***ORIGINAL PAPER***

# A NOTE ON NEW GENERALIZATIONS OF k-HORADAM SEQUENCES AND THE POWER SEQUENCES OF THESE GENERALIZATIONS

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***Abstract.*** *In this article, firstly, we have described new generalizations of generalized $k-$Horadam sequence and we named the generalizations as another generalized $k-$Horadam sequence $\left\{H'_{k,n}\right\}_{n\in\mathbb{N}}$, a different generalized $k-$Horadam sequence $\left\{q_{k,n}\right\}$ and an altered generalized $k-$Horadam sequence $\left\{Q_{k,n}\right\}$, respectively. Then, we have studied properties of these new generalizations and we have obtained generating function and extended Binet formula for each generalization. Also, we have introduced a power sequence for an altered generalized $k-$Horadam sequence in order to be used in different applications like number theory, cryptography, coding theory and engineering.*

***Keywords:*** *generalized $k-$Horadam sequence; generating function; generalized Fibonacci sequence; extended Binet's formula; the power Fibonacci sequence module $m$.*

## 1. INTRODUCTION

The Fibonacci sequence, $\left\{F_n\right\}_{n=0}^{\infty}$, is a sequence of numbers, beginning with the integer couple 0 and 1, in which the value of any element is computed by taking the sum of the two consecutive numbers. If so, for $n\geq 2$, $F_n = F_{n-1} + F_{n-2}$ [1]. This number sequence, which was previously found by Indian mathematicians in the 6th century, but the sequence was introduced by Fibonacci as a result of calculating the problem related to the reproduction of rabbits in the book called Fibonacci's Liber Abaci in 1202. Fibonacci has not done any work using these sequences. In fact, the first researches on these sequences were made about 600 years later. However, the subsequent research has increased substantially. There have been many studies in the literature dealing with the quadratic number sequences. Some authors have obtained generalization of the Fibonacci sequence by changing only the first two terms of the sequence or with minor changes only the recurrence relation, while others have obtained generalizations of the Fibonacci sequence by changing both of them. Some of these sequences are chronologically as follows:

Lucas, Pell, Pell Lucas, Jacobsthal and Jacobsthal–Lucas sequences, $k-$Fibonacci and $k-$Lucas, generalized Fibonacci sequence with two real parameters used non linear recurrence relation, generalized $k-$Fibonacci and generalized $k-$Lucas, generalized

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Fibonacci sequence with four real parameters used non linear recurrence relation, generalized $k-$Horadam [2-19].

All of these sequences are based on the Fibonacci sequence. Fibonaci sequence has many impressive features. And, in 2012, it is obtained that the power Fibonacci sequence module $m$ [6] which is based on Fibonacci sequence and is one of the reccurent sequences [2]. There are quite a lot implementations of these numbers different areas like engineering, nature and cryptography and coding theory.

In this study, firstly, we have defined a new generalization $\left\{H'_{k,n}\right\}_{n\in\mathbb{N}}$ built by altering only the initial conditions of generalized $k-$Horadam sequence. Then, for this new generalization, we have obtained generating function and Binet formula similarly in [17]. After that, we have also introduced different generalizations ( $\left\{q_{k,n}\right\}$ , $\left\{Q_{k,n}\right\}$ ) built by altering both initial conditions and recurrence relations of generalized $k-$Horadam sequence. Also, for new generalizations, we have obtained generating function and Binet formula. In both generalizations, it is used non linear recurrence relations. Then, we have studied on these generalizations. All the generalizations described in this work depend on scaler-valued polynomials used. Unlike other generalizations, the initial terms also consist of scaler-valued polynomials. Finally, we have introduced a power sequence which can be used in different applications like number theory, cryptography, coding theory and engineering for one of the generalizations of generalized $k-$Horadam sequence.

## 2. MATERIALS AND METHODS

In this study, generalized Fibonacci sequence, generalized $k-$Horadam sequence are used as material, and obtaining generating functions and Binet formulas are used as method. The definitions of these concepts used are as follows:

**Definition 2.1.** $f(k), g(k)$ are scaler-valued polynomials where $k$ is a positive real number. For all $n\geq 0$ and $f^2(k)+4g(k)\geq 0$, generalized $k-$Horadam sequence $\left\{H_{k,n}\right\}_{n\in\mathbb{N}}$ is described by $H_{k,n+2}=f(k)H_{k,n+1}+g(k)H_{k,n}$ with initial conditions $H_{k,0}=a$, $H_{k,1}=b$ [17].

**Definition 2.2.** For any two nonzero real numbers $a$ and $b$, the generalized Fibonacci sequence say $\left\{q_n\right\}$ is defined recursively by

$$q_n=\begin{cases} aq_{n-1}+q_{n-2} & ;\ if \quad n\ is\ even \\ bq_{n-1}+q_{n-2} & ;\ if \quad n\ is\ odd \end{cases}$$ for $n\geq 2$, with initial conditions $q_0=0, q_1=1$ [3].

**Definition 2.3.** A generating function is a way of encoding an infinite sequence of numbers $\left(a_n\right)$ by treating them as the coefficients of a formal power series. This series is called the generating function of the sequence [1].

**Definition 2.4.** Binet's formula is an explicit formula used to find the $n$th term of the Fibonacci sequence. It is so named because it was derived by mathematician Jacques Philippe Marie Binet, though it was already known by Abraham de Moivre [1].

## 3. RESULTS AND DISCUSSION

In this chapter, we described new generalizations( $\left\{H'_{k,n}\right\}_{n\in\mathbb{N}}$ , $\left\{q_{k,n}\right\}$, $\left\{Q_{k,n}\right\}$) of generalized $k-$Horadam sequence whose initial terms are also scaler-valued polynomials of generalized $k-$Horadam sequence. Thus, we have composed the sequences which is more general than generalized $k-$Horadam sequence. In addition, we have obtained generating functions and Binet formulas for the new generalizations. Finally, we have introduced a new power sequence used in different applications with one of the new generalizations.

