ORIGINAL PAPER

SOLITARY WAVE SOLUTIONS OF SOME CONFORMABLE TIME-FRACTIONAL COUPLED SYSTEMS VIA AN ANALYTIC APPROACH

ANIQA ZULFIQAR[1], JAMSHAD AHMAD[1]

Manuscript received: 09.12.2020; Accepted paper: 18.05.2021
Published online: 30.06.2021.

Abstract. *The main purpose of this research is to inquire the new solitary wave solution of the coupled time-fractional models to validate the influence and proficiency of the planned variational iteration method (VIM) using conformable derivative definition. Applications to four demanding nonlinear problems like Hirota-Satsuma coupled KdV equations, modified Boussinesq (MB) equation, approximate long wave (ALW) equation and Drinfeld-Sokolov-Wilson (DSW) equation demonstrate the efficiency and the robustness of the method. An analysis of the consequences with effects of relevant parameters and comparison with the exact solution presented with the help of graphs tables and gives the further understanding of numerical results by others. The convergence of the method is illustrated numerical and their physical significance is discussed.***

Keywords: *Conformable fractional derivative; conformable variational iteration method; modified Boussinesq equation; Hirota-Satsuma coupled KdV equation; long wave equation; Drinfeld-Sokolov-Wilson (DSW) equation; solitary wave solution.*

1. INTRODUCTION

The nonlinear coupled mathematical models are used to model most of the natural problems, such as ocean engineering, optical fibers, chemical-physics, fluid dynamics, biology, plasma physics and other fields of engineering. For diagnosing these mathematical models as well as in addition follow these physical models in realistic mathematical studies, it is essential to discover their approximate and exact solutions that support in understanding the phenomena. In fractional calculus a lot of numerical, analytical and approximate methods are developed for handling nonlinear models [1-7].

There are miscellaneous researches correlated to this fractional derivative. Atangana et al. [8] give a few definitions of the conformable derivative (CD). The importance and applications of conformable fractional derivatives have been addressed by many researchers [9-11]. The Time-space fractional heat differential equations are being resolved with the aid of the CD delivered by Cenesiz and Kurt [12]. New exact solutions of conformable Burgers' type equations obtained by Çenesiz et al. [13]. Some new solutions were presented of Boussinesq and combined KdV-mKdV equations, Drinfeld-Sokolov-Wilson system in shallow water waves and generalized Hirota-Satsuma system of equations by Tasbozan et al. [14-16].

The HS equations ware familiarized in [17] and these equations examine the collaboration of two long waves with diverse dispersion associations and happen as a specific case of the Toda lattice equation, and these models possessive many potential applications in

[1] University of Gujrat, Department of Mathematics, 50700 Punjab, Pakistan.
E-mail: jamshadahmadm@gmail.com; aniqa_zulfiqa@yahoo.com.

many nonlinear science fields. The solutions of these equations are studied by using many methods such as modified simple equation method [18], new iterative method [19], homotopy perturbation method (HPM) [20] and many other schemes.

Whitham, Broer and Kaup [21-24] put the Whitham-Broer-Kaup (WBK) equations as an expansion of shallow water waves and are of the form

$$\begin{aligned} D_t^\alpha u &= -uu_x - v_x - bu_{xx},\ 0<\alpha\le 1, \\ D_t^\alpha v &= -uv_x - vu_x - au_{xxx} + bv_{xx}, \end{aligned} \tag{1}$$

Drinfeld, Solokov and Wilson derived an equation known as DSW equation which is based on the model of water waves [25-26]. Generalized time fractional DSW equation is written in the form of

$$\begin{aligned} D_t^\alpha u &= -pvv_x,\ 0<\alpha\le 1, \\ D_t^\beta v &= -qv_{xxx} - ruv_x - su_x v,\ 0<\beta\le 1, \end{aligned} \tag{2}$$

where $p, q, r\ and\ s$ are nonzero parameters and α and β are the order of fractional derivative. By taking α = β the Eq. (2) reduce to Eq. (3), which was presented in [27].

$$\begin{aligned} D_t^\alpha u &= -pvv_x,\ 0<\alpha\le 1, \\ D_t^\alpha v &= -qv_{xxx} - ruv_x - su_x v. \end{aligned} \tag{3}$$

These equations are premeditated by numerous authors through different techniques like, Laplace ADM [28], q-homotopy analysis transform method [29] and other techniques. The paper is organized as follows: Numerical implementation of methods for solving Hirota-Satsuma coupled KdV equations, modified Boussinesq (MB) equation, approximate long wave (ALW) equation and Drinfeld-Sokolov-Wilson (DSW) equation in section 1. Results and discussion is given in section 2. Lastly, a conclusion section is given in section 3.
In literature, primary definitions of fractional calculus are given which can be practiced in the calculation. It is famous that there are precise definitions of fractional Integral and fractional derivatives, which includes, Grünwald-Letnikov, Riesz, Riemann-Liouville (RL), Caputo, Hadamard and Erdélyi-Kober and lots of others [30-32]. The fractional CVIM has been carried out in many models with the aid of many authors [33-35]. In this work we shall mainly focus on behavior of CVIM. The conformable fractional derivative (CFD) takes into consideration for this examine because it is simple for the calculation. He [36-38] presented the standard VIM and implemented on many differential equations. CVIM is built on the CD for fractional ODEs and CVIM for FPDEs is presented in [39].

2. MATERIALS AND METHODS

2.1. MATERIALS

Some primary definitions of fractional calculus are recalled which can be practiced in the calculation.

Definition 1. The fractional integral operator of RL of order $\alpha \geq 0$of a function $f \in C\mu, \mu \geq -1$ is defined as

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} (x-\upsilon)^{\alpha-1} d\upsilon, \ \alpha > 0, \quad x > 0 \tag{4}$$

$$J^{0} f(x) = f(x). \tag{5}$$

Definition 2. The Caputo's fractional derivative is [32]

$$D^{\alpha} t^{\xi} = \begin{cases} \dfrac{\Gamma(\xi+1)}{\Gamma(\xi+1-\alpha)} t^{\xi-\alpha}, \ \xi \in \mathrm{N}_0, \xi \geq \lceil \alpha \rceil \\ 0, \qquad\qquad \xi \in \mathrm{N}_0, \xi < \lceil \alpha \rceil, \end{cases} \tag{6}$$

$$D^{\alpha} c = 0, \ \ c \in \Re..$$

Definition 3. Let f be an n time differentiable at x then the CFD is well-defined by [40]

$$(T_{\alpha} f)(x) = \lim_{\varepsilon \to 0} \frac{f^{([\alpha]-1)}\left(x + \varepsilon x^{[\alpha]-\alpha}\right) - f^{([\alpha]-1)}(x)}{\varepsilon}, \ \ x > 0, \ \alpha \in (0,1]. \tag{7}$$

