}^{V}W\right)\left({}^{C}\varphi\,{}^{C}U\right)\\
&\quad-\left({}^{C}g\left({}^{C}\varphi\,{}^{C}U,\ {}^{C}W\right)\right){}^{V}P-\left({}^{C}g\left({}^{V}\varphi\,{}^{C}U,\ {}^{C}W\right)\right){}^{C}P,\ {}^{C}Z)\\
&\quad+{}^{C}g({}^{C}\mathrm{W},\ {}^{C}\nabla_{{}^{C}U}\,{}^{C}Z+\left({}^{C}\omega\,{}^{C}Z\right)\left({}^{C}\varphi\,{}^{V}U\right)+\left({}^{C}\omega\,{}^{V}Z\right)\left({}^{C}\varphi\,{}^{C}U\right)\\
&\quad-\left({}^{C}g\left({}^{C}\varphi\,{}^{C}U,\ {}^{C}Z\right)\right){}^{V}P-\left({}^{C}g\left({}^{V}\varphi\,{}^{C}U,\ {}^{C}Z\right)\right){}^{C}P)
\end{aligned}
$$

$$
\begin{aligned}
&= {}^{C}g\left({}^{C}\nabla_{{}^{C}U}\,{}^{C}W,\ {}^{C}Z\right)+{}^{C}g\left({}^{C}\mathrm{W},\ {}^{C}\nabla_{{}^{C}U}\,{}^{C}Z\right)+\left({}^{C}\omega\,{}^{C}W\right)\left({}^{C}g\left({}^{V}\varphi\,{}^{C}U,\ {}^{C}Z\right)\right)\\
&\quad+\left({}^{C}\omega\,{}^{V}W\right)\left({}^{C}g\left({}^{C}\varphi\,{}^{C}U,\ {}^{C}Z\right)\right)-{}^{C}g\left({}^{C}\varphi\,{}^{C}U,\ {}^{C}W\right){}^{V}\left(\omega(Z)\right)\\
&\quad-{}^{C}g\left({}^{V}\varphi\,{}^{C}U,\ {}^{C}W\right){}^{C}\left(\omega(Z)\right)+\left({}^{C}\omega\,{}^{C}Z\right){}^{C}g\left({}^{C}\mathrm{W},{}^{V}\varphi\,{}^{C}U\right)\\
&\quad+\left({}^{C}\omega\,{}^{V}Z\right){}^{C}g\left({}^{C}\mathrm{W},{}^{C}\varphi\,{}^{C}U\right)-{}^{C}g\left({}^{C}\varphi\,{}^{C}U,\ {}^{C}Z\right){}^{V}\left(\omega(W)\right)\\
&\quad-{}^{C}g\left({}^{V}\varphi\,{}^{C}U,\ {}^{C}Z\right){}^{C}\left(\omega(W)\right)\\
&= {}^{C}g\left({}^{C}\nabla_{{}^{C}U}\,{}^{C}W,\ {}^{C}Z\right)+{}^{C}g\left({}^{C}\mathrm{W},\ {}^{C}\nabla_{{}^{C}U}\,{}^{C}Z\right).
\end{aligned}
\tag{20}
$$

and we write following equation

$$
\begin{aligned}
\left({}^{C}U\right){}^{C}g\left({}^{C}W,\ {}^{C}Z\right)&=\left({}^{C}\tilde{\nabla}_{{}^{C}U}\,{}^{C}g\right)\left({}^{C}W,\ {}^{C}Z\right)\\
&\quad+{}^{C}g\left({}^{C}\tilde{\nabla}_{{}^{C}U}\,{}^{C}W,\ {}^{C}Z\right)+{}^{C}g\left({}^{C}W,\ {}^{C}\tilde{\nabla}_{{}^{C}U}\,{}^{C}Z\right).
\end{aligned}
\tag{21}
$$