### *3.1. NEW GENERALİZATİONS OF $k-$HORADAM SEQUENCES*

**Definition 3.1.1.** $f(k), g(k)$, $i(k), j(k)$ are scaler-valued polynomials where $k$ is a positive real number. For all $n \geq 0$ and $f^2(k)+4g(k)\geq 0$, another generalized $k-$Horadam sequence $\left\{H'_{k,n}\right\}_{n\in\mathbb{N}}$ is described by $H'_{k,n+2} = f(k)H'_{k,n+1} + g(k)H'_{k,n}$ with initial conditions $H'_{k,0} = i(k)$, $H'_{k,1} = j(k)$.

For another generalized $k-$Horadam sequence, characteristic equation is following $\lambda^2 = f(k)\lambda + g(k)$. This equation has two real roots as $l_1 = \dfrac{f(k)+\sqrt{f^2(k)+4g(k)}}{2}$ and $l_2 = \dfrac{f(k)-\sqrt{f^2(k)+4g(k)}}{2}$ $(l_1 \geq l_2)$.

It is easily seen that if $i(k)=a$ and $j(k)=b$, $a,b\in\mathbb{R}$, generalized $k-$Horadam sequence in [17] is obtained.

**Theorem 3.1.2.** For each $n\in\mathbb{N}$, the Binet formula is given as $H'_{k,n} = \dfrac{Ml_1^n - Nl_2^n}{l_1 - l_2}$ , where $M = j(k) - i(k)l_2$ and $N = j(k) - i(k)l_1$ and the generating function of another generalized $k-$Horadam sequence $\left\{H'_{k,n}\right\}_{n\in\mathbb{N}}$ is given as $\sum_{t=0}^{\infty} H'_{k,t}x^t = \dfrac{H'_{k,0} + x\left(H'_{k,1} - f(k)H'_{k,0}\right)}{1 - f(k)x - g(k)x^2}$ .

*Proof:* Binet formula and generating function can be easily obtained for another generalized $k-$Horadam sequence if $a = i(k)$, $b = j(k)$, $r_1 = l_1$ and $r_2 = l_2$ in [17].

**Definition 3.1.3.** $f(k), g(k)$, $i(k), j(k)$ are scaler-valued polynomials where $k$ is a positive real number. A different generalization of $k-$Horadam sequence $\left\{H''^{(f(k),g(k))}_{k,n}\right\}_{n\in\mathbb{N}}$ is described by

$$H''^{(f(k),g(k))}_{k,n} = \begin{cases} f(k)H''^{(f(k),g(k))}_{k,n-1} + H''^{(f(k),g(k))}_{k,n-2} & ;if \quad n \text{ is even} \\ \\ g(k)H''^{(f(k),g(k))}_{k,n-1} + H''^{(f(k),g(k))}_{k,n-2} & ;if \quad n \text{ is odd} \end{cases}$$

with initial conditions $H''^{(f(k),g(k))}_{k,0} = i(k)$ , $H''^{(f(k),g(k))}_{k,1} = j(k)$.

In order to refrain clumsy notation, let us denote $H''^{(f(k),g(k))}_{k,n}$ by $q_{k,n}$. Hence, the sequence $\{q_{k,n}\}$ provides

$$q_{k,0} = i(k)\,, q_{k,1} = j(k), \text{ for } n \geq 2, \; q_{k,n} = \begin{cases} f(k)q_{k,n-1} + q_{k,n-2} & ;if \quad n \text{ is even} \\ \\ g(k)q_{k,n-1} + q_{k,n-2} & ;if \quad n \text{ is odd} \end{cases}.$$

**Example 3.1.4.** Let $\{q_{k,n}\}$ be a scaler valued polynomials sequence providing the recurrence relation $q_{k,n} = \begin{cases} (k+5)q_{k,n-1} + q_{k,n-2} & ; if \quad n \text{ is even} \\ \\ (k^3-1)q_{k,n-1} + q_{k,n-2} & ; if \quad n \text{ is odd} \end{cases}$ for $n \geq 2$, with initial conditions $q_{k,0} = k$ , $q_{k,1} = k+2$. Thus,

$$q_{k,n} = k, k+2, k^4+2k^3-2, k^5+7k^4+10k^3-k-8, k^8+7k^7+10k^6-k^5-7k^4-16k^3+k+6, \ldots$$

is a different generalized $k-$Horadam sequence.

**Theorem 3.1.5.** The generating function for the different generalization of $k-$Horadam sequence $\{q_{k,n}\}$ is given as

$$\sum_{n=0}^{\infty} q_{k,n}x^n = \frac{q_{k,0} + q_{k,1}x + \left(f(k)q_{k,1} - f(k)g(k)q_{k,0} - q_{k,0}\right)x^2 + \left(g(k)q_{k,0} - q_{k,1}\right)x^3}{1 - \left(f(k)g(k)+2\right)x^2 + x^4}.$$

*Proof:* Let $H''(x)$ be a generating function for the $\{q_{k,n}\}$ sequence. Then we obtain

$$H''(x) = q_{k,0} + q_{k,1}x + q_{k,2}x^2 + \ldots + q_{k,n}x^n + \ldots = \sum_{t=0}^{\infty} q_{k,t}\, x^t$$