If the above limit exists, then f is called α-differentiable. Let $\alpha \in (0,1]$ and f, g be α-differentiable at a point $x > 0$, then T_{α} satisfies the following properties:

(i) $T_{\alpha}(af + bg) = aT_{\alpha}(f) + bT_{\alpha}(g), \ \ \textit{for all} \ \ a, b \in \Re$

(ii) $T_{\alpha}(x^{p}) = px^{p-\alpha}, \quad \textit{for all} \ \ p \in \Re$

(iii) $T_{\alpha}(\lambda) = 0$, for all constant functions $f(x) = \lambda$

(iv) $T_{\alpha}(fg) = fT_{\alpha}(g) + gT_{\alpha}(f)$

(v) $T_{\alpha}(f/g) = \{gT_{\alpha}(f) - fT_{\alpha}(g)\}/g^{2}$

In addition, f is differentiable $T_{\alpha}(f)(x) = x^{1-\alpha} \dfrac{df}{dx}(x)$.

2.2. METHOD

Firstly the CVIM is discussed for the solution of the following non-linear FPDE

$$T_t^{\alpha} w(x,t) + L(w(x,t)) + N(w(x,t)) = h(x,t), \ n < \alpha \leq n+1,$$

(8)

where L and N is linear and non-linear operator respectively, h(x) is source term and T_t^{α} is CFD of order α. To solve differential Eq. (8) via CFVIM write in the form

$$t^{\lceil \alpha \rceil - \alpha} \frac{\partial^{([\alpha])} w(x,t)}{\partial t^{([\alpha])}} + L(w(x,t)) + N(w(x,t)) = h(x,t). \tag{9}$$

As in CVIM, the CF for Equation (8) can be erected as

$$w_{n+1}(x,t) = w_n(x,t) + \int_0^t \mu(\tau)\left(\tau^{\lceil\alpha\rceil-\alpha}\frac{\partial^{([\alpha])}w_n(x,\tau)}{\partial\tau^{([\alpha])}} + L(w_n(x,\tau)) + N(\tilde{w}_n(x,\tau)) - h(x,\tau)\right)d\tau, \quad (10)$$

Finally, the solution is

$$w(x,t) = \lim_{n\to\infty} w_n(x,t). \quad (11)$$

3. RESULTS AND DISCUSSION

In this section, the CVIM has been attributed to seeking the traveling wave solution of the Hirota-Satsuma coupled KdV equations, modified Boussinesq (MB) equation, approximate long wave (ALW) equation and Drinfeld-Sokolov-Wilson (DSW) equation.

3.1. HS COUPLED KDV EQUATION

$$\begin{aligned} &T_t^{\alpha} u = \frac{1}{2}u_{xxx} - 3uu_x + 3(vw)_x, \\ &T_t^{\alpha} v = -v_{xxx} + 3uv_x, \qquad\qquad 0<\alpha<1, \\ &T_t^{\alpha} w = -w_{xxx} + 3uw_x. \end{aligned} \quad (12)$$

subject to

$$\begin{aligned} &u(x,0) = \frac{\beta-2k^2}{3} + 2k^2\tanh^2(kx), \\ &v(x,0) = \frac{-4k^2c_0(\beta+k^2)}{3c_1{}^2} + \frac{-4k^2(\beta+k^2)\tanh(kx)}{3c_1}, \\ &w(x,0) = c_0 + c_1\tanh^2(kx). \end{aligned} \quad (13)$$

where $k, c_0, c_1 \neq 0$ and β are arbitrary constants. If $c=-\beta$ then these are the traveling wave solutions [41]. The corresponding exact solution is given by

$$\begin{aligned} &u(x,t) = \frac{\beta-2k^2}{3} + 2k^2\tanh^2(k(x-ct)), \\ &v(x,t) = \frac{-4k^2c_0(\beta+k^2)}{3c_1{}^2} + \frac{-4k^2(\beta+k^2)\tanh(k(x-ct))}{3c_1}, \\ &w(x,t) = c_0 + c_1\tanh^2(k(x-ct)). \end{aligned} \quad (14)$$

Following the procedure described above, consequently the independent approximate solutions are

$$
\begin{aligned}
u_0 &= \frac{\beta - 2k^2}{3} + 2k^2 \tanh^2(kx), \\
v_0 &= \frac{-4k^2 c_0(\beta + k^2)}{3{c_1}^2} + \frac{-4k^2(\beta + k^2)\tanh(kx)}{3c_1}, \\
w_0 &= c_0 + c_1 \tanh(kx).
\end{aligned}
\tag{15}
$$

$$
u_1 = \frac{\beta-2k^2}{3} + 2k^2\tanh^2(kx) - t\left(\begin{array}{l}
12k^3\operatorname{sech}^2(kx)\tanh(kx)\left(\dfrac{\beta-2k^2}{3}+2k^2\tanh^2(kx)\right)+ \\
\dfrac{1}{2}\left(32k^5\operatorname{sech}^4(kx)\tanh(kx)-16k^5\operatorname{sech}^2(kx)\tanh^3(kx)\right) \\
-3\left(\begin{array}{l}
k\operatorname{sech}^2(kx)c_1\left(-\dfrac{4k^2c_0(\beta+k^2)}{3c_1^2}-\dfrac{4k^2(\beta+k^2)\tanh(kx)}{3c_1}\right)- \\
\dfrac{4k^3(\beta+k^2)\operatorname{sech}^2(kx)(c_0+c_1\tanh(kx))}{3c_1}
\end{array}\right)
\end{array}\right),
$$

