ORIGINAL PAPER

(V, λ)-ORDER SUMMABLE IN RIESZ SPACES

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Manuscript received: 03.02.2021; Accepted paper: 11.09.2021
Published online: 30.09.2021.

Abstract. *Statistical convergence is an active area, and it appears in a wide variety of topics. However, it has not been studied extensively in Riesz spaces. There are a few studies about the statistical convergence on Riesz spaces, but they only focus on the relationship between statistical and order convergences of sequences in Riesz spaces. In this paper, we introduce the notion of (V,λ)-order summable by using the concept of λ-statistical monotone and the λ-statistical order convergent sequences in Riesz spaces. Moreover, we give some relations between (V,λ)-order summable and λ-statistical order convergence.*
Keywords: *(V,λ)-order summable; λ-statistical monotone; λ-statistical order convergence; statistical convergence; Riesz space; order convergence.*

1. INTRODUCTION AND PRELIMINARIES

Statistical convergence is a generalization of the ordinary convergence of a real sequence. The idea of statistical convergence was firstly introduced by Zygmund [1] in the first edition of his monograph in 1935. Fast [2] and Steinhaus [3] independently improved this idea in the same year 1951. Several generalizations and applications of this concept have been investigated by several authors in series of papers [3-9]. But, statistical convergence on Riesz spaces has not been studied extensively. Only a few studies have been conducted on this recently; see for example [10-13]. They give relations between the order convergence and the statistical convergence on Riesz spaces. Therefore, we aim to introduce a concept of the statistical convergence on Riesz spaces by using the λ-density which is a useful and classical tool of statistical convergence.

The natural density of subsets of $\mathbb{N}$ plays a critical role in statistical convergence. Take a subset A of natural numbers $\mathbb{N}$. If the limit $\lim_{n\to+\infty}\frac{1}{n}|\{k\leq n: k\in A\}|$ exists then this unique limit is called *the natural density of* A, abbreviated by $\delta(A)$, where $|\{k\leq n: k\in A\}|$ is the number of members of A. Also, a sequence (x_k) *statistically converges* to L provide that

$$\lim_{n\to+\infty}\frac{1}{n}\ |\{k\leq n: |x_k-L|\geq\varepsilon\}|=0$$

for each $\varepsilon>0$. Then it is written by $S-\lim x_k=L$. If $L=0$ then (x_k) is a statistically null sequence. Throughout this paper, the vertical bar of sets will stand for *the cardinality* of the given sets.

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Let's consider a non-decreasing sequence (λ_n) of positive scalars such that $\lambda_1 = 1$ and $\lambda_{n+1} \leq \lambda_n + 1$. By using this sequence, we can obtain a new sequence of intervals $I_n := [n - \lambda_n + 1, n]$. A sequence (x_n) is said to be *λ-statistically convergent* to L if, for every $\varepsilon > 0$, we have

$$\lim_{n\to+\infty} \frac{1}{\lambda_n} |\{k \in I_n : |x_k - L| \geq \varepsilon\}| = 0.$$

Thus, we write $S_\lambda - \lim x_n = L$. Moreover, the *λ-densitiy* of a subset M of $\mathbb{N}$ is abbreviated by $\delta_\lambda(M) = \lim_{n\to+\infty} \frac{1}{\lambda_n} |\{k \in I_n : k \in M\}|$ [14].

Now, we turn our attention to Riesz space that is another concept of functional analysis, and it was introduced by F. Riesz in [15]. Riesz space is an ordered vector space that has many applications; see for example [15-24]. A real-valued vector space E with an order relation "$\leq$" is called *ordered vector space* if, for every $x, y \in E$, we have

1. $x \leq y$ implies $x + z \leq y + z$ for all $z \in E$,
2. $x \leq y$ implies $\lambda x \leq \lambda y$ for every $0 \leq \lambda \in \mathbb{R}$.