Note that,

$$g(k)xH^{''}(x)=g(k)q_{k,0}x+g(k)q_{k,1}x^2+g(k)q_{k,2}x^3+...+g(k)q_{k,n}x^{n+1}+...$$

and

$$x^2H^{''}(x)=q_{k,0}x^2+q_{k,1}x^3+q_{k,2}x^4+...+q_{k,n}x^{n+2}+...$$

Because $q_{k,2n+1}=g(k)q_{k,2n}+q_{k,2n-1}$ , we obtain

$$(1-g(k)x-x^2)H^{''}(x)=q_{k,0}+q_{k,1}x+\sum_{t=1}^{\infty}(q_{k,2t}-g(k)q_{k,2t-1}-q_{k,2t-2})x^{2t} \ .$$

Also because $q_{k,2n}=f(k)q_{k,2n-1}+q_{k,2n-2}$ , we obtain

$$(1-g(k)x-x^2)H^{''}(x)=q_{k,0}+q_{k,1}x+\sum_{t=1}^{\infty}(f(k)-g(k))q_{k,2t-1}x^{2t}$$

$$(1-g(k)x-x^2)H^{''}(x)=q_{k,0}+q_{k,1}x+(f(k)-g(k))x\sum_{t=1}^{\infty}q_{k,2t-1}x^{2t-1} \ .$$

Now, let $h^{''}(x)=\sum_{t=1}^{\infty}q_{k,2t-1}x^{2t-1}$ . Because

$$\begin{aligned} q_{k,2n+1}&=g(k)q_{k,2n}+q_{k,2n-1} \\ &=g(k)(f(k)q_{k,2n-1}+q_{k,2n-2})+q_{k,2n-1} \\ &=(f(k)g(k)+2)q_{k,2n-1}+q_{k,2n-3} \end{aligned} \ ,$$

we have

$$\begin{aligned}(1-(f(k)g(k)+2)x^2-x^4)h^{''}(x)&=q_{k,1}x+(g(k)q_{k,0}-q_{k,1})x^3+\sum_{t=3}^{\infty}(q_{k,2t-1}-(f(k)g(k)+2)q_{k,2t-3}+q_{k,2t-5})x^{2t-1} \\ &=q_{k,1}x+(g(k)q_{k,0}-q_{k,1})x^3.\end{aligned}$$

Thus, $h^{''}(x)=\dfrac{q_{k,1}x+(g(k)q_{k,0}-q_{k,1})x^3}{(1-(f(k)g(k)+2)x^2+x^4)}$ . So, we form

$$(1-g(k)x-x^2)H^{''}(x)=q_{k,0}+(q_{k,1}-g(k)q_{k,0})x+(f(k)-g(k))x\frac{q_{k,1}x+(g(k)q_{k,0}-q_{k,1})x^3}{(1-(f(k)g(k)+2)x^2+x^4)} \ .$$

Consequently, desired result is obtained following:

$$H^{''}(x)=\frac{q_{k,0}+q_{k,1}x+\left(f(k)q_{k,1}-f(k)g(k)q_{k,0}-q_{k,0}\right)x^2+\left(g(k)q_{k,0}-q_{k,1}\right)x^3}{1-\left(f(k)g(k)+2\right)x^2+x^4}.$$

□

**Theorem 3.1.6.** The Binet formula for different generalized $k-$Horadam sequence $\{q_{k,n}\}$ are given by

$$q_{k,n}=\left(\frac{f(k)^{1-\xi(n)}}{(f(k)g(k))^{\left\lfloor\frac{n+2}{2}\right\rfloor}}\right)\frac{\gamma\theta^{n}(j(k)\theta+i(k)g(k))-\theta\gamma^{n}(i(k)g(k)+j(k)\gamma)}{\gamma-\theta}.$$

*Proof:* Firstly, for $\gamma$ and $\theta$ are roots of the quadratic equation $x^2-f(k)g(k)x-f(k)g(k)=0$ and $\xi(n)=\begin{cases}0; & if \quad n \text{ is even}\\ 1; & if \quad n \text{ is odd}\end{cases}$ is the piecewise function.

For the sequence $\{q_{k,n}\}$, it is easily obtained that the generating function is given as according to Theorem 3.1.5.

$$H^{''}(x)=\frac{q_{k,0}+q_{k,1}x+\left(f(k)q_{k,1}-f(k)g(k)q_{k,0}-q_{k,0}\right)x^2+\left(g(k)q_{k,0}-q_{k,1}\right)x^3}{1-\left(f(k)g(k)+2\right)x^2+x^4}$$

We have rebuilt by using the partial fraction decomposition,

$$H^{''}(x)=\frac{1}{\gamma-\theta}\left[\frac{f(k)(\gamma+1)(j(k)-g(k)i(k))-\gamma i(k)+i(k)g(k)(\gamma+1)x-\gamma j(k)x}{x^2-(\gamma+1)}+\frac{f(k)(\theta+1)(g(k)i(k)-j(k))+\theta i(k)-i(k)g(k)(\theta+1)x+\theta j(k)x}{x^2-(\theta+1)}\right].$$

$$\frac{A-Bz}{z^2-C}=\sum_{n=0}^{\infty}BC^{-n-1}z^{2n+1}-\sum_{n=0}^{\infty}AC^{-n-1}z^{2n}.$$

So, the generating function $H^{''}(x)$ is determined as follows:

$$H^{''}(x)=\frac{1}{\gamma-\theta}\left[\frac{f(k)(\gamma+1)(j(k)-g(k)i(k))-\gamma i(k)+i(k)g(k)(\gamma+1)x-\gamma j(k)x}{x^2-(\gamma+1)}+\frac{f(k)(\theta+1)(g(k)i(k)-j(k))+\theta i(k)-i(k)g(k)(\theta+1)x+\theta j(k)x}{x^2-(\theta+1)}\right]$$

$$=\frac{1}{\gamma-\theta}\left[\sum_{n=0}^{\infty}\frac{(\gamma j(k)-i(k)g(k)(\gamma+1))(\theta+1)^{n+1}+(j(k)g(k)(\theta+1)x-\theta j(k))(\gamma+1)^{n+1}}{(\gamma+1)^{n+1}(\theta+1)^{n+1}}x^{2n+1}+\sum_{n=0}^{\infty}\frac{(\gamma i(k)-f(k)(\gamma+1)(j(k)-g(k)i(k)))(\theta+1)^{n+1}+(f(k)(\theta+1)(j(k)-g(k)i(k))-\theta i(k))(\gamma+1)^{n+1}}{(\gamma+1)^{n+1}(\theta+1)^{n+1}}x^{2n}\right].$$

It is easily obtained that $\gamma$ and $\theta$ provides following equations.