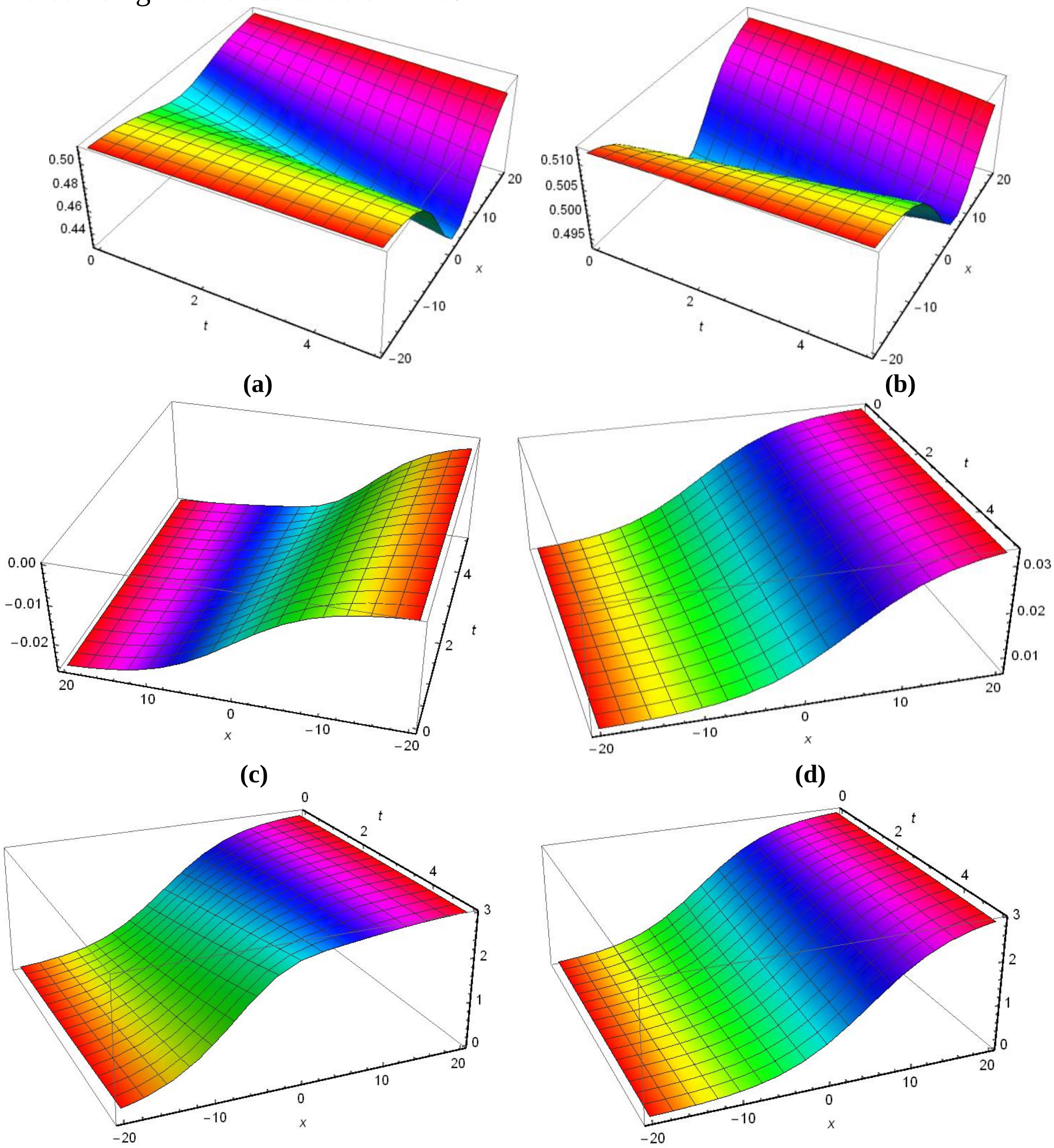
$$
v_1 = \frac{-4k^2c_0(\beta+k^2)}{3c_1^2} + \frac{-4k^2(\beta+k^2)\tanh(kx)}{3c_1} + t\left(\begin{array}{l}
\dfrac{8k^5(\beta+k^2)\operatorname{sech}^4(kx)}{3c_1}- \\
\dfrac{16k^5(\beta+k^2)\operatorname{sech}^2(kx)\tanh^2(kx)}{3c_1}+ \\
\dfrac{4k^3(\beta+k^2)\operatorname{sech}^2(kx)\left(\frac{1}{3}(\beta-2k^2)+2k^2\tanh^2(kx)\right)}{c_1}
\end{array}\right),
$$

$$
w_1 = c_0 + c_1\tanh(kx) - t\left(\begin{array}{l}
-2k^3\operatorname{sech}^4(kx)c_1 + 4k^3\operatorname{sech}^2(kx)c_1\tanh^2(kx) \\
-3k\operatorname{sech}^2(kx)c_1\left(\dfrac{1}{3}(-2k^2+\beta)+2k^2\tanh^2(kx)\right)
\end{array}\right).
\tag{16}
$$

$$
u_2 = \frac{1}{3}\left(\begin{array}{l}
4k^2+\beta-6k\left(1+2k^2t^2\left(8k^4+8k^2\beta+\beta^2\right)\right)\sec h^2(kx)+18k^4t^2\left(8k^4+8k^2\beta+\beta^2\right)\sec h^4(kx) \\
+\dfrac{1}{(-2+\alpha)c_1^2}4k^3t^{1-\alpha}\sec h^2(kx)\left(\begin{array}{l}
\\
96k^4t^{2+\alpha}(-2+\alpha)\left(k^2+\beta\right)^2\sec h^2(kx)c_0^2\tanh(kx) \\
-\left(\begin{array}{l}
3\left(t+2t^{\alpha}\right)(-2+\alpha)\left(4k^2+3\beta\right)+4k^2t^{2+\alpha}(-2+\alpha) \\
\sec h^2(kx)\left(\begin{array}{l}
-96k^6-144k^4\beta-55k^2\beta^2-\beta^3+9k^2 \\
\left(4k^2+3\beta\right)^2\sec h^2(kx)
\end{array}\right)
\end{array}\right) \\
c_1^2\tanh(kx)+6\left(k^2+\beta\right)c_0c_1\left(\begin{array}{l}
-t-2t^{\alpha}(-2+\alpha) \\
\left(\begin{array}{l}
1+k^3t \\
\left(\begin{array}{l}
-2kt\left(4k^2+3\beta\right)\left(-3+2\cosh(2kx)\right) \\
\sec h^4(kx)+\left(1-3\sec h^2(kx)\right)\tanh(kx)
\end{array}\right)
\end{array}\right)
\end{array}\right)
\end{array}\right)
\end{array}\right)
$$

