ORIGINAL PAPER

ON GENERALIZED (*k*, *r*) – GAUSS PELL NUMBERS

BAHAR KULOĞLU[1], ENGİN ÖZKAN[2*]

Manuscript received: 16.01.2021; Accepted paper: 28.08.2021;
Published online: 30.09.2021.

Abstract. *We define the generalized (k, r) – Gauss Pell numbers by using the definition of a distance between numbers. Then we examine their properties and give some important identities for these numbers. In addition, we present the generating functions for these numbers and the sum of the terms of the generalized $(k,r) -$ Gauss Pell numbers.*

Keywords: $k -$ *Pell numbers; Gauss Pell numbers;* $r -$*distance Gauss Pell numbers; generating functions.*

1. INTRODUCTION

Many studies have been conducted on number sequences, especially on Fibonacci number sequences. Some of these are [1-10]. We see some generalizations of Pell numbers in [11-16]. Also, we see some works on generalized Gauss numbers in [17-19]. In [18], Özkan and Taştan presented Gauss Fibonacci polynomials and Gauss Lucas polynomials and proved some theorems related to these polynomials.

Falcon and Plaza [2] presented the definition of *k*-Fibonacci number. Falcon [1] applied the definition of *r*-distance to the *k*-Fibonacci numbers in such a way that it generalized earlier results [20, 10]. Then Panwar et al. gave several identities for generalized (*k*, *r*)-Fibonacci Numbers [7].

In this paper, we apply the definition of *r*-distance to $k -$ Gauss Pell numbers. We present some properties of these sequences. In addition, the generating functions, some important identities, sum of the terms of the generalized $(k,r) -$ Pell numbers are given. Now we recall the definition of $k -$Pell sequence and its Binet Formula.

Definition 1.1. For $k \in \mathbb{N}$, the $k -$Pell sequence is defined by

$$P_{k,n} = 2P_{k,n-1} + kP_{k,n-2}, \text{for } n \geq 2$$

with $P_{k,0} = 0, P_{k,1} = 1$ [9].

Let's give the first few terms for $k -$Pell numbers as follows:

$$P_{k,0} = 0$$

$$P_{k,1} = 1$$

[1] Erzincan Binali Yıldırım University, Graduate School of Natural and Applied Sciences, Erzincan, Turkey. E-mail: bahar_kuloglu@hotmail.com.
[2] Erzincan Binali Yıldırım University, Faculty of Arts and Sciences, Department of Mathematics, Erzincan, Turkey.
**Corresponding author*: eozkanmath@gmail.com; eozkan@erzincan.edu.tr.

$$P_{k,2} = 2$$

$$P_{k,3} = k + 4$$

$$P_{k,4} = 4k + 8$$

$$P_{k,5} = k^2 + 12k + 16$$

$$P_{k,6} = 6k^2 + 32k + 32$$

$$P_{k,7} = k^3 + 24k^2 + 80k + 64$$

If $k = 1$ the classical Pell sequence is obtained. $P_1 = \{0,1,2,5,12,29,70,169, \ldots\}$. $r^2 - 2r - k = 0$ is characteristic equation for k −Pell sequence. The n th k −Pell sequence is given by

$$P_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}$$

where r_1, r_2 are the root of characteristic equation, $r^2 - 2r - k = 0$ [9].

Definition 1.2. Let $k \geq 2, n \geq 0$ be integers. The distance Fibonacci numbers $Fd(k,n)$ are defined by

$$Fd(k,n) = Fd(k, n-k+1) + Fd(k, n-k) \text{ for } n \geq k$$

and $Fd(k,n) = 1$ for $0 \leq n \leq k-1$ [8].

2. MAIN RESULTS

Definition 2.1. For the natural numbers $k \geq 1, n \geq 0$ and $r \geq 1$, we define the generalized (k,r) −Gauss Pell numbers $GP_{k,n}(r)$ by

$$GP_{k,n}(r) = 2GP_{k,n-r}(r) + kGP_{k,n-2}(r) \text{ for } n \geq r \quad (1)$$

with $GP_{k,n}(r) = i, n = 0,1,2,3, \ldots, r-1$ except $GP_{k,1}(1) = k$.