$(i)\,(\gamma+1)(\theta+1)=1$,

$(ii)\,\gamma+\theta=f(k)g(k)$,

$(iii)\,\gamma.\theta \;=-f(k)g(k)$,

$(iv)\,\gamma+1=\dfrac{\gamma^2}{f(k)g(k)}$,

$(v)\,\theta+1=\dfrac{\theta^2}{f(k)g(k)}$,

$(vi)\,-\theta(\gamma+1)=\gamma$,

$(vii)\,-\gamma(\theta+1)=\theta$.

We have obtained by using the equations above $H^{''}(x)$.

$$H^{''}(x)=\left(\frac{1}{f(k)g(k)}\right)^{n+1}\left[\begin{array}{l}\displaystyle\sum_{n=0}^{\infty}\frac{(\gamma j(k)-i(k)g(k)(\gamma+1))\theta^{2n+2}+(i(k)g(k)(\theta+1)x-\theta j(k))\gamma^{2n+2}}{\gamma-\theta}x^{2n+1}+\\ \displaystyle\sum_{n=0}^{\infty}\frac{(\gamma i(k)-f(k)(\gamma+1)(j(k)-g(k)i(k)))\theta^{2n+2}+(f(k)(\theta+1)(j(k)-g(k)i(k))-\theta i(k))\gamma^{2n+2}}{\gamma-\theta}x^{2n}\end{array}\right]$$

$$H^{''}(x)=\left(\frac{1}{f(k)g(k)}\right)^{n+1}\left[\begin{array}{l}\displaystyle\sum_{n=0}^{\infty}\frac{\gamma\theta^{2n+1}(j(k)\theta+i(k)g(k))-\theta\gamma^{2n+1}(j(k)g(k)+j(k)\gamma)}{\gamma-\theta}x^{2n+1}+\\ \displaystyle\sum_{n=0}^{\infty}\frac{\gamma\theta\theta^{2n}[i(k)\theta+(f(k)(j(k)-g(k)i(k)))]-\theta\gamma\gamma^{2n}[f(k)(j(k)-g(k)i(k))+i(k)\gamma]}{\gamma-\theta}x^{2n}\end{array}\right]$$

$$H^{''}(x)=\left[\begin{array}{l}\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{f(k)g(k)}\right)^{n+1}\frac{\gamma\theta^{2n+1}(j(k)\theta+i(k)g(k))-\theta\gamma^{2n+1}(i(k)g(k)+j(k)\gamma)}{\gamma-\theta}x^{2n+1}+\\ \displaystyle\sum_{n=0}^{\infty}f(k)\left(\frac{1}{f(k)g(k)}\right)^{n+1}\frac{\gamma\theta^{2n}(j(k)\theta+i(k)g(k))-\theta\gamma^{2n}(i(k)g(k)+j(k)\gamma)}{\gamma-\theta}x^{2n}\end{array}\right]$$

.

Combining these summations, we obtain

$$H^{''}(x)=\sum_{n=0}^{\infty}f(k)^{1-\xi(n)}\left(\frac{1}{f(k)g(k)}\right)^{\left\lfloor\frac{n+2}{2}\right\rfloor}\frac{\gamma\theta^{n}(j(k)\theta+i(k)g(k))-\theta\gamma^{n}(i(k)g(k)+j(k)\gamma)}{\gamma-\theta}x^{n}=\sum_{n=0}^{\infty}q_{k,n}x^{n}\;.$$

Hence, for all $n \geq 0$, we obtain

$$q_{k,n} = \left( \frac{f(k)^{1-\xi(n)}}{(f(k)g(k))^{\left\lfloor \frac{n+2}{2} \right\rfloor}} \right) \frac{\gamma\theta^n (j(k)\theta + i(k)g(k)) - \theta\gamma^n (i(k)g(k) + j(k)\gamma)}{\gamma - \theta} .$$

Now, we will describe a new sequence that are obtained by altering both the recurrence relation and initial conditions of the generalized $k-$Horadam sequence. This new generalization depends on six scaler-valued polynomials. This generalization is obtained by making a small change in different generalization of $k-$Horadam sequence we have defined above.