$$v_2=\frac{1}{3c_1^2}4k^2\left(k^2+\beta\right)\left(c_0\left(-1-4k^4t^2\right)\left(k^2+\beta\right)\operatorname{sech}^4(kx)\left(-3+4kt\tanh(kx)\right)\right)+\frac{1}{-2+\alpha}t^{-\alpha}c_1$$
$$\left(8k^5t^{3+\alpha}(-2+\alpha)\operatorname{sech}^4(kx)\left(kt\beta\left(4k^2+3\beta\right)-3\left(k^2+\beta\right)\tanh(kx)\right)+kt\beta\operatorname{sech}^2(kx)\begin{pmatrix}-t+t^{\alpha}(-2+\alpha)\\ \left(-2+kt\beta\tanh(kx)\right)\end{pmatrix}\right),$$
$$w_2=c_0\begin{pmatrix}1+4k^4t^2\left(k^2+\beta\right)\operatorname{sech}^4(kx)\left(-3+kt\beta\tanh(kx)\right)\frac{1}{-2+\alpha}t^{-\alpha}c_1\\ \begin{pmatrix}-8k^5t^{3+\alpha}(-2+\alpha)\beta\left(4k^2+3\beta\right)\operatorname{sech}^6(kx)+t^{\alpha}(-2+\alpha)\tanh(kx)\\ +8k^4t^{2+\alpha}(-2+\alpha)\operatorname{sech}^4(kx)\left(kt\beta\left(4k^2+3\beta\right)-3\left(k^2+\beta\right)\tanh(kx)\right)\\ +kt\beta\operatorname{sech}^2(kx)\left(-t+t^{\alpha}(-2+\alpha)\left(-2+kt\beta\tanh(kx)\right)\right)\end{pmatrix}\end{pmatrix} \tag{17}$$

Continuing in the similar manners, the higher order approximate solution can be calculated using Mathematica version 10.4



(a) (b)

(c) (d)

(e) (f)

Figure 1. Comparison of three dimensional surface plots of solitary wave solution (17) for differential values of relevant parameters and α with the exact solution (14).

Table 1. Numerical analysis of solution (17) and solution (14) when α =0.5, α =0.75, α =1 and $k = 0.1, \beta = 1.5, c_1 = 1.5, c_0 = 1.5$ for u(x, t).

T	x	CVIM Solution			Exact Solution
		α =0.5	α =0.75	α =1	
0.2	0	0.489240	0.489813	0.490936	0.493351
	0.5	0.488993	0.489606	0.490811	0.493341
	1	0.488870	0.489521	0.490799	0.493431
0.4	0	0.485781	0.486818	0.488576	0.493405
	0.5	0.485279	0.486391	0.488276	0.493335
	1	0.484923	0.486104	0.488104	0.493365
0.6	0	0.482746	0.484107	0.486251	0.493494
	0.5	0.482018	0.483477	0.485777	0.493365
	1	0.481456	0.483004	0.485446	0.493335
0.8	0	0.480060	0.481609	0.483962	0.493619
	0.5	0.479131	0.480792	0.483315	0.493431
	1	0.478382	0.480146	0.482823	0.493341

Table 2. Comparison of CVIM solution and exact when α =0.5, α =0.75, α =1 and $k = 0.1, \beta = 1.5, c_1 = 1.5, c_0 = 1.5$ for v(x, t) of Eq. (17) and Eq. (14).

T	X	CVIM Solution			Exact Solution
		α =0.5	α =0.75	α =1	
0.2	0	−0.014107	−0.014011	−0.013824	−0.013020
	0.5	−0.014775	−0.014680	−0.014493	−0.012351
	1	−0.015436	−0.015342	−0.015156	−0.011687
0.4	0	−0.014689	−0.014517	−0.014224	−0.012618
	0.5	−0.015354	−0.015182	−0.014889	−0.011952
	1	−0.016009	−0.015838	−0.015548	−0.011292
0.6	0	−0.015206	−0.014979	−0.014621	−0.012217
	0.5	−0.015866	−0.015640	−0.015283	−0.011555
	1	−0.016513	−0.016289	−0.015935	−0.010902
0.8	0	−0.015668	−0.015409	−0.015017	−0.011819
	0.5	−0.016322	−0.016064	−0.015673	−0.011162
	1	−0.016961	−0.016706	−0.016318	−0.010516

Table 3. Comparison of CVIM solution (17) and exact solution (14) when α =0.5, α =0.75, α =1 and $k = 0.1, \beta = 1.5, c_1 = 1.5, c_0 = 1.5$ for w(x, t).

T	X	CVIM Solution			Exact Solution
		α =0.5	α =0.75	α =1	
0.2	0	1.576475	1.565817	1.544891	1.502999
	0.5	1.651144	1.640512	1.619639	1.574935
	1	1.725064	1.714512	1.693794	1.649410
0.4	0	1.641618	1.622306	1.589565	1.502999
	0.5	1.715893	1.696629	1.663970	1.574935
	1	1.789103	1.769983	1.737568	1.649410
0.6	0	1.699308	1.673969	1.634022	1.502999
	0.5	1.773054	1.747779	1.707931	1.574935
	1	1.845463	1.820376	1.780825	1.649410
0.8	0	1.750929	1.722074	1.678260	1.502999
	0.5	1.824010	1.795226	1.751523	1.574935
	1	1.895516	1.866947	1.823569	1.649410

3.2. THE MB EQUATIONS

$$\begin{aligned} T_t^{\alpha} u &= -u u_x - v_x, \qquad 0 < \alpha \le 1, \\ T_t^{\alpha} v &= -u v_x - v u_x - u_{xxx}. \end{aligned} \tag{18}$$

subject to

$$\begin{aligned} u(x,0) &= \omega - 2l \coth\left(l(x+c)\right), \\ v(x,0) &= -2l^2 \operatorname{csc}h^2\left(l(x+c)\right), \end{aligned} \tag{19}$$

The corresponding exact solution is given by

$$\begin{aligned} u(x,t) &= \omega - 2l \coth\left(l(x+c-\omega t)\right), \\ v(x,t) &= -2l^2 \operatorname{csc}h^2\left(l(x+c-\omega t)\right), \end{aligned} \tag{20}$$

According to the procedure described above, the consequently approximate independent solutions are