So, if $GP_k(r) = \{GP_{k,n}(r) : n \in \mathbb{N}\}$, then we can give the following values for some r.

$$GP_k(1) = \{i, k, 2k + ki, 4k + 2ki + k^2, 4k^2 + 8k + i(k^2 + 4k), k^3 + 12k^2 + 16k + i(4k^2 + 8k), 6k^3 + 32k^2 + 32k + i(k^3 + 12k^2 + 16k), \ldots\}$$

$$GP_k(2) = \{i, i, i(k+2), i(k+2), i(k+2)^2, i(k+2)^2, i(k+2)^3, i(k+2)^3, i(k+2)^4, i(k+2)^4, \ldots\}$$

$$GP_k(3) = \{i, i, i, i(k+2), i(k+2), i(k^2 + 2k + 2), i(k+2)^2, i(k^3 + 2k^2 + 4k + 4)\}$$

$$GP_k(4) = \{i, i, i, i, i(k+2), i(k+2), i(k^2 + 2k + 2), i(k^2 + 2k + 2), i(k^3 + 2k^2 + 4k + 4), i(k^3 + 2k^2 + 4k + 4), \ldots\}$$

$$GP_k(5) = \{i, i, i, i, i, i(k+2), i(k+2), i(k^2+2k+2), i(k^2+2k+2), i(k^3+2k^2+2k+2), i(k^3+2k^2+4k+4), \ldots\}$$

Now, let us give generalized (k, r) −Gauss Pell numbers $GP_{k,n}(r)$ for some special values as $k = 1,2,3, \ldots$

For $k = 1$, we get the sequences in Table 1.

Table 1. $(\mathbf{1}, \boldsymbol{r})$ **−Gauss Pell numbers.**

n	0	1	2	3	4	5	6	7	8
$GP_{1,n}(1)$	i	1	2+i	5+2i	12+5i	29+12i	70+29i	…	…
$GP_{1,n}(2)$	i	i	3i	3i	9i	9i	27i	27i	81i
$GP_{1,n}(3)$	i	i	i	3i	3i	5i	9i	11i	15i
$GP_{1,n}(4)$	i	i	i	i	3i	3i	5i	5i	11i
$GP_{1,n}(5)$	i	i	i	i	i	3i	3i	5i	5i

Here, $GP_{1,n}(1) = \{i, 1, 2+i, 5+2i, 12+5i, \ldots\}$ shows Gaussian Pell Numbers.

Theorem 2.2. For $r \geq 2$, it is

$$GP_{k,r}(r) = i(k+2)$$

Proof: From equation (1), the proof can be seen easily.

Theorem 2.3. If r is even, then we have

$$GP_{k,2n}(2m) = GP_{k,2n+1}(2m)$$

Proof: We prove the result by induction on *n*.

For $n = 0$, by definition of r −distance, we have $GP_{k,0}(2m) = i$ and $GP_{k,1}(2m) = i$.

Let us suppose this formula is true until $2n+1$. Then

$$GP_{k,2n+2}(2m) = 2GP_{k,2n+2-2m}(2m) + kGP_{k,2n}(2m)$$
$$= 2GP_{k,2(n+1-m)}(2m) + kGP_{k,2n}(2m)$$

$$GP_{k,2n+3}(2m) = 2GP_{k,2n+3-2m}(2m) + kGP_{k,2n+1}(2m)$$
$$= 2GP_{k,2(n+1-m)+1}(2m) + kGP_{k,2n+1}(2m)$$

and both expressions are equal because

$$GP_{k,2n+1}(2m) = GP_{k,2n} \rightarrow GP_{k,2(n+1-m)+1}(2m) = GP_{k,2(n+1-m)}(2m).$$