**Definition 3.1.7.** $f(k), g(k), u(k), v(k), i(k), j(k)$ are scaler-valued polynomials where $k$ is a positive real number. Altered generalized $k-$Horadam sequence $\{Q_{k,n}\}_{n=0}^{\infty}$ is described recursively by

$$Q_{k,n} = \begin{cases} f(k)Q_{k,n-1} + u(k)Q_{k,n-2} & ; if \quad n \text{ is even} \\ g(k)Q_{k,n-1} + v(k)Q_{k,n-2} & ; if \quad n \text{ is odd} \end{cases}$$

for $n \geq 2$, with initial conditions

$$Q_{k,0} = i(k), Q_{k,1} = j(k).$$

**Example 3.1.8.** Let $\{Q_{k,n}\}$ be a scaler-valued polynomials sequence providing the recurrence relation $Q_{k,n} = \begin{cases} (k-9)Q_{k,n-1} + (k^3)Q_{k,n-2} & ; if \quad n \text{ is even} \\ (k^2-2)Q_{k,n-1} + (k+3)Q_{k,n-2} & ; if \quad n \text{ is odd} \end{cases}$ for $n \geq 2$, with initial conditions $Q_{k,0} = k+1$, $Q_{k,1} = k+7$.

$$Q_{k,n} = k+1, k+7, k^3+8k^2+2k-11, 2k^4+6k^3-70k^2-25k+99, 2k^6+6k^5-73k^4-26k^3+265k^2+45k-231, ...$$

is an altered generalized $k-$Horadam sequence.

**Theorem 3.1.9.** For the altered generalized $k-$Horadam sequence $\{Q_{k,n}\}$, the generating function is given by

$$\begin{aligned} \sum_{n=0}^{\infty} Q_{k,n} x^n &= \frac{Q_{k,0} + Q_{k,1}x + \left( f(k)Q_{k,1} - (f(k)g(k) + v(k))Q_{k,0} \right) x^2 + \left( g(k)Q_{k,2} - (f(k)g(k) + u(k))Q_{k,1} \right) x^3}{1 - \left( f(k)g(k) + u(k) + v(k) \right) x^2 + u(k)v(k)x^4} \\ &= \frac{Q_{k,0} + Q_{k,1}x + \left( f(k)Q_{k,1} - (f(k)g(k) + v(k))Q_{k,0} \right) x^2 + \left( u(k)(g(k)Q_{k,0} - Q_{k,1}) \right) x^3}{1 - \left( f(k)g(k) + u(k) + v(k) \right) x^2 + u(k)v(k)x^4} . \end{aligned}$$

*Proof:* It is easily obtained that the sequence satisfies the following equation:

$$Q_{k,t}=\left(f(k)g(k)+u(k)+v(k)\right)Q_{k,t-2}-u(k)v(k)Q_{k,t-4}\,;n\geq 4\,.$$

A generating function for the $\{Q_{k,n}\}$ sequence is $H^{'''}(x)$ . So, it is obtained

$$H^{'''}(x)=Q_{k,0}+Q_{k,1}x+Q_{k,2}x^2+...+Q_{k,n}x^n+...=\sum_{t=0}^{\infty}Q_{k,t}x^t\,\cdot$$

If attention is paid,

$$\begin{aligned}(1-f(k)g(k)+u(k)+v(k))x^2+u(k)v(k)x^4)H^{'''}(x)&=\sum_{t=0}^{\infty}Q_{k,t}x^t-(f(k)g(k)+u(k)+v(k))\sum_{t=0}^{\infty}Q_{k,t}x^{t+2}+u(k)v(k)\sum_{t=0}^{\infty}Q_{k,t}x^{t+4}\\&=\sum_{t=0}^{\infty}Q_{k,t}x^t-(f(k)g(k)+u(k)+v(k))\sum_{t=2}^{\infty}Q_{k,t-2}x^t+u(k)v(k)\sum_{t=4}^{\infty}Q_{k,t-4}x^t\\&=Q_{k,0}+Q_{k,1}x+(Q_{k,2}-(f(k)g(k)+u(k)+v(k))Q_{k,0})x^2+(Q_{k,3}-(f(k)g(k)+u(k)+v(k))Q_{k,1})x^3\\&+\sum_{t=4}^{\infty}(Q_{k,t}-(f(k)g(k)+u(k)+v(k))Q_{k,t-2}+u(k)v(k))Q_{k,t-4})x^t\end{aligned}$$

Because $Q_{k,n}=\left(f(k)g(k)+u(k)+v(k)\right)Q_{k,n-2}-u(k)v(k)Q_{k,n-4}\,;n\geq 4\,,$

$$\begin{aligned}(1-f(k)g(k)+u(k)+v(k))x^2+u(k)v(k)x^4)H^{'''}(x)&=Q_{k,0}+Q_{k,1}x+(Q_{k,2}-(f(k)g(k)+u(k)+v(k))Q_{k,0})x^2\\&+(Q_{k,3}-(f(k)g(k)+u(k)+v(k))Q_{k,1})x^3\end{aligned}$$

$$H^{'''}(x)=\sum_{t=0}^{\infty}Q_{k,t}x^t=\frac{Q_{k,0}+Q_{k,1}x+\left(f(k)Q_{k,1}-(f(k)g(k)+v(k))Q_{k,0}\right)x^2+\left(g(k)Q_{k,2}-(f(k)g(k)+u(k))Q_{k,1}\right)x^3}{1-\left(f(k)g(k)+u(k)+v(k)\right)x^2+u(k)v(k)x^4}.$$

□

**Theorem 3.1.10.** Binet formula for altered generalized $k-$Horadam sequence $\{Q_{k,n}\}$ is given by $Q_{k,n}=\left(\dfrac{f(k)^{1-\xi(n)}}{(f(k)g(k))^{\left\lfloor\frac{n+2}{2}\right\rfloor}}\right)\dfrac{\psi\tau^n(j(k)\tau+i(k)g(k))-\tau\psi^n(i(k)g(k)+j(k)\psi)}{\psi-\tau}\,.$

*Proof:* Firstly, let $\psi$ and $\tau$ are roots of the quadratic equation $x^2-(f(k)g(k)+u(k)-v(k))x-f(k)g(k)v(k)=0$ and $\xi(n)=\begin{cases}0; & if \quad n \text{ is even}\\ 1; & if \quad n \text{ is odd}\end{cases}$ is the piecewise function.