Writing the relation (20) into relation (21), we get

$$
\begin{aligned}
\left({}^{C}\tilde{\nabla}_{{}^{C}U}\,{}^{C}g\right)\left({}^{C}W,\ {}^{C}Z\right)&={}^{C}\left(Ug(W,Z)\right)-{}^{C}\left(g(\nabla_{U}W,Z)\right)-{}^{C}\left(g(W,\nabla_{U}Z)\right)\\
&={}^{C}\left((\nabla_{U}g)(W,Z)\right)\\
&=0.
\end{aligned}
\tag{22}
$$

From the relations (19) and (22) it follows that ${}^{C}\tilde{\nabla}$ is a quarter-symmetric metric connection with respect to ${}^{C}\nabla$ on $\left(TM_{2n},{}^{C}\varphi,{}^{C}g,{}^{C}\nabla\right)$.

Now we show that the relation (3) is valid on almost Norden Manifold $\left(M_{2n},\varphi,g\right)$ endowed with the quarter-symmetric metric connection ∇.

Theorem 2. Let g be a Norden metric on the almost Norden manifold endowed with the quarter-symmetric metric connection $\tilde{\nabla}$. Then

$$F(U,W,Z)=F(U,Z,W)$$

$$F(U,\varphi W,\varphi Z)=F(U,W,Z)$$

where $F(U,W,Z)=g\left(\left(\tilde{\nabla}_U\ \varphi\right)W,Z\right)$.

Proof: We will show that the relation (3) defined Riemannian connection ∇ is provided for the quarter-symmetric metric connection $\tilde{\nabla}$ on almost Norden manifold $\left(M_{2n},\varphi,g\right)$. Using the relation (1) and

$$Ug(W,Z)=\left(\tilde{\nabla}_U\ g\right)(W,Z)+g\left(\tilde{\nabla}_U\ W,Z\right)+g\left(W,\tilde{\nabla}_U\ Z\right) \tag{23}$$

from $Ug(\varphi W,Z)=Ug(W,\varphi Z)$, we obtain

$$\begin{aligned}
\left(\tilde{\nabla}_U\ g\right)(\varphi W,Z)+g\left(\tilde{\nabla}_U\ \varphi W,Z\right)&=\left(\tilde{\nabla}_U\ g\right)(W,\varphi Z)+g\left(\tilde{\nabla}_U W,\varphi Z\right)\\
+g\left(\varphi W,\ \tilde{\nabla}_U\ Z\right)&\qquad +g\left(W,\tilde{\nabla}_U\ \varphi Z\right)
\end{aligned}$$

$$\begin{aligned}
g\left(\tilde{\nabla}_U\ \varphi W,Z\right)+g\left(\varphi W,\tilde{\nabla}_U\ Z\right)&=g\left(\tilde{\nabla}_U W,\varphi Z\right)+g\left(W,\tilde{\nabla}_U\ \varphi Z\right)\\
g\left(\varphi W,\tilde{\nabla}_U\ Z\right)-g\left(W,\tilde{\nabla}_U\ \varphi Z\right)&=g\left(\tilde{\nabla}_U W,\varphi Z\right)-g\left(\tilde{\nabla}_U\ \varphi W,Z\right)\\
g\left(W,\varphi\left(\tilde{\nabla}_U\ Z\right)\right)-g\left(W,\tilde{\nabla}_U\ \varphi Z\right)&=g\left(\varphi\left(\tilde{\nabla}_U W\right),Z\right)-g\left(\tilde{\nabla}_U\ \varphi W,Z\right)\\
g\left(W,\varphi\left(\tilde{\nabla}_U\ Z\right)-\tilde{\nabla}_U\ \varphi Z\right)&=g\left(\varphi\left(\tilde{\nabla}_U W\right)-\tilde{\nabla}_U\ \varphi W,Z\right)\\
g\left(W,\left(\tilde{\nabla}_U\ \varphi\right)Z\right)&=g\left(\left(\tilde{\nabla}_U\ \varphi\right)W,Z\right)
\end{aligned} \tag{24}$$