Theorem 2.4. $\sum_{j=1}^{n} k^{\frac{n-j}{2}}(GP_{k,j}(r) + GP_{k,j-1}(r)) = \frac{GP_{k,n+r}(r) + GP_{k,n+r-1}(r) - 2ik^{\frac{n+1}{2}}}{2}.$

Proof: By direct calculation, we have

$$GP_{k,n+r}(r) + GP_{k,n+r-1}(r) = 2GP_{k,n}(r) + kGP_{k,n+r-2} + 2P_{k,n-1}(r) + kGP_{k,n+r-3}$$

$$= 2GP_{k,n}(r) + 2GP_{k,n-1}(r) + k\left(2GP_{k,n-2}(r) + kGP_{k,n+r-4}(r)\right)$$
$$+ k\left(2GP_{k,n-3}(r) + kGP_{k,n+r-5}(r)\right)$$

$$= 2GP_{k,n}(r) + 2GP_{k,n-1}(r) + 2kGP_{k,n-2}(r) + 2kGP_{k,n-3}(r) + k^2\left(2GP_{k,n-4}(r) + kGP_{k,n+r-6}(r)\right) + k^2\left(2GP_{k,n-5}(r) + kGP_{k,n+r-7}(r)\right)$$

$$= 2GP_{k,n}(r) + 2GP_{k,n-1}(r) + 2kGP_{k,n-2}(r) + 2kGP_{k,n-3}(r) + 2k^2GP_{k,n-4}(r) + 2k^2GP_{k,n-5}(r) + k^3GP_{k,n+r-6}(r) + k^3GP_{k,n+r-7}(r)$$

$$= 2(GP_{k,n}(r) + GP_{k,n-1}(r) + kGP_{k,n-2}(r) + kGP_{k,n-3}(r) + k^2GP_{k,n-4}(r) + k^2GP_{k,n-5}(r) + k^3GP_{k,n-6}(r) + k^3GP_{k,n-7}(r) + \cdots + k^{\frac{n-1}{2}}GP_{k,1}(r) + k^{\frac{n-1}{2}}GP_{k,0}(r) + k^{\frac{n+1}{2}}GP_{k,r-1}(r) + k^{\frac{n+1}{2}}GP_{k,r-2}(r)$$

$$= 2\sum_{j=1}^{n}\left(GP_{k,j}(r)k^{\frac{n-j}{2}} + GP_{k,j-1}(r)k^{\frac{n-j}{2}}\right) + 2ik^{\frac{n+1}{2}} = GP_{k,n+r}(r) + GP_{k,n+r-1}(r)$$

$$= \sum_{j=1}^{n} k^{\frac{n-j}{2}}(GP_{k,j}(r) + GP_{k,j-1}(r)) = \frac{GP_{k,n+r}(r) + GP_{k,n+r-1}(r) - 2ik^{\frac{n+1}{2}}}{2}$$

As is desired.

Theorem 2.5. The sum of the even and odd terms of the above sequences are, respectively,

i. $\sum_{j=0}^{n} k^{n-j}GP_{k,2j}(r) = \frac{GP_{k,2n+r}(r) - ik^{n+1}}{2}$

ii. $\sum_{j=0}^{n} k^{n-j}GP_{k,2j+1}(r) = \frac{GP_{k,2n+r+1}(r) - ik^{n+1}}{2}$

Proof: i. By Equation (1), we have

$$GP_{k,2n+r}(r) = 2GP_{k,2n}(r) + kGP_{k,2n+r-2}(r)$$

$$= 2GP_{k,2n}(r) + k\left(2GP_{k,2n-2}(r) + kGP_{k,2n+r-4}(r)\right)$$

$$= 2GP_{k,2n}(r) + 2kGP_{k,2n-2}(r) + k^2\left(2GP_{k,2n-4}(r) + kGP_{k,2n+r-6}(r)\right)$$

$$= 2GP_{k,2n}(r) + 2kGP_{k,2n-2}(r) + 2k^2GP_{k,2n-4}(r) + \cdots + 2k^nGP_{k,0}(r) + k^{n+1}GP_{k,r-2}(r)$$

$$= 2\sum_{j=0}^{n} k^{n-j}GP_{k,2j} + ik^{n+1} = GP_{k,2n+r}(r)$$

$$= \sum_{j=0}^{n} k^{n-j}GP_{k,2j}(r) = \frac{GP_{k,2n+r}(r) - ik^{n+1}}{2}$$

ii. The second formula can be proven in a similar way.