We know that the generating function for $\{Q_{k,n}\}$ is given by according to Theorem 3.1.9.

$$H^{'''}(x)=\frac{Q_{k,0}+Q_{k,1}x+\left(f(k)Q_{k,1}-(f(k)g(k)+v(k))Q_{k,0}\right)x^2+\left(g(k)Q_{k,2}-(f(k)g(k)+u(k))Q_{k,1}\right)x^3}{1-\left(f(k)g(k)+u(k)+v(k)\right)x^2+u(k)v(k)x^4}.$$

Using the partial fraction decomposition, we obtain

$$H^{'''}(x)=\frac{1}{u(k)v(k)(\psi-\tau)}\left[\begin{array}{l}\dfrac{(\psi+v(k))(f(k)(j(k)-g(k)i(k))-i(k)v(k))+i(k)u(k)v(k)-(u(k)[\psi(j(k)-g(k)i(k))-i(k)g(k)v(k)])x}{x^2-\dfrac{\psi+v(k)}{u(k)v(k)}}+\\ \dfrac{(f(k)(g(k)i(k)-j(k))+v(k)j(k))(\tau+v(k))-i(k)u(k)v(k)-(u(k)[i(k)g(k)(\tau+v(k))-j(k)\tau])x)}{x^2-\dfrac{\tau+v(k)}{u(k)v(k)}}\end{array}\right].$$

Also,

$$\frac{A-Bz}{z^2-C}=\sum_{n=0}^{\infty}BC^{-n-1}z^{2n+1}-\sum_{n=0}^{\infty}AC^{-n-1}z^{2n}.$$

So, the generating function $H^{'''}(x)$ is determined as

$$H^{'''}(x)=\frac{1}{u(k)v(k)(\psi-\tau)}\left[\begin{array}{l}\displaystyle\sum_{n=0}^{\infty}\begin{array}{l}(u(k)[\psi(j(k)-g(k)i(k))-i(k)g(k)v(k)])\left(\dfrac{\psi+v(k)}{u(k)v(k)}\right)^{-n-1}x^{2n+1}+\\ (u(k)[i(k)g(k)(\tau+v(k))-j(k)\tau])\left(\dfrac{\tau+v(k)}{u(k)v(k)}\right)^{-n-1}x^{2n+1}+\end{array}\\ \displaystyle\sum_{n=0}^{\infty}\begin{array}{l}((\psi+v(k))(f(k)(g(k)i(k)-j(k))+i(k)v(k))-i(k)u(k)v(k))\left(\dfrac{\psi+v(k)}{u(k)v(k)}\right)^{-n-1}x^{2n}+\\ ((f(k)(j(k)-g(k)i(k))-v(k)i(k))(\tau+v(k))+i(k)u(k)v(k))\left(\dfrac{\tau+v(k)}{u(k)v(k)}\right)^{-n-1}x^{2n}\end{array}\end{array}\right].$$

We are simplify using the following equations which are obtained with $\psi$ and $\tau$.

$(i)\,\psi+\tau=f(k)g(k)+u(k)-v(k)$
$(ii)\,\psi.\tau=-f(k)g(k)v(k)$
$(iii)\,(\psi+v(k))(\tau+v(k))=u(k)v(k)$
$(iv)\,f(k)g(k)(\tau+v(k))=\tau(\tau+v(k)-u(k))$
$(v)\,f(k)g(k)(\psi+v(k))=\psi(\psi+v(k)-u(k))$

Using these equations, we obtain

$$H^{'''}(x)=\frac{1}{u(k)v(k)(\psi-\tau)}\left[\begin{array}{l}\left(\displaystyle\sum_{n=0}^{\infty}\begin{array}{l}(u(k)[\tau(i(k)g(k)-j(k))+i(k)g(k)v(k)])\left(\psi+v(k)\right)^{n+1}x^{2n+1}-\\ (u(k)[\psi(i(k)g(k)-j(k))+i(k)g(k)v(k)])\left(\tau+v(k)\right)^{n+1}x^{2n+1}\end{array}\right)+\\ \left(-\displaystyle\sum_{n=0}^{\infty}\begin{array}{l}((\tau+v(k))(f(k)(i(k)g(k)-j(k)))+i(k)v(k)))-i(k)u(k)v(k))\left(\psi+v(k)\right)^{n+1}x^{2n}-\\ ((\psi+v(k))(f(k)(i(k)g(k)-j(k))+i(k)v(k))-i(k)u(k)v(k))\left(\tau+v(k)\right)^{n+1}x^{2n}\end{array}\right)\end{array}\right]$$

$$H^{'''}(x) = \frac{1}{(\psi-\tau)}\left(\sum_{n=0}^{\infty}\begin{array}{l}[(\tau - f(k)g(k))(i(k)g(k)-j(k))+i(k)g(k)(\psi+v(k))]\left(\psi+v(k)\right)^{n} x^{2n+1} - \\ [(\psi - f(k)g(k))(i(k)g(k)-j(k))+i(k)g(k)(\tau+v(k))]\left(\tau+v(k)\right)^{n} x^{2n+1}]\end{array}\right)$$

$$+\frac{1}{(\psi-\tau)}\left(-\sum_{n=0}^{\infty}\begin{array}{l}((f(k)(i(k)g(k)-j(k)))+i(k)v(k)))-i(k)(\psi+v(k)))\left(\psi+v(k)\right)^{n} x^{2n} - \\ ((f(k)(i(k)g(k)-j(k))+i(k)v(k))-i(k)(\tau+v(k)))\left(\tau+v(k)\right)^{n} x^{2n}\end{array}\right)$$