The relation (24) shows that $F(U,W,Z)=F(U,Z,W)$ where $F(U,W,Z)=g\left(\left(\tilde{\nabla}_U\ \varphi\right)W,Z\right)$. After we obtain

$$\begin{aligned}
F(U,\varphi W,\varphi Z)&=g\left(\left(\nabla_U\varphi\right)\varphi W,\varphi Z\right)\\
&=g\left(\nabla_U\varphi^2 W,\varphi Z\right)-g\left(\varphi\left(\nabla_U\varphi W\right),\varphi Z\right)\\
&=-g\left(\nabla_U W,\varphi Z\right)+g\left(\nabla_U\varphi W,Z\right)\\
&=-g\left(\varphi\left(\nabla_U W\right),Z\right)+g\left(\nabla_U\varphi W,Z\right)\\
&=g\left(\left(\nabla_U\varphi\right)W,Z\right)\\
&=F(U,W,Z).
\end{aligned} \tag{25}$$

The relation (25) shows that $F(U,\varphi W,\varphi Z)=F(U,W,Z)$ where $F(U,W,Z)=g\left(\left(\tilde{\nabla}_U\ \varphi\right)W,Z\right)$.

Theorem 3. Let $\left(M_{2n}\ ,\varphi,g,\tilde{\nabla}\right)$ be an almost Norden manifold endowed with quarter-symmetric metric connection. The Nijenhuis tensor N_φ with respect to quarter-symmetric metric connection $\tilde{\nabla}$ coincides with the Nijenhuis tensor N_φ with respect to the Riemannian connection ∇.

Proof: The Nijenhuis tensor $N_\varphi(U,W)$ on an almost Norden manifold with respect to the Riemannian connection is given in relation (5). Another form of Nijenhuis tensor with respect to Riemannian connection is written by

$$\begin{aligned} N_\varphi(U,W) &= [\varphi U,\varphi W]-[U,W]-\varphi[\varphi U,W]-\varphi[U,\varphi W] \\ &= \nabla_{\varphi U}\ \varphi W-\nabla_{\varphi W}\ \varphi U-\nabla_U\ W+\nabla_W\ U-\varphi\nabla_{\varphi U}\ W \\ &\quad +\varphi\nabla_W\ \varphi U-\varphi\nabla_U\ \varphi W+\varphi\nabla_{\varphi W}\ U. \end{aligned} \tag{26}$$

In the same way, we define that the Nijenhuis tensor $N_\varphi(U,W)$ on an almost Norden manifold with respect to the quarter-symmetric metric connection ∇ is written by

$$\begin{aligned} N_\varphi(U,W) &= \nabla_{\varphi U}\ \varphi W-\nabla_{\varphi W}\ \varphi U-\nabla_U\ W+\nabla_W\ U \\ &\quad -\varphi\nabla_{\varphi U}\ W+\varphi\nabla_W\ \varphi U-\varphi\nabla_U\ \varphi W+\varphi\nabla_{\varphi W}\ U. \end{aligned} \tag{27}$$

By using the relation (11), we obtain

$$\begin{aligned} N_\varphi(U,W) &= \nabla_{\varphi U}\ \varphi W-\nabla_{\varphi W}\ \varphi U-\nabla_U\ W+\nabla_W\ U \\ &\quad -\varphi\nabla_{\varphi U}\ W+\varphi\nabla_W\ \varphi U-\varphi\nabla_U\ \varphi W+\varphi\nabla_{\varphi W}\ U \\ &\quad +\omega(\varphi W)\varphi(\varphi U)-g\left(\varphi(\varphi U),\varphi W\right)P \\ &\quad -\omega(\varphi U)\varphi(\varphi W)+g\left(\varphi(\varphi W),\varphi U\right)P \\ &\quad -\omega(W)\varphi U+g\left(\varphi U,W\right)P+\omega(U)\varphi W-g\left(\varphi W,U\right)P \\ &\quad -\varphi\omega(W)\varphi(\varphi U)+\varphi g\left(\varphi(\varphi U),W\right)P \\ &\quad +\varphi\omega(\varphi U)\varphi W-\varphi g\left(\varphi W,\varphi U\right)P \\ &\quad -\varphi\omega(\varphi W)\varphi U+\varphi g\left(\varphi U,\varphi W\right)P \\ &\quad +\varphi\omega(U)\varphi(\varphi W)-\varphi g\left(\varphi(\varphi W),U\right)P \\ &= \nabla_{\varphi U}\ \varphi W-\nabla_{\varphi W}\ \varphi U-\nabla_U\ W+\nabla_W\ U \\ &\quad -\varphi\nabla_{\varphi U}\ W+\varphi\nabla_W\ \varphi U-\varphi\nabla_U\ \varphi W+\varphi\nabla_{\varphi W}\ U \\ &= N_\varphi(U,W). \end{aligned} \tag{28}$$