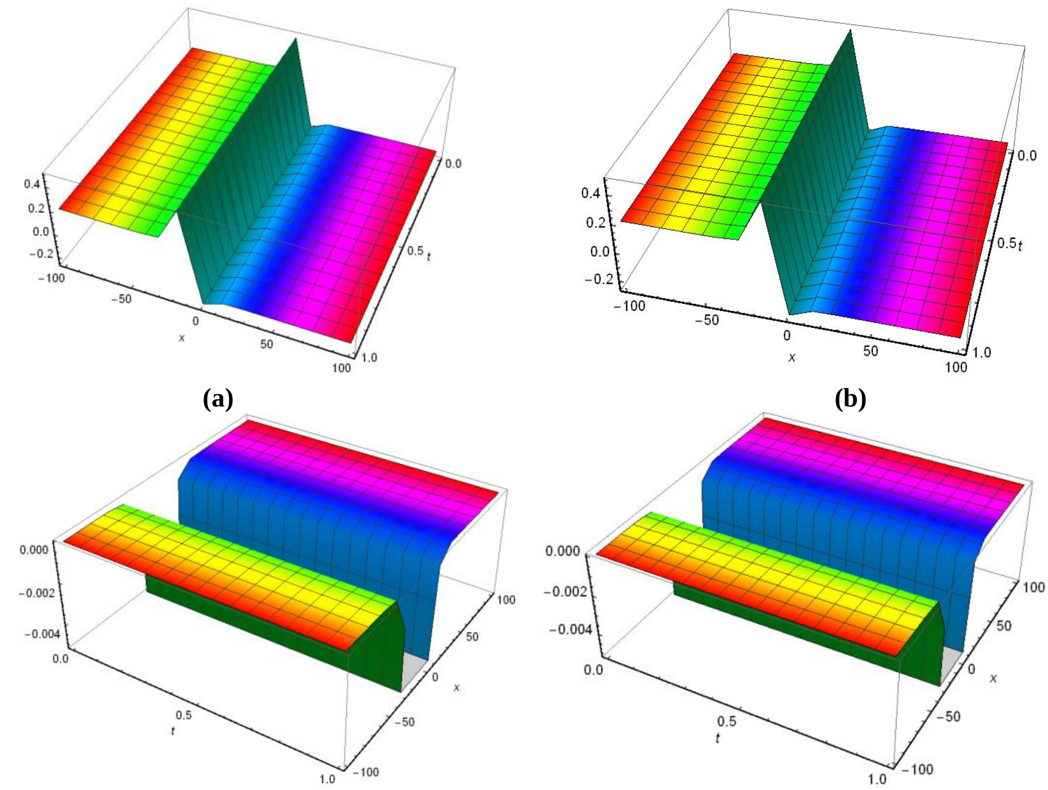
$$\begin{aligned} u_0 &= \omega - 2l \coth\left(l(x+c)\right), \\ v_0 &= -2l^2 \operatorname{csc}h^2\left(l(x+c)\right), \end{aligned} \tag{21}$$

$$\begin{aligned} u_1 &= \omega - 2l \operatorname{csc}h^2\left(l(x+c)\right)\left(2lt\omega + \sinh\left(2l(x+c)\right)\right), \\ v_1 &= l^2 \operatorname{csc}h^4\left(l(x+c)\right)\left(1 + 8l^2 t + (-1+4l^2 t)\cosh\left(2l(x+c)\right) - 2lt\omega \sinh\left(l(x+c)\right)\right). \end{aligned} \tag{22}$$

$$u_2 = \omega - \frac{1}{-2+\alpha}\left(2l^2 t^{1-\alpha}\left(t + 2t^{\alpha}(-2+\alpha)\right)\omega \operatorname{csc}h^2\left(l(c+x)\right)\right) + \frac{2}{3} l \coth\left(l(c+x)\right)$$
$$\left(-3 + l^2 t^2 \operatorname{csc}h^2\left(l(c+x)\right)\left(12l^2 - 3\omega^2 + 4l^2\left(9 + t\omega^2\right)\operatorname{csc}h^2\left(l(c+x)\right)\right)\right),$$

$$v_2 = \frac{1}{12} l^2 \operatorname{csc}h^4\left(l(x+c)\right)\left(\begin{array}{l} 12\left(-1 + 4l^2 t\left(1 + \dfrac{t^{1-\alpha}}{-2+\alpha}\right)\right)\cosh\left(2l(c+x)\right) + \dfrac{1}{-2+\alpha t} t^{-\alpha} \\ \left(\begin{array}{l} 24lt^2\left(4l - \omega \sinh\left(2l(c+x)\right)\right)4t^{\alpha}(-2+\alpha) \\ \left(\begin{array}{l} 3 + 4l^2 t\left(12 - 3t\omega^2 + 8l^2 t\left(-6 + t\omega^2\right)\right) + 2lt \\ \left(\begin{array}{l} -3l\left(-2 + 4l^2 t + t\omega^2\right)\cosh\left(2l(c+x)\right) + \\ 2l^2 t\left(\begin{array}{l} 4\left(9 - 4l^2 t\right)\omega \coth\left(l(c+x)\right) + l \\ \left(-45 + 10t\omega^2 - 36ltw \coth\left(l(c+x)\right)\operatorname{csc}h^2\left(l(c+x)\right)\right) \\ + 6\left(-1 + 2l^2 t\right)\omega \sinh\left(2l(c+x)\right) \end{array}\right) \end{array}\right) \end{array}\right) \end{array}\right) \end{array}\right). \tag{23}$$

The higher order approximate solution can be calculated using Mathematica version 10.4



(a) (b)

(c) (d)

Figure 2. Comparison of three dimensional surface plots of solitary wave solution (23) for differential values of parameters and α with the exact solution (20).

Table 4. Analysis of solution (20) and (23) when α =0.5, α =0.75, α =1 and $\omega = 0.005, l = 0.1, c = 10$ for u(x, t).

t	X	CVIM Solution			Exact Solution
		α =0.5	α =0.75	α =1	
0.2	0	−0.257622	−0.257619	−0.257612	−0.257622
	0.5	−0.250832	−0.250829	−0.250823	−0.250831
	1	−0.244857	−0.244855	−0.244849	−0.244856
0.4	0	−0.257614	−0.257608	−0.257597	−0.257636
	0.5	−0.250828	−0.250822	−0.250813	−0.250843
	1	−0.244856	−0.244851	−0.244843	−0.244867
0.6	0	−0.257585	−0.257577	−0.257564	−0.257651
	0.5	−0.250806	−0.250799	−0.250788	−0.250856
	1	−0.244840	−0.244834	−0.244824	−0.244878
0.8	0	−0.257534	−0.257525	−0.257511	−0.257665
	0.5	−0.250768	−0.250760	−0.250748	−0.250869
	1	−0.244811	−0.244804	−0.244793	−0.244889