$$H^{'''}(x) = \frac{1}{(f(k)g(k))^{n}(\psi-\tau)}\left[\begin{array}{l}\left(\sum_{n=0}^{\infty}\begin{array}{l}[(\tau - f(k)g(k))(i(k)g(k)-j(k))+i(k)g(k)(\psi+v(k))]\psi^{n}\left(\psi+v(k)-u(k)\right)^{n} x^{2n+1} - \\ [(\psi - f(k)g(k))(i(k)g(k)-j(k))+i(k)g(k)(\tau+v(k))]\tau^{n}\left(\tau+v(k)-u(k)\right)^{n} x^{2n+1}\end{array}\right)+ \\ \left(-\sum_{n=0}^{\infty}\begin{array}{l}((f(k)(i(k)g(k)-j(k)))+i(k)v(k)))-i(k)(\psi+v(k)))\psi^{n}\left(\psi+v(k)-u(k)\right)^{n} x^{2n} - \\ ((f(k)(i(k)g(k)-j(k))+i(k)v(k))-i(k)(\tau+v(k)))\tau^{n}\left(\tau+v(k)-u(k)\right)^{n} x^{2n}\end{array}\right)\end{array}\right]$$

$$H^{'''}(x) = \frac{1}{(f(k)g(k))^{n}(\psi-\tau)}\left[\begin{array}{l}\left(\sum_{n=0}^{\infty}\begin{array}{l}[(j(k)\psi^{n}\left(\psi+v(k)-u(k)\right)^{n+1}+i(k)g(k)u(k)\psi^{n}\left(\psi+v(k)-u(k)\right)^{n}]x^{2n+1} - \\ \\ [(j(k)\tau^{n}\left(\tau+v(k)-u(k)\right)^{n+1}+i(k)g(k)u(k)\tau^{n}\left(\tau+v(k)-u(k)\right)^{n}]x^{2n+1}\end{array}\right)+ \\ \left(-\sum_{n=0}^{\infty}\begin{array}{l}((f(k)(i(k)g(k)-j(k)))+i(k)v(k)))-i(k)(\psi+v(k)))\psi^{n}\left(\psi+v(k)-u(k)\right)^{n} x^{2n} - \\ \\ ((f(k)(i(k)g(k)-j(k))+i(k)v(k))-i(k)(\tau+v(k)))\tau^{n}\left(\tau+v(k)-u(k)\right)^{n} x^{2n}\end{array}\right)\end{array}\right]$$

$$H^{'''}(x) = \frac{1}{(f(k)g(k))^{n}(\psi-\tau)}\left[\begin{array}{l}\left(\sum_{n=0}^{\infty}\begin{array}{l}[(j(k)\psi^{n}\left(\psi+v(k)-u(k)\right)^{n+1}x^{2n+1}+i(k)g(k)u(k)\psi^{n}\left(\psi+v(k)-u(k)\right)^{n}x^{2n+1}] - \\ [(j(k)\tau^{n}\left(\tau+v(k)-u(k)\right)^{n+1}x^{2n+1}+i(k)g(k)u(k)\tau^{n}\left(\tau+v(k)-u(k)\right)^{n}x^{2n+1}]\end{array}\right)+ \\ \\ \left(-\sum_{n=0}^{\infty}\begin{array}{l}((f(k)(i(k)g(k)-j(k)))+i(k)v(k)))\psi^{n}\left(\psi+v(k)-u(k)\right)^{n}x^{2n}-i(k)(\psi+v(k))\psi^{n}\left(\psi+v(k)-u(k)\right)^{n}x^{2n} - \\ ((f(k)(i(k)g(k)-j(k))+i(k)v(k))\tau^{n}\left(\tau+v(k)-u(k)\right)^{n}x^{2n}-i(k)(\tau+v(k))\tau^{n}\left(\tau+v(k)-u(k)\right)^{n}x^{2n}]\end{array}\right)\end{array}\right]$$

$$H^{'''}(x) = \left(\begin{array}{l}\sum_{n=0}^{\infty}\left(\frac{j(k)}{(f(k)g(k))^{n}}\left[\frac{(\psi^{n}\left(\psi+v(k)-u(k)\right)^{n+1}-(\tau^{n}\left(\tau+v(k)-u(k)\right)^{n+1})}{(\psi-\tau)}\right]x^{2n+1}\right)+ \\ \sum_{n=0}^{\infty}\left(\frac{i(k)g(k)u(k)}{(f(k)g(k))^{n}}\left[\frac{\psi^{n}\left(\psi+v(k)-u(k)\right)^{n}-\tau^{n}\left(\tau+v(k)-u(k)\right)^{n}}{(\psi-\tau)}\right]x^{2n+1}\right)\end{array}\right)$$

$$+\left(\begin{array}{l}\sum_{n=0}^{\infty}\frac{i(k)}{(f(k)g(k))^{n}}\left[\frac{(\psi+v(k))\psi^{n}\left(\psi+v(k)-u(k)\right)^{n}-((\tau+v(k))\tau^{n}\left(\tau+v(k)-u(k)\right)^{n})}{(\psi-\tau)}\right]x^{2n}+ \\ \sum_{n=0}^{\infty}\frac{(f(k)(j(k)-i(k)g(k)))-i(k)v(k))}{(f(k)g(k))^{n}}\left[\frac{\psi^{n}\left(\psi+v(k)-u(k)\right)^{n}-\tau^{n}\left(\tau+v(k)-u(k)\right)^{n}}{(\psi-\tau)}\right]x^{2n}\end{array}\right)$$

$$H'''(x) = \frac{1}{(f(k)\,g(k))^{n}} \left( \left( \sum_{n=0}^{\infty} \left[ \frac{j(k)\,(\psi+v(k))-u(k)(j(k)-i(k)g(k))\,\psi^{n}\left(\psi+v(k)-u(k)\right)^{n}-([j(k)\,(\tau+v(k))-u(k)\,(j(k)-i(k)g(k))]\,\tau^{n}\left(\tau+v(k)-u(k)\right)^{n})}{(\psi-\tau)} \right] x^{2n+1} \right) + \left( \sum_{n=0}^{\infty} \left[ \frac{i(k)\,\psi+f(k)(j(k)-i(k)g(k))\psi^{n}\left(\psi+v(k)-u(k)\right)^{n}-i(k)\,\tau+f(k)\,(j(k)-i(k)g(k))\,(\tau^{n}\left(\tau+v(k)-u(k)\right)^{n})}{(\psi-\tau)} \right] x^{2n} \right) \right).$$