An ordered vector space E is called *Riesz space* or *vector lattice* if, for any two vectors $x, y \in E$, the infimum and the supremum

$$x \wedge y = \inf\{x, y\} \quad \text{and} \quad x \vee y = \sup\{x, y\}$$

exist in E, respectively. For an element x in a Riesz space E, *the positive part*, *the negative part*, and *module* of x are respectively

$$x^+ := x \vee 0, \quad x^- := (-x) \vee 0 \quad \textit{and} \quad |x| := x \vee (-x).$$

In the present paper, the vertical bar $|\cdot|$ of elements in Riesz spaces will stand for the module of given elements. It is clear that the positive and negative parts of vectors are positive. The order convergence is crucial for the concept of Riesz spaces. Thus, we continue with its definition.

Definition 1.1. A sequence (x_n) in a Riesz space E is called *order convergent* to $x \in E$ whenever there exists another sequence $(y_n) \downarrow 0$, i.e., $\inf y_n = 0$ and $y_n \downarrow$, such that $|x_n - x| \leq y_n$ holds for all $n \in \mathbb{N}$.

The statistical convergence in Riesz spaces was introduced and studied by using the notion of statistical monotonicity which was introduced in [25] for real sequences. We take the following notion from [13].

Definition 1.2. A sequence (x_n) in a Riesz space E is called *statistically monotone decreasing* if there exists a set $K = \{n_1 < n_2 < \cdots\}$ in $\mathbb{N}$ such that $\delta(K) = 1$ and (x_{n_k}) is decreasing. In this case, we write $x_n \downarrow^{st}$. Moreover, if we have $\inf(x_{n_k}) = x$ for some $x \in E$ then (x_n) is said to be *statistically monotone convergent* to x, and abbreviated as $x_n \downarrow^{st} x$.

2. λ-STATISTICAL CONVERGENCE

In this section, we give the notion of λ-statistical monotone and λ-statistical convergencent sequence in Riesz spaces with the order convergence, taken from [26].

Definition 2.1. Let E be a Riesz space and (x_n) be a sequence in E. Then

a) (x_n) is said to be *λ-statistically decreasing* if there exists a subset $M = \{n_1 < n_2, \cdots\}$ of the natural numbers $\mathbb{N}$ with $\delta_\lambda(M) = 1$ such that the sequence (x_{n_m}) is monotone decreasing, i.e., (x_n) is monotone decreasing on M. In this case, we write $(x_n) \downarrow^{\lambda_{st}}$.

b) (x_n) is called *λ-statistical order convergent* to $x \in E$ if there exist another sequence $(y_n) \downarrow^{\lambda_{st}} 0$ in E and a subset M of $\mathbb{N}$ with $\delta_\lambda(M) = 1$ such that $|x_{n_m} - x| \le y_{n_m}$ holds for each $n_m \in M$. We abbreviate it as $x_n \xrightarrow{\lambda_{st}o} x$.

One can observe that if there exists another sequence $(y_n) \downarrow^{\lambda_{st}} 0$ such that the δ_λ-density of the set $\{n \in \mathbb{N}: |x_n - x| \nleq y_n\}$ is equal to zero then $x_n \xrightarrow{\lambda_{st}o} x$.

Remark 2.2.

i. Every monotone sequence is λ-statistical monotone in Riesz spaces.

ii. Every order convergent decreasing sequence λ_{st}-converges to its order limit in Riesz spaces.

iii. The λ-statistically monotone convergence implies the λ-statistical order convergence.

In general, λ-statistical monotone sequence does not need to be a monotone sequence. To see this, we consider the following example.

Example 2.3. Consider the Riesz space $E := \mathbb{N}$. Suppose (x_k) is a sequence in E denoted by

$$x_k := \begin{cases} 1, & n - [\sqrt{\lambda_n}] + 1 \le k \le n \\ k, & \text{otherwise} \end{cases}.$$

So, it can be seen that (x_k) is a λ-statistical increasing sequence, but it is not monotone increasing.