Theorem 2.6. Generating function of the sequence $GP_k(r)$=$\{GP_{k,n}(r)\}$ is

$$Gp_k(r,x) = \frac{i(1+x)}{1-kx^2-2x^r}$$

Proof: By direct calculation, we have

$$\sum_{n=0}^{\infty} GP_{k,n}(r)x^n = GP_{k,0}(r) + GP_{k,1}(r)x + \sum_{n=2}^{\infty}\big(2GP_{k,n-r}(r) + kGP_{k,n-2}(r)\big)x^n$$

$$= i + xi + 2x^r \sum_{n=2}^{\infty} GP_{k,n-r}(r)x^{n-r} + kx^2 \sum_{n=2}^{\infty} GP_{k,n-2}(r)x^{n-2}$$

$$= i + xi + 2x^r \sum_{p=0}^{\infty} GP_{k,p}(r)x^p + kx^2 \sum_{s=0}^{\infty} GP_{k,s}(r)x^s$$

$$(1-kx^2-2x^r)\sum_{n=0}^{\infty} GP_{k,n}(r)x^n = i(1+x)$$

Finally, considering that for $r \geq 2$, it is $GP_{k,0}(r) = GP_{k,1}(r) = i$, we obtain the indicated generating function. For $r = 1$, the characteristic equation of genaralized Gauss Pell, associated to the recurrence relation (1)

$$\alpha^2 - 2\alpha - k = 0 \qquad (5)$$

with two distinct roots α_1 and α_2, we get

$$\alpha_1 = 1 - \sqrt{1+k}, \qquad \alpha_2 = 1 + \sqrt{1+k}$$

$$\alpha_1 + \alpha_2 = 2$$

$$\alpha_1\alpha_2 = -k$$

$$\alpha_1 - \alpha_2 = -2\sqrt{1+k}$$

Now, we give Binet formula for $GP_{k,n}(1)$.

Theorem 2.7. (Binet's formula) The n^{th} $(k,1)$ $-$Gauss Pell numbers is given by

$$GP_{k,n}(1) = \frac{{\alpha_1}^{n+1}\alpha_2 - \alpha_1{\alpha_2}^{n+1}}{\alpha_2-\alpha_1} + i\,\frac{\alpha_2{\alpha_1}^n - \alpha_1{\alpha_2}^n}{\alpha_2-\alpha_1} \qquad (6)$$

Proof: By Equation (5), we have

$$GP_{k,n}(1) = c_1{\alpha_1}^n + c_2{\alpha_2}^n$$

For $n = 0,\ n = 1$, solving linear equation, we obtain

$$c_1 = \frac{i\alpha_2 - k}{\alpha_2-\alpha_1},\ c_2 = \frac{k - i\alpha_1}{\alpha_2-\alpha_1}$$

which are unique.

So, we get

$$GP_{k,n}(1) = \frac{{\alpha_1}^{n+1}\alpha_2 - \alpha_1{\alpha_2}^{n+1}}{\alpha_2-\alpha_1} + i\,\frac{\alpha_2{\alpha_1}^n - \alpha_1{\alpha_2}^n}{\alpha_2-\alpha_1}$$

which concludes the proof.