We see that the Nijenhuis tensor N_φ coincides with the Nijenhuis tensor N_φ.

We know that $(M_{2n},\varphi,g,\nabla)$ is a Norden manifold if the Nijenhuis tensor N_φ vanishes. The Nijenhuis tensor N_φ being zero indicates that φ is integrable. In other words, the almost Norden manifold $(M_{2n},\varphi,g,\nabla)$ is a Norden manifold if φ is integrable. Hence, we have:

Theorem 4. An almost Norden manifold endowed with the quarter-symmetric metric connection ∇ is a Norden manifold if and only if it is a Norden manifold with respect to to the Riemannian connection ∇.

The Nijenhuis tensor ${}^{C}N$ with respect to the Riemannian connection ${}^{C}\nabla$ on $(TM_{2n},{}^{C}\varphi,{}^{C}g,{}^{C}\nabla)$ is given by

$$\begin{aligned}{}^{C}N_\varphi({}^{C}U,{}^{C}W)={}&{}^{C}\nabla_{{}^{C}(\varphi U)}{}^{C}(\varphi W)-{}^{C}\nabla_{{}^{C}(\varphi W)}{}^{C}(\varphi U)-{}^{C}\nabla_{{}^{C}U}{}^{C}W+{}^{C}\nabla_{{}^{C}W}{}^{C}U\\&-{}^{C}\varphi\,{}^{C}\nabla_{{}^{C}(\varphi U)}{}^{C}W+{}^{C}\varphi\,{}^{C}\nabla_{{}^{C}W}{}^{C}(\varphi U)-{}^{C}\varphi\,{}^{C}\nabla_{{}^{C}U}{}^{C}(\varphi W)\\&+{}^{C}\varphi\,{}^{C}\nabla_{{}^{C}W}{}^{C}(\varphi U)\;.\end{aligned}\tag{29}$$

Next, we are going to give the relationship between Nijenhuis tensor ${}^{C}N_\varphi$ with respect to the quarter-symmetric metric connection ${}^{C}\nabla$ and Nijenhuis tensor ${}^{C}N$ with respect to the Riemannian connection ${}^{C}\nabla$.

Theorem 5. $(TM_{2n},{}^{C}\varphi,{}^{C}g,{}^{C}\nabla)$ be the tangent bundle of an almost Norden manifold with the quarter-symmetric metric connection. The Nijenhuis tensor ${}^{C}N_\varphi$ with respect to the quarter-symmetric metric connection ${}^{C}\nabla$ coincides with the Nijenhuis tensor ${}^{C}N$ with respect to the Riemannian connection ${}^{C}\nabla$.

Proof: The Nijenhuis tensor ${}^{C}N_\varphi$ on tangent bundle of an almost Norden manifold with the respect to the quarter-symmetric metric connection ${}^{C}\nabla$ is given by

$$\begin{aligned}{}^{C}N_\varphi({}^{C}U,{}^{C}W)={}&{}^{C}\nabla_{{}^{C}(\varphi U)}{}^{C}(\varphi W)-{}^{C}\nabla_{{}^{C}(\varphi W)}{}^{C}(\varphi U)-{}^{C}\nabla_{{}^{C}U}{}^{C}W+{}^{C}\nabla_{{}^{C}W}{}^{C}U\\&-{}^{C}\varphi\,{}^{C}\nabla_{{}^{C}(\varphi U)}{}^{C}W+{}^{C}\varphi\,{}^{C}\nabla_{{}^{C}W}{}^{C}(\varphi U)-{}^{C}\varphi\,{}^{C}\nabla_{{}^{C}U}{}^{C}(\varphi W)\\&+{}^{C}\varphi\,{}^{C}\nabla_{{}^{C}W}{}^{C}(\varphi U).\end{aligned}\tag{30}$$