Proposition 2.4. If (x_n) is a λ-statistically increasing sequence in a Riesz space then the λ-density of $\{n \in \mathbb{N}: x_n \nleq x_{n+1}\}$ is equal to zero.

Proof: Suppose that (x_n) is a λ-statistically increasing sequence in a Riesz space E. Hence, there is a subset $M = \{n_1 < n_2, \cdots\}$ of $\mathbb{N}$ such that $\delta_\lambda(M) = 1$, and (x_n) is monotone increasing on M, i.e., $x_n \le x_n + 1$ for all $n \in M$. Thus, we can obtain

$$\{n \in \mathbb{N}: x_n \nleq x_{n+1}\} \subseteq \mathbb{N} - M.$$

Therefore, one can get $\delta_\lambda(\{n \in \mathbb{N}: x_n \nleq x_{n+1}\}) = 0$ because of $\delta_\lambda(\mathbb{N} - M) = 0$.
In the following work, we give some basic observations.

Theorem 2.5. Let (x_n) and (y_n) be nets in a Riesz space E. Then

i. $(x_n) \downarrow^{\lambda_{st}} x$ and $(y_n) \downarrow^{\lambda_{st}} y$ implies $(x_n \vee y_n) \downarrow^{\lambda_{st}} x \vee y$;

ii. $x_n \downarrow^{\lambda_{st}} x$ if and only if $(x_n - x) \downarrow^{\lambda_{st}} 0$;

iii. $(x_n) \downarrow^{\lambda_{st}} x$ implies $(\alpha x_n) \downarrow^{\lambda_{st}} \alpha x$ for every $\alpha \in \mathbb{R}$;

iv. $(x_n) \downarrow^{\lambda_{st}} x$ and $(y_n) \downarrow^{\lambda_{st}} y$ implies $(x_n + y_n) \downarrow^{\lambda_{st}} (x + y)$;

v. $(x_{n_k}) \downarrow^{\lambda_{st}} x$ is hold for any subsequence (x_{n_k}) of $(x_n) \downarrow^{\lambda_{st}} x$ whenever (x_{n_k}) is decreasing and $\delta_\lambda(\{n_1, n_2, n_3, \cdots\}) = 1$;

vi. $(x_n) \downarrow^{\lambda_{st}} x$ and $(x_n) \downarrow^{\lambda_{st}} y$ implies $x = y$;

vii. $0 \le (x_n) \downarrow^{\lambda_{st}} x$ implies $x \in E_+$;

viii. if $0 \le y_n \le x_n$ for all $n \in \mathbb{N}$, $(x_n) \downarrow^{\lambda_{st}} 0$, and (y_n) is decreasing then we have $(y_n) \downarrow^{\lambda_{st}} 0$;

ix. if $(x_n) \downarrow^{\lambda_{st}} x$, $(y_n) \downarrow^{\lambda_{st}} y$, and $x_n \ge y_n$ for all $n \in \mathbb{N}$ then $x \ge y$.

Proof: We show only some main results, the other cases are analogous

(i) Since $(x_n) \downarrow^{\lambda_{st}} x$ and $(y_n) \downarrow^{\lambda_{st}} y$, there exist subsets M_1 and M_2 of $\mathbb{N}$ such that $\delta_\lambda(M_1) = \delta_\lambda(M_2) = 1$, and exist sequences (x_n) and (y_n) such that they are monotone decreasing to x and y on M_1 and M_2, respectively. Let's consider the set $M = M_1 \cap M_2$. Then, following from the inequality $\delta_\lambda(M_1) + \delta_\lambda(M_2) \le 1 + \delta_\lambda(M_1 \cap M_2)$, we have $\delta_\lambda(M_1 \cap M_2) = 1$. On the other hand, $(x_n \vee y_n)$ is monotone decreasing on M because both (x_n) and (y_n) are monotone decreasing on M. Now, by applying [21, Thm.12.4], we can obtain

$$|x_n \vee y_n - x \vee y| \le |x_n \vee y_n - y_n \vee x| + |x \vee y_n - x \vee y|$$

$$\le |x_n - x| + |y_n - y|.$$

Thus, we have $\inf(x_n \vee y_n) = x \vee y$ on M because of $\inf(x_n - x) = 0$ and $\inf(y_n - y) = 0$ on M_1 and M_2, respectively, and so, both also hold on M. Hence, we get the desired, $(x_n \vee y_n) \downarrow^{\lambda_{st}} x \vee y$, result.