Theorem 2.8. There is a relationship between the $(k,r)-$Gauss Pell numbers and $(k,r)-$Pell numbers as following:

$$GP_{k,n}(r)+GP_{k,n+1}(r)=P_{k,n+1}(r)+iP_{k,n}(r)$$

Proof: We give the proof by induction on n and fixed $r=1$.
For $n=1, r=1$ we have

$$GP_{k,1}(1)+GP_{k,2}(1)=k+2k+ki=3k+ki=P_{k,2}(1)+iP_{k,1}(1)$$

Assume it is true for $n=t$ and $r=1$, that is,

$$GP_{k,t}(1)+GP_{k,t+1}(1)=P_{k,t+1}(1)+iP_{k,t}(1)$$

Now, prove that it is true for $n=t+1$ and $r=1$, that is,

$$GP_{k,t+1}(1)+GP_{k,t+2}(1)=2GP_{k,t}(1)+kGP_{k,t-1}(1)+2GP_{k,t+1}(1)+kGP_{k,t}(1)$$

$$=2GP_{k,t}(1)+2GP_{k,t+1}(1)+k\left(GP_{k,t-1}(1)+GP_{k,t}(1)\right)$$

$$=2\left(P_{k,t+1}(1)+iP_{k,t}(1)\right)+k(P_{k,t}(1)+iP_{k,t-1}(1))$$

$$=2P_{k,t+1}(1)+kP_{k,t}(1)+\ 2iP_{k,t}(1)+kiP_{k,t-1}(1)$$

$$=P_{k,t+2}+iP_{k,t+1}$$

as desired.

Theorem 2.9. The $(k,r)-$Gauss Pell numbers for $k=1\ \ r\geq 1$ve $n\geq 2$

$$GP_{k,n}(r)-2\sum_{j=1}^{n-1}GP_{k,j}(r)=1+i.$$

Proof: By Equation (4), for $n=2,3,4\ldots$, we get

$$GP_{k,2}(r)-GP_{k,0}(r)=2GP_{k,2-r}(r)$$

$$GP_{k,3}(r)-GP_{k,1}(r)=2GP_{k,3-r}(r)$$

$$GP_{k,4}(r)-GP_{k,2}(r)=2GP_{k,4-r}(r)$$

$$GP_{k,5}(r)-GP_{k,3}(r)=2GP_{k,5-r}(r)$$

$$GP_{k,6}(r)-GP_{k,4}(r)=2GP_{k,6-r}(r)$$

$$\vdots$$

$$GP_{k,n}(r)-GP_{k,n-2}(r)=2GP_{k,n-r}(r)$$

By summing equation side by side, we find for $r = 1$

$$GP_{k,n}(r) - 1 - i = 2(GP_{k,2-r}(r) + GP_{k,3-r}(r) + \cdots GP_{k,n-r}(r))$$
$$GP_{k,n}(r) - 2\sum_{j=1}^{n-1} GP_{k,j}(r) = 1 + i$$

for $r = 2,3,4, \ldots$.

$$GP_{k,n}(r) - 2\sum_{j=1}^{n-1} GP_{k,j}(r) = 2i$$

Theorem 2.10. (Cassini identity)

$$GP_{k,n-1}(1).GP_{k,n+1}(1) - GP^2{}_{k,n}(1) = (-GP_{k,1}(1))^n(-2i + GP_{k,1}(1) + 1).$$