Table 5. Comparison when α =0.5, α =0.75,α =1 and $\omega = 0.005, l = 0.1, c = 10$for v(x, t) of Eq. (20) and Eq. (23)

t	X	CVIM Solution			Exact Solution
		α =0.5	α =0.75	α =1	
0.2	0	−0.014079	−0.014136	−0.014246	−0.014485
	0.5	−0.012391	−0.012437	−0.012528	−0.012725
	1	−0.010936	−0.010975	−0.011050	−0.011214
0.4	0	−0.013741	−0.013844	−0.014017	−0.014489
	0.5	−0.012112	−0.012196	−0.012339	−0.012728
	1	−0.010705	−0.010775	−0.010893	−0.011217

t	X	CVIM Solution			Exact Solution
		α =0.5	α =0.75	α =1	
0.6	0	−0.013448	−0.013582	−0.013793	−0.014493
	0.5	−0.011870	−0.011979	−0.012154	−0.012731
	1	−0.010503	−0.010595	−0.010739	−0.0112194
0.8	0	−0.013190	−0.013343	−0.013574	−0.014496
	0.5	−0.011657	−0.011782	−0.011973	−0.012734
	1	−0.010326	−0.010430	−0.010588	−0.011222

3.3. THE LONG WAVE (ALW) EQUATIONS

$$\begin{aligned} T_t^{\alpha}u &= -uu_x - v_x - \frac{1}{2}v_{xx}, \ \ 0<\alpha\le 1, \\ T_t^{\alpha}v &= -uv_x - vu_x + \frac{1}{2}v_{xx}. \end{aligned} \tag{24}$$

subject to

$$\begin{aligned} u(x,0) &= \omega - l\coth(l(x+c)), \\ v(x,0) &= -l^2 \operatorname{csc}h^2(l(x+c)). \end{aligned} \tag{25}$$

The corresponding exact solution is

$$\begin{aligned} u(x,t) &= \omega - 2l\coth(l(x+c-\omega t)), \\ v(x,t) &= -2l^2 \operatorname{csc}h^2(l(x+c-\omega t)), \end{aligned} \tag{26}$$

According to the procedure described above, the consequently approximate independent solutions are

$$\begin{aligned} u_0 &= \omega - l\coth(l(x+c)), \\ v_0 &= -l^2 \operatorname{csc}h^2(l(x+c)). \end{aligned} \tag{27}$$

$$\begin{aligned} u_1 &= \omega - l^2 t\omega \operatorname{csc}h^2(l(x+c)) - \coth(l(x+c))\left(l + 2l^3 t \operatorname{csc}h^2(l(x+c))\right), \\ v_1 &= -l^2 \operatorname{csc}h^2(l(x+c))\begin{pmatrix} -1 - 2lt\omega\coth(l(x+c)) + 2l^2 t\coth^2(l(x+c)) \\ +l^2 t \operatorname{csc}h^2(l(x+c)) \end{pmatrix}. \end{aligned} \tag{28}$$

$$u_2 = \omega + \frac{1}{3(-2+\alpha)} l t^{-\alpha} \left(\begin{array}{l} lt\omega \operatorname{csc}h^2(l(x+c)) \begin{pmatrix} -3t + t^{\alpha}(-2+\alpha) \\ \begin{pmatrix} -6(1+3l^2 t) + l^2 t \operatorname{csc}h^2(l(x+c)) \\ \left(-27 + 8l^2 t + 10 l^2 t \operatorname{csc}h^2(l(x+c))\right) \end{pmatrix} \end{pmatrix} \\ +\coth(l(x+c)) \begin{pmatrix} -3t^{\alpha}(-2+\alpha) + l^2 t \operatorname{csc}h^2(l(x+c)) \\ \begin{pmatrix} -6t + t^{\alpha}(-2+\alpha) \\ \begin{pmatrix} 3(-4 + 7l^2 t - t\omega^2) + l^2 t \operatorname{csc}h^2(l(x+c)) \\ \left(57 + 8l^2 t + 2t\omega^2 + 12 l^2 t \operatorname{csc}h^2(l(x+c))\right) \end{pmatrix} \end{pmatrix} \end{pmatrix} \end{array} \right),$$

$$v_2=\frac{1}{6}l^2\operatorname{csc}h^2\left(l\left(x+c\right)\right)\left(-6+12l^2t+\frac{1}{-2+\alpha}lt^{1-\alpha}\left(\begin{array}{l} t^{\alpha}\left(-2+\alpha\right) \\ \left(\begin{array}{l} -12l\left(-1+t\left(l^2+\omega^2\right)\right)+24\left(-1+l^2t\right)\omega\coth\left(l\left(x+c\right)\right)+ \\ 2l\left(\begin{array}{l} 18-l^2t\left(15+16l^2t\right)+t\left(-9+8l^2t\right)\omega^2 \\ +4lt\left(9+2l^2t\right)\omega\coth\left(l\left(x+c\right)\right) \end{array}\right)\operatorname{csc}h^2\left(l\left(x+c\right)\right) \\ +l^3t\left(-15-112l^2t+20t\omega^2+12lt\omega\coth\left(l\left(x+c\right)\right)\right) \\ \operatorname{csc}h^4\left(l\left(x+c\right)\right)-84l^5t^2\operatorname{csc}h^6\left(l\left(x+c\right)\right) \end{array}\right) \\ +6t\left(-2\omega\coth\left(l\left(x+c\right)\right)+l\left(2+3\operatorname{csc}h^2\left(l\left(x+c\right)\right)\right)\right) \end{array}\right)\right). \tag{29}$$

The higher order approximate solution can be calculated using Mathematica version 10.4.