(v) Assume $(x_n) \downarrow^{\lambda_{st}} x$ in E. Then there exists a subset M of $\mathbb{N}$ with $\delta_\lambda(M) = 1$ such that the sequence (x_{n_m}) is monotone decreasing to x. Thus, by applying Proposition 2, it is clear $(x_{n_m}) \downarrow^{\lambda_{st}} x$. However, we should show the argument for arbitrary subsequences. By the way, let's consider a decreasing subsequence (x_{n_k}) of (x_n) such that $\delta_\lambda(K) = 1$ for $K = \{n_1, n_2, n_3, \cdots\}$. Assume $K \ne M$. Otherwise, the proof is obvious. Also, if K does not exist then there is nothing to prove. Now, we prove $(x_{n_k}) \downarrow^{\lambda_{st}} x$. Since (x_{n_m}) is monotone decreasing to x, we have $(x_{n_m}) \geq x$ for all $m \in M$. Also, we can see that M and L are almost equal because the λ-density of the set $J = M \cap K$ is equal to one. Hence, we can find a subsequence $(x_{n_{k_j}})$ of (x_{n_k}) such that x is lower bound of it, and also, it is clear that $(x_{n_{k_j}})$ is monotone decreasing and the λ-density of its index set is equal to one. Take another lower bound $w \in E$ of $(x_{n_{k_j}})$, i.e. $x_{n_{k_j}} \geq w$ for all $j \in \mathbb{N}$. Fix an index j. Then, since M and L are almost equal, one can find an index $m_j \in M$ so that $x_{m_j} = x_{n_{k_j}} \geq w$. Thus, we get $f \geq w$ because x is the infimum of (x_{n_m}). As a result, we see that x is the infimum of $(x_{n_{k_j}})$, i.e., $(x_{n_k}) \downarrow^{\lambda_{st}} x$.

(vi) Suppose that $(x_n) \downarrow^{\lambda_{st}} x$ and $(x_n) \downarrow^{\lambda_{st}} y$ hold in E. Then there exist subsets M and K of $\mathbb{N}$ with $\delta_\lambda(M) = \delta_\lambda(K) = 1$ such that the subsequences (x_{n_m}) and (x_{n_k}) are monotone decreasing to x and y, respectively. Now, if we choose $J = M \cap K$ then we have $\delta_\lambda(J) = 1$. Thus, we can consider a subsequence (x_{n_j}). So, (x_{n_j}) is monotone decreasing to both x and y because (x_{n_j}) is a subsequence of both (x_{n_m}) and (x_{n_k}). Therefore, we obtain $x = y$ because order limit is unique.

Proposition 2.6. The order convergence implies the λ-statistical order convergence in Riesz spaces.

Proof: Suppose $x_n \xrightarrow{o} x$ in a Riesz space E. Then there exists another sequence $(y_n) \downarrow 0$ in E such that $|x_n - x| \leq y_n$ holds for all $n \in \mathbb{N}$. Now, by using Proposition 2, we can get $(y_n) \downarrow^{\lambda_{st}} 0$. So, there is a subset M such that $\delta_\lambda(M) = 1$ and $(y_{n_m}) \downarrow 0$. Moreover, we have $|x_{n_m} - x| \leq y_{n_m}$, and so, we get $x_n \xrightarrow{\lambda_{st}o} x$.

Now, motivated by Riesz space theory, we give several basic and useful results.