Proof: By using the Binet's formula, we get

$$\left(\frac{({\alpha_1}^n\alpha_2 - \alpha_1{\alpha_2}^n) + i({\alpha_1}^{n-1}\alpha_2 - \alpha_1{\alpha_2}^{n-1})}{\alpha_2 - \alpha_1}\right)\left(\frac{({\alpha_1}^{n+2}\alpha_2 - \alpha_1{\alpha_2}^{n+2}) + i({\alpha_1}^{n+1}\alpha_2 - \alpha_1{\alpha_2}^{n+1})}{\alpha_2 - \alpha_1}\right)$$
$$-\left(\frac{({\alpha_1}^{n+1}\alpha_2 - \alpha_1{\alpha_2}^{n+1}) + i({\alpha_1}^n\alpha_2 - \alpha_1{\alpha_2}^n)}{\alpha_2 - \alpha_1}\right)^2$$
$$= \frac{({\alpha_1}^n\alpha_2 - \alpha_1{\alpha_2}^n)\,({\alpha_1}^{n+2}\alpha_2 - \alpha_1{\alpha_2}^{n+2})}{(\alpha_2 - \alpha_1)^2} + \frac{i({\alpha_1}^n\alpha_2 - \alpha_1{\alpha_2}^n)({\alpha_1}^{n+1}\alpha_2 - \alpha_1{\alpha_2}^{n+1})}{(\alpha_2 - \alpha_1)^2}$$
$$+\frac{i({\alpha_1}^{n-1}\alpha_2 - \alpha_1{\alpha_2}^{n-1})({\alpha_1}^{n+2}\alpha_2 - \alpha_1{\alpha_2}^{n+2})}{(\alpha_2 - \alpha_1)^2} - \frac{({\alpha_1}^{n-1}\alpha_2 - \alpha_1{\alpha_2}^{n-1})({\alpha_1}^{n+1}\alpha_2 - \alpha_1{\alpha_2}^{n+1})}{(\alpha_2 - \alpha_1)^2}$$
$$-\frac{({\alpha_1}^{2n+2}{\alpha_2}^2 + {\alpha_1}^2{\alpha_2}^{2n+2} - 2{\alpha_1}^{n+2}{\alpha_2}^{n+2})}{(\alpha_2 - \alpha_1)^2} + \frac{({\alpha_1}^{2n}{\alpha_2}^2 + {\alpha_1}^2{\alpha_2}^{2n} - 2{\alpha_1}^{n+1}{\alpha_2}^{n+1})}{(\alpha_2 - \alpha_1)^2}$$
$$-\frac{2i({\alpha_1}^{n+1}\alpha_2 - \alpha_1{\alpha_2}^{n+1})({\alpha_1}^n\alpha_2 - \alpha_1{\alpha_2}^n)}{(\alpha_2 - \alpha_1)^2}$$

Now, after some algebra, we get

$$\frac{i[{\alpha_1}^n{\alpha_2}^n({\alpha_1}^2\alpha_2 + \alpha_1{\alpha_2}^2 - {\alpha_1}^3 - {\alpha_2}^3)] - {\alpha_1}^n{\alpha_2}^n[\alpha_1{\alpha_2}^3 + {\alpha_1}^3\alpha_2 - {\alpha_1}^2 - {\alpha_2}^2 - 2{\alpha_1}^2{\alpha_2}^2 + 2\alpha_1\alpha_2]}{(\alpha_2 - \alpha_1)^2}$$
$$\frac{i({\alpha_1}^n{\alpha_2}^n({\alpha_1}^2 - {\alpha_2}^2)(\alpha_2 - \alpha_1) - {\alpha_1}^n{\alpha_2}^n(\alpha_1\alpha_2 - 1)(\alpha_1 - \alpha_2)^2}{(\alpha_2 - \alpha_1)^2}$$
$$= -i({\alpha_1}^n{\alpha_2}^n)(\alpha_2 + \alpha_1) - ({\alpha_1}^n{\alpha_2}^n)(\alpha_1\alpha_2 - 1)$$
$$= -i({\alpha_1}^n{\alpha_2}^n)2 - ({\alpha_1}^n{\alpha_2}^n)(\alpha_1\alpha_2 - 1)$$
$$= (-GP_{k,1}(1))^n(-2i + GP_{k,1}(1) + 1)$$

which concludes the proof.