By using the relations (11) and (14), we have

$${}^{C}N_\varphi({}^{C}U,{}^{C}W)={}^{C}N_\varphi({}^{C}U,{}^{C}W).\tag{31}$$

The almost Norden manifold $\left(TM_{2n}, {}^{C}\varphi, {}^{C}g, {}^{C}\nabla\right)$ is a Norden manifold if ${}^{C}\varphi$ is integrable. Hence, we have

Corollary 6. The tangent bundle of an almost Norden manifold $\left(M_{2n},\varphi,g,\nabla\right)$, where ∇ is a quarter-symmetric connection, is a Norden manifold with respect to ${}^{C}\nabla$ if and only if it is a Norden manifold with respect to the Riemannian connection ${}^{C}\nabla$.

G defined by $G(U,W)=g(\varphi U,W)$ for any $U,W\in\Im_0^1(M_{2n})$ is called the twin Norden metric on a Norden manifold $(M_{2n},\varphi,g,\nabla)$ [21]. The complete lift of twin Norden metric G is denoted by

$$
{}^{C}G\left({}^{C}U, {}^{C}W\right)={}^{C}g\left({}^{C}(\varphi U), {}^{C}W\right). \tag{32}
$$

Proposition 7. Let G be a twin Norden metric on a Norden manifold $(M_{2n},\varphi,g,\nabla)$. Then

$$
\left({}^{C}\nabla_{{}^{C}Z}\,{}^{C}G\right)\left({}^{C}U, {}^{C}W\right)=0
$$

for any $U,W,Z\in\Im_0^1(M_{2n})$.

Proof: For twin Norden metric G, we have

$$
\begin{aligned}
\left({}^{C}\nabla_{{}^{C}Z}\,{}^{C}G\right)\left({}^{C}U, {}^{C}W\right) &= \left({}^{C}Z\right){}^{C}G\left({}^{C}U, {}^{C}W\right)-{}^{C}G\left({}^{C}\nabla_{{}^{C}Z}\,{}^{C}U, {}^{C}W\right)\\
&\quad -{}^{C}G\left({}^{C}U, {}^{C}\nabla_{{}^{C}Z}\,{}^{C}W\right)\\
&=\left({}^{C}Z\right){}^{C}G\left({}^{C}U, {}^{C}W\right)-{}^{C}G\left({}^{C}\nabla_{{}^{C}Z}\,{}^{C}U, {}^{C}W\right)\\
&\quad -{}^{C}G\left({}^{C}U, {}^{C}\nabla_{{}^{C}Z}\,{}^{C}W\right)\\
&=\left({}^{C}\nabla_{{}^{C}Z}\,{}^{C}G\right)\left({}^{C}U, {}^{C}W\right)\\
&=\left({}^{C}\nabla_{{}^{C}Z}\,{}^{C}g\right)\left({}^{C}(\varphi U), {}^{C}W\right)\\
&=0.
\end{aligned} \tag{33}
$$

After similar computations to those from the proof of Theorem 1, the relation in the statement is proved.

4. CONCLUSION

In this paper, we obtain the complete lift of quarter-symmetric metric connection on tangent bundle of almost Norden manifold. We show that the Nijenhuis tensor with respect to the quarter-symmetric metric connection coincides with the Nijenhuis tensor with respect to the Riemannian connection. Under the condition of integrability of structure we show that the almost Norden manifold endowed with the quarter-symmetric metric connection becomes Norden manifold.

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