Theorem 2.7. Let E be Riesz spaces. Then the following facts hold:

i. $x_n \xrightarrow{\lambda_{st}o} x$ if and only if $(x_n - x) \xrightarrow{\lambda_{st}o} 0$ if and only if $|x_n - x| \xrightarrow{\lambda_{st}o} 0$;

ii. The lattice operations are λ-statistically order continuous;

iii. The $\lambda_{st}o$-limit is linear

iv. The $\lambda_{st}o$-convergence has a unique limit;

v. The positive cone E_+ is closed under the $\lambda_{st}o$-convergence in E.

Proof: (iii) The part of the scalar multiplication is clear. Thus, we show the additive part. Consider two sequences $(x_n) \xrightarrow{\lambda_{st}o} x$ and $(y_n) \xrightarrow{\lambda_{st}o} y$ in E. Then there exist sequences $(u_n) \downarrow^{\lambda_{st}} 0$ and $(v_n) \downarrow^{\lambda_{st}} 0$ such that $\delta_\lambda(\{n \in \mathbb{N}: |x_n - x| \nleq u_n\}) = 0$ and $\delta_\lambda(\{n \in \mathbb{N}: |y_n - y| \nleq v_n\}) = 0$. Also, one can obtain $\{n \in \mathbb{N}: |(x_n + y_n) - (x + y)| \nleq v_n + u_n\} \subseteq \{n \in \mathbb{N}: |x_n - x| \nleq u_n\} \cup \{n \in \mathbb{N}: |y_n - y| \nleq v_n\}$. Hence, we get the result because we have $(u_n + v_n) \downarrow^{\lambda_{st}} 0$ by using Proposition 2.10.(iv) and $\delta_\lambda(\{n \in \mathbb{N}: |(x_n + y_n) - (x + y)| \nleq v_n + u_n\}) = 0$.

Proposition 2.8. Every monotone λ-statistical order convergent sequence is order convergent to its $\lambda_{st}o$-limit in Riesz spaces.

Proof: It is enough to show that if $E \ni x_n \uparrow$ and $x_n \xrightarrow{\lambda_{st}o} x$ then $x_n \uparrow x$. Take an arbitrary index n_0. Then $x_n - x_{n_0} \in X_+$ for $n \geq n_0$. By using (iii) and (v) of Theorem 3, we have $x_n - x_{n_0} \xrightarrow{\lambda_{st}o} x - x_{n_0} \in E_+$. Thus, we get $x \geq x_{n_0}$ for any n. Since n_0 is arbitrary, then x is an upper bound of x_{n_0}. Now, assume $y \geq x_n$ for all n. Then, again by Theorem 3, $y - x_n \xrightarrow{\lambda_{st}o} y - x \in E_+$, or $y \geq x$. Thus, $x_n \uparrow x$.

3. (V, λ)-ORDER SUMMABLE

In this section, we introduce the notion of (V, λ)-order summable. Moreover, we give a relation between (V, λ)-order summable and λ-statistical order convergence. Remind that the generalized De la Vallêe-Poussin is defined by

$$t_n(x) := \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

for a non-decreasing sequence of positive numbers (λ_n) such that $\lambda_{n+1} \leq \lambda_n + 1$ and $\lambda_1 = 1$, where $I_n = [n - \lambda_n + 1, n]$. Also, a sequence $x := (x_k)$ in real numbers is called (V, λ)-*summable* to a number L if $t_n(x) \to L$ as $n \to +\infty$; see [27]. Mursaleen investigated relation between the λ-statistical convergence and (V, λ)-summability; see [14]. Hence, motivated by the above definition, we give the following notion.

Definition 3.1. Let $x := (x_k)$ be a sequence in a Riesz space E. Then x is called (V, λ)-*order summable* to a vector $w \in E$ if

$$t_n(x) \xrightarrow{o} w$$

holds in E. In this case, we write $x_k \xrightarrow{(V,\lambda)^o} w$.