Theorem 2.11. $\lim_{n\to\infty}\frac{GP_{k,n}(1)}{GP_{k,n-1}(1)} = \alpha_2.$

*Proof***:** Using Binet's formula

$$= \frac{{\alpha_1}^{n+1}\alpha_2 - \alpha_1{\alpha_2}^{n+1} + i\alpha_2{\alpha_1}^n - \alpha_1{\alpha_2}^n}{{\alpha_1}^n\alpha_2 - \alpha_1{\alpha_2}^n + i\alpha_2{\alpha_1}^{n-1} - \alpha_1{\alpha_2}^{n-1}}$$

$$= \frac{\left(\left(\frac{\alpha_1}{\alpha_2}\right)^n \alpha_1 - \alpha_1\alpha_2\right) + i\left(\alpha_2\left(\frac{\alpha_1}{\alpha_2}\right)^2 - \alpha_1\right)}{\left(\left(\frac{\alpha_1}{\alpha_2}\right)^n \alpha_2 - \alpha_1\right) + i\left(\alpha_2\left(\frac{\alpha_1}{\alpha_2}\right)^n \alpha_1^{-1} - \alpha_1\alpha_2^{-1}\right)}$$

$$= \frac{\alpha_1\alpha_2 + i\alpha_1}{\alpha_1 + i\frac{\alpha_1}{\alpha_2}} = \frac{\alpha_1\alpha_2(\alpha_2 + i)}{\alpha_1(\alpha_2 + i)} = \alpha_2$$

as desired.

3. CONCLUSION

In this study, the generalized (k, r) – Gauss Pell numbers have been introduced and studied. Several properties of these numbers were deduced. In addition, the generating functions, some important identity, and sum of the terms of the generalized (k, r) – Pell numbers were given. Also, we showed another way of finding the generalized (k, r) – Gauss Pell sequence from the generating function with graphs. These identities could be used to develop new identities of the numbers.

Acknowledgments: *The authors appreciate the editor and anonymous referees for their careful corrections, helpful suggestions, and valuable comments on the original version of this paper.*

REFERENCES

[1] Falcon, S., *General Mathematics Notes*, **25**(2), 148, 2014.
[2] Falcon, S., Plaza, A., *Chaos, Solitons &Fractals*, **32**(5), 1615, 2007.
[3] Mikkawy, M., Sogabe, T., *Applied Mathematics and Computation*, **215**, 4456, 2010.
[4] Özkan, E., *Indian Journal of Pure and Applied Mathematics*, **38**(4), 241, 2007.
[5] Dikici, R., Özkan, E., *Applied Mathematics and Computation*, **144**(2-3), 323, 2003.
[6] Özkan, E., Altun, İ., *Communications in Algebra*, **47**(10), 4020, 2019.
[7] Panwar, A., Sisodiya, K., Rathore, G.P.S., *International Journal of Mathematics Trends and Technology*, **51**(2), 146, 2017.
[8] Bednarz, U., Wloch, A., Wolowiec-Musial, M., *Commentationes Mathematicae*, **53**(1), 35, 2013.
[9] Catarino, P., *International Journal of Mathematical Analysis*, **7**(38), 1877, 2013.
[10] Wloch, I., Bednarz, U., Brod, D., Wloch, A., Wolowic-Musial, M., *Discrete Applied Mathematics*, **161**, 2695, 2013.
[11] Belbachir, H., Bencherif, F., *Ars Combinatoria*, **102**, 305, 2011.
[12] Bród, D., *Annales Mathematicae Silesianae*, **33**, 66, 2019.
[13] Fayeab, B., Lucab, F., *Annales Mathematicae et Informaticae*, **45**, 55, 2015.
[14] Horadam, A. F., *Fibonacci Quarterly*, **9**(3), 245, 1971.
[15] Horadam, A. F., *Ulam Quarterly*, **3**, 34, 1994.
[16] Panda, G. K., Sahukar, M. K., *Proceedings - Mathematical Sciences*, **130**(25), 1, 2020.
[17] Berzsenyi, G., *The Fibonacci Quarterly*, **15**, 223, 1977.
[18] Özkan, E., Taştan, M., *Communications in Algebra*, **48**(3), 952, 2020.
[19] Taş, S., *Balıkesir Üniversitesi Fen Bilimleri Enstitüsü Dergisi*, **21**(1), 184, 2019.
[20] Brod, D., Piejko, K., Wloch, I., *Ars Combinatoria*, **112**, 397, 2013.