If these summations are combined, we obtain

$$H'''(x) = \sum_{n=0}^{\infty} \left[ \frac{1}{(f(k)\,g(k))^{n}} \left( i(k)^{1-\zeta(n)}\, j(k)^{\zeta(n)} \psi + (-1)^{n} u(k)^{1-\zeta(n)} f(k)^{\zeta(n)} (j(k)-i(k)g(k)) + j(k)v(k)\tau \right) \frac{\psi^{n}\left(\psi+v(k)-u(k)\right)^{n} - \tau^{n}\left(\tau+v(k)-u(k)\right)^{n}}{(\psi-\tau)} \right] x^{n}$$

$$= \sum_{n=0}^{\infty} Q_{k,n}\, x^{n}.$$

Hence, for all $n \geq 0$, we have

$$Q_{k,n} = \left( \frac{(i(k))^{1-\zeta(n)}\, j(k)^{\zeta(n)}\psi + (-1)^{n} u(k)^{1-\zeta(n)} f(k)^{\zeta(n)} (j(k)-i(k)g(k)) + j(k)v(k)\tau)}{(f(k)g(k))^{n}} \right) \frac{\psi^{n}\left(\psi+v(k)-u(k)\right)^{n} - \tau^{n}\left(\tau+v(k)-u(k)\right)^{n}}{(\psi-\tau)}.$$

### *3.2. THE ALTERED GENERALİZED POWER $k-$HORADAM SEQUENCE IN SCALAR-VALUED POLYNOMIAL MODULE $s(k)$*

In this study, we also introduced the power sequence of new generalizations defined above.

**Definition 3.2.1.** $f(k), g(k)$, $u(k)$, $v(k)$, $r(k)$ and $s(k)$ are scaler-valued polynomials where $k$ is a positive real number and let $\left\{ Q^{*}_{G_{k,n}} \right\}_{n=0}^{\infty}$ be scaler-valued polynomials sequence satisfying the recurrence relation

$$Q^{*}_{G_{k,n}} = \begin{cases} f(k)Q^{*}_{G_{k,n-1}} + u(k)Q^{*}_{G_{k,n-2}} & ; \quad if\ n\ is\ even \\ g(k)Q^{*}_{G_{k,n-1}} + v(k)Q^{*}_{G_{k,n-2}} & ; \quad if\ n\ is\ odd \end{cases}.$$

If $Q^{*}_{G_{k,n}} \equiv 1, r(k), r(k)^{2}, r(k)^{3}, \ldots (\operatorname{mod} s(k))$ for some modulus $s(k)$, then $Q^{*}_{G_{k,n}}$ is called an altered generalized power $k-$Horadam sequence in scalar-valued polynomial module $s(k)$.

**Example 3.2.2.** Let $\left\{ Q^{*}_{G_{k,n}} \right\}_{n=0}^{\infty}$ be a scaler-valued polynomials sequence providing the recurrence relation

$$Q^{*}_{G_{k,n}} = \begin{cases} (k+3)Q^{*}_{G_{k,n-1}} + Q^{*}_{G_{k,n-2}} & ; \quad if\ n\ is\ even \\ kQ^{*}_{G_{k,n-1}} + (4k+16)Q^{*}_{G_{k,n-2}} & ; \quad if\ n\ is\ odd \end{cases}$$

with initial conditions

$$Q^{*}_{G_0} = 1,\ Q^{*}_{G_1} = k+4 .$$

In this case

$$Q^{*}_{G_{k,n}} \equiv 1, k+4, k^2+8k+16, k^3+12k^2+48k+64, 16k^3+96k+248k+256, \ldots (\mathrm{mod}\,(k^4+8k))$$

is an altered generalized power $k-$Horadam sequence in scalar-valued polynomial module $k^4+8k$.

In addition, it is easily seen that if $u(k)=1$ and $v(k)=1$ are taken in definition of the altered generalized power $k-$Horadam sequence in scalar-valued polynomial module, it is obtained a different generalized power $k-$Horadam sequence in scalar-valued polynomial module. But for $u(k)=v(k)=1$, a power sequence instance in this form in scalar-valued polynomial module can't be found. So, a different generalized power $k-$Horadam sequence in scalar-valued polynomial module can't been.

## 4. CONCLUSIONS

In this article, firstly, we have defined a new generalization built by altering only the initial conditions of generalized $k-$Horadam sequence. Then, for this new generalization, we have obtained generating function and Binet formula similarly in [17]. After that, we have also introduced different generalizations built by altering both initial conditions and recurrence relations of generalized $k-$Horadam sequence. Also, for new generalizations, we have obtained generating functions and Binet formulas. In both generalizations, it is used non linear recurrence relations. Then, we have studied on these generalizations. All the generalizations described in this work depend on scaler-valued polynomials used. Unlike other generalizations, the initial terms also consist of scaler-valued polynomials.

Finally, we have introduced a power sequence which can be used in different applications like number theory, cryptography, coding theory and engineering for one of the generalizations of generalized $k-$Horadam sequence.

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