Definition 3.2. Let us consider the set of all λ-statistical order convergent sequences in a Riesz space as $S_\lambda^o(E)$. Then one can define the strongly (V, λ)-order summable as follow

$$[V, \lambda]^o := \{x = (x_k) : \exists w \in E,\ \frac{1}{\lambda_n} \sum_{k \in I_n} |x_k - w| \xrightarrow{o} 0\}$$

Theorem 3.3. Let $\lambda \in \Lambda$ and E be a Riesz space. If $x_k \xrightarrow{[V,\lambda]^o} w$ then $x_k \xrightarrow{S_\lambda^o(E)} w$.

Proof: Suppose that $x_k \xrightarrow{[V,\lambda]^o} w$ holds. Then we have

$$\sum_{k \in I_n} |x_k - w| \geq \sum_{\substack{k \in I_n \\ |x_k - w| \geq y_n}} |x_k - w| \geq |\{k \in I_n : |x_k - w| \geq y_n\}| y_n$$

holds for each $n \in \mathbb{N}$. Hence, we obtain that $x_k \xrightarrow{[V,\lambda]^o} w$ implies $x_k \xrightarrow{S_\lambda^o(E)} w$.

The converse of Theorem 4.2. is not true in general. To see this, we give the following Example.

Example 3.3. Let's consider the Riesz space $E := \mathbb{R}$. Take a sequence $x = (x_k)$ denoted by

$$x_k := \begin{cases} k^2, & x \leq n - [\sqrt{\lambda_n}] + 1 \leq k \leq n \\ \frac{1}{k+1}, & otherwise \end{cases}$$

for all k. Then choose $y_n := n$ whenever $n - [\sqrt{\lambda_n}] + 1 \leq k \leq n$ and $\frac{1}{n}$ for the otherwise. Thus, one can see that $y_n \downarrow^{\lambda_{st}} 0$. Now, consider the subset $M = \{1, n \in \mathbb{N} : n \notin [n - [\sqrt{\lambda_n}] + 1, n]\}$. Then we have $\delta_\lambda(M) = 1$ and $|x_{n_m}| \leq y_{n_m}$ for every $m \in M$. So, we obtain $x_k \xrightarrow{\lambda_{st} o} 0$. But, $\frac{1}{\lambda_n} \sum_{k \in I_n} |x_k - 0|$ goes to infinity, i.e., $x_k \nrightarrow 0$ in view of $[V, \lambda]^o$-convergence.

Theorem 3.4. Let E be a σ-order complete Riesz space. Then $S^o(E) \subseteq S^o_\lambda(E)$ if and only if $\liminf_{n \to +\infty} \frac{\lambda_n}{n} > 0$

Proof: For a given sequence $(y_n) \downarrow^{\lambda_{st}} 0$, we can observe

$$\{k \in I_{n_m} : |x_k - w| \geq y_{n_m}\} \subseteq \{k \leq n_m : |x_k - w| \geq y_{n_m}\}$$

for each $n_m \in M$, where M is the subset in Definition 3.1. Thus, we have the following inequality

$$|\{k \leq n_m : |x_k - w| \geq y_{n_m}\}| \geq |\{k \in I_{n_m} : |x_k - w| \geq y_{n_m}\}|$$

$$\geq \lambda_{n_m} |\{k \in I_{n_m} : |x_k - w| \geq y_{n_m}\}|$$

for every $m \in \mathbb{N}$. By taking the limit, we obtain that the statistical order convergence implies the λ-statistical convergence.

For the converse, assume that $\liminf_{n \to +\infty} \frac{\lambda_n}{n} = 0$. Then there exists a subset M of $\mathbb{N}$ such that $\frac{\lambda_{n_m}}{n_k} < \frac{1}{m}$ for all $m \in M$; see [2, p.510]. Now, define a sequence $x = x_k$ by 1 whenever $k \in I_{n_m}$ and by 0 for the otherwise. So, it is clear that $x \in S^o(E)$. But, by considering Theorem 4.2., we obtain $x \notin S^o_\lambda(E)$.

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