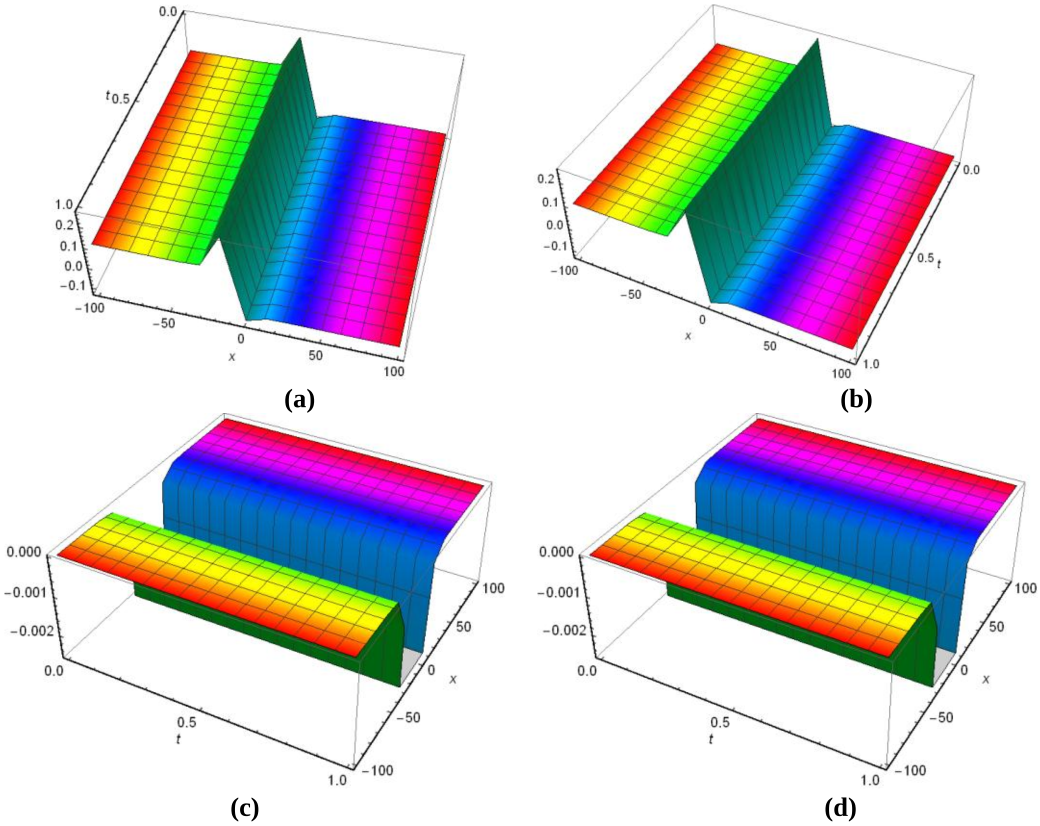


Figure 3. Comparison of three dimensional surface plots of (29) for differential values of parameters and α with the exact solution (26)

Table 6. Numerical analysis of solution (29) and (26) when α =0.5, α =0.75, α =1 and $\omega = 0.005, l = 0.1, c = 10$ for u(x, t).

t	X	CVIM Solution α =0.5	CVIM Solution α =0.75	CVIM Solution α =1	Exact Solution
0.2	0	−0.126955	−0.126864	−0.126683	−0.126311
	0.5	−0.123468	−0.123389	−0.123235	−0.122915
	1	−0.120403	−0.120336	−0.120203	−0.119928
0.4	0	−0.127496	−0.127329	−0.127048	−0.126318
	0.5	−0.123932	−0.123789	−0.123548	−0.122922
	1	−0.120805	−0.120682	−0.120474	−0.119933
0.6	0	−0.127959	−0.127740	−0.127397	−0.126325
	0.5	−0.124331	−0.124144	−0.123849	−0.122928
	1	−0.121150	−0.120989	−0.120736	−0.119939
0.8	0	−0.128356	−0.128107	−0.127730	−0.126332
	0.5	−0.124675	−0.124462	−0.124139	−0.122934
	1	−0.121449	−0.121266	−0.120988	−0.119945

Table 7. Numerical analysis of CVIM solution (29) and exact solution (26) when α =0.5, α =0.75, α =1 and $\omega = 0.005, l = 0.1, c = 10$ for v(x, t).

t	X	CVIM Solution α =0.5	CVIM Solution α =0.75	CVIM Solution α =1	Exact Solution
0.2	0	−0.007141	−0.007155	−0.007182	−0.007243
	0.5	−0.006279	−0.006290	−0.006312	−0.006362
	1	−0.005538	−0.005547	−0.005566	−0.005607
0.4	0	−0.007057	−0.007082	−0.007124	−0.007244
	0.5	−0.006209	−0.006230	−0.006265	−0.006364
	1	−0.005480	−0.005497	−0.005526	−0.005608
0.6	0	−0.006982	−0.007015	−0.007067	−0.007246
	0.5	−0.006148	−0.006175	−0.006218	−0.006366
	1	−0.005429	−0.005452	−0.005486	−0.005610
0.8	0	−0.006915	−0.006952	−0.007009	−0.007248
	0.5	−0.006093	−0.006124	−0.006171	−0.006367
	1	−0.005383	−0.005409	−0.005448	−0.005611

3.4. THE DSW EQUATION

$$\begin{aligned} &T_t^{\alpha} u = -pvv_x, \ 0<\alpha\le 1, \\ &T_t^{\alpha} v = -qv_{xxx} - ruv_x - su_x v. \end{aligned} \tag{30}$$

where p, q, r *and* s are nonzero parameters [42], subject to conditions

$$\begin{aligned} &u(x,0) = 3\operatorname{sech}^2(x), \\ &v(x,0) = 2\operatorname{sech}(x). \end{aligned} \tag{31}$$

The corresponding exact solution of equation (30) is given by

$$\begin{aligned} &u(x,t) = 3\operatorname{sech}^2(x-2t), \\ &v(x,t) = 2\operatorname{sech}(x-2t). \end{aligned} \tag{32}$$

According to the procedure described above, the consequently approximate independent solutions are

$$
\begin{aligned}
u_0 &= 3\sec h^2(x),\\
v_0 &= 2\sec h(x),
\end{aligned}
\tag{33}
$$

$$
\begin{aligned}
u_1 &= 3\sec h^2(x) + 4pt\sec h^2(x)\tanh(x),\\
v_1 &= 2\sec h(x) - t\left(-6r\sec h^3(x)\tanh(x) - 12s\sec h^3(x)\tanh(x)\right),
\end{aligned}
\tag{34}
$$

$$
u_2 = 3\sec h^2(x) + 4pt\sec h^2(x)\tanh(x) - 2p\sec h^2(x)\begin{pmatrix} -\dfrac{2t^{2-\alpha}\tanh(x)}{-2+\alpha} + \\ t\begin{pmatrix} -2\tanh(x) + 6(r+2s)^2 t^2\sec h^6(x)\tanh(x) \\ -12(r+2s)^2 t\sec h(x)^2\tanh(x)^2 \\ -3(r+2s)t\sec h^4(x)\begin{pmatrix} -1+6(r+2s) \\ t\tanh^3(x)\end{pmatrix}\end{pmatrix}\end{pmatrix},
$$

$$
\begin{aligned}
v_2 &= 2\sec h(x) - t\left(-6r\sec h^3(x)\tanh(x) - 12s\sec h^3(x)\tanh(x)\right) - t\sec h^3(x)\\
&\begin{pmatrix} (r+2s)t\sec h^4(x)(9r+8p(r+s)t\tanh(x)) - \dfrac{1}{-2+\alpha}\begin{pmatrix} 2(r+2s)t^{-\alpha}\tanh(x) \\ (3(t+t^{\alpha}(-2+\alpha)\tanh(x)))\end{pmatrix} \\ t\sec h^2(x)(-4ps + 9(3r^2+8rs+4s^2)\tanh^2(x) + 8p(3r^2+8rs+4s^2)t\tanh^3(x))\end{pmatrix},
\end{aligned}
\tag{35}
$$

The higher order approximate solution can be calculated using Mathematica version 10.4.

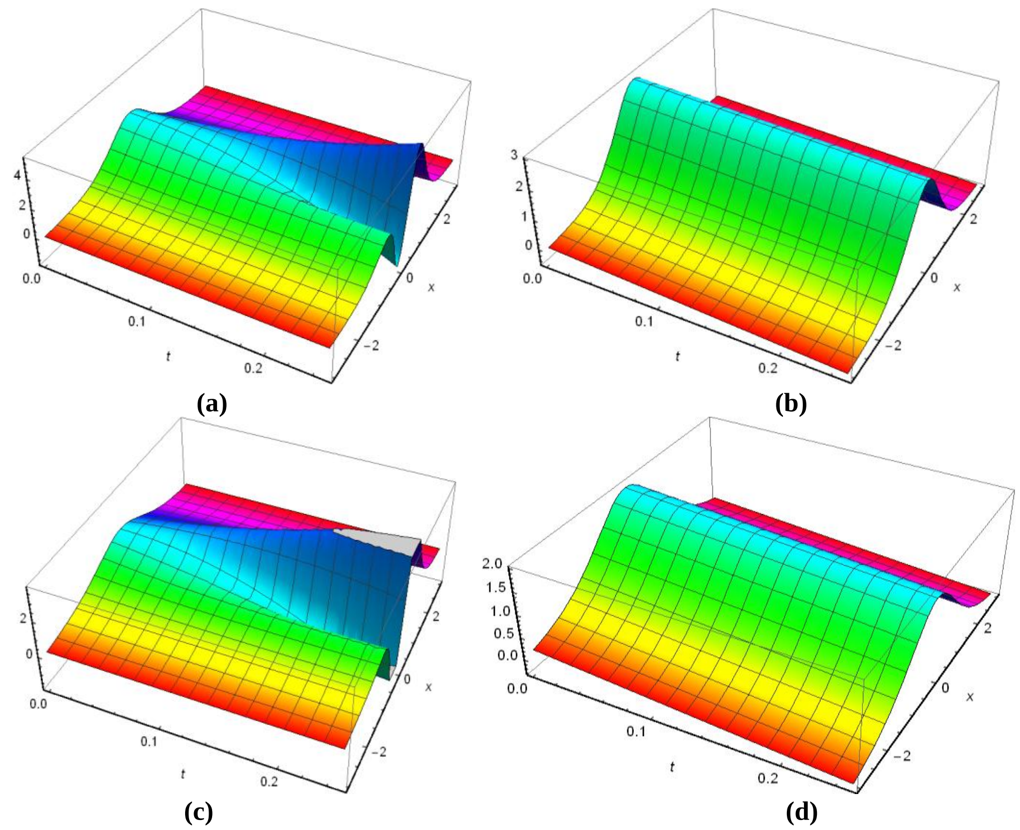


Figure 4. Comparison of three dimensional surface plots of solitary wave solution in (35) for differential values of relevant parameters and α with the exact solution (32).

Table 8. Comparison of CVIM solution (35) and exact solution (32) when α =0.5, α =0.75, α =1 and $p=3, q=2, r=2, s=1$ for u(x, t).

t	X	CVIM Solution			Exact Solution
		α =0.5	α =0.75	α =1	
0.01	0 0.5 1	2.992872 2.384642 1.299998	2.992862 2.394642 1.299998	2.992875 2.348642 1.299921	2.988791 2.339342 1.239921
0.001	0 0.5 1	2.999931 2.367985 1.267541	2.999934 2.367985 1.267546	2.999881 2.363715 1.263791	2.998781 2.357342 1.257921
0.0001	0 0.5 1	2.999993 2.360215 1.260695	2.999993 2.360185 1.260665	2.999983 2.363712 1.260315	2.999831 2.359142 1.259724
0.00001	0 0.5 1	2.999998 2.359432 1.259967	2.999998 2.359432 1.259967	2.999998 2.359392 1.259965	2.999981 2.359322 1.259671

Table 9: Comparison of CVIM solution (35) and exact solution (32) when α =0.5, α =0.75, α =1 and $p=3, q=2, r=2, s=1$ for v(x, t).

t	X	CVIM Solution			Exact Solution
		α =0.5	α =0.75	α =1	
0.01	0 0.5 1	1.991641 1.789742 1.296560	1.991641 1.789742 1.296571	1.991641 1.788642 1.296110	1.982341 1.753642 1.276110
0.001	0 0.5 1	1.999928 1.788952 1.297398	1.999928 1.788015 1.297351	1.999981 1.781382 1.297391	1.998234 1.771642 1.294110
0.0001	0 0.5 1	1.999998 1.775183 1.296915	1.999998 1.775128 1.297065	1.999998 1.774421 1.296615	1.999831 1.773442 1.295912
0.00001	0 0.5 1	1.999998 1.773791 1.296162	1.999998 1.773791 1.296216	1.999998 1.773721 1.296162	1.999981 1.773622 1.296091

3.5. DISCUSSION

The four coupled systems of fractional PDEs (18), (24), (30), and (35) with described ICs are solved with the help of symbolic software Mathematica. The comparison with the previous studied exact solution is discussed in Tables 1-8 for various values of relevant parameters. Through graphical demonstrations, we observe that the soliton is a wave that keeps its shape, preserve after colliding by some other similar wave. Four examples are presented in this work to explore the effectiveness of CVIM. We can see that from obtaining results, the evaluated method gives incredible exactness in comparison to the techniques presented in the literature. Fig. 1 (a)-(f) corresponding to CVIM's results and the exact solution for Solitary wave solution $u(x,t)$, $v(x,t)$ and $w(x,t)$ at $\mathrm{k}=0.1, \beta=1.5, \mathrm{c}_1=1.5, \mathrm{c}_0=1.5$ and α =1 for HSCKdV equation. Fig. 1 (a) and (b) shows the surfaces of approximate solution and exact solution respectively of Eq. (1) which is of bell shaped for $u(x,t)$. Fig1 (c) and (d) are the response to CVIM's results and the exact solution for $v(x,t)$ which is of kink-type. Fig 1. (e) and (f) show the CVIM's results and the exact solution for $w(x,t)$ which is also of kink- type. It is significant to observe that some of the acquired solutions give greater comparability with earlier established solutions.

Fig. 2 (a)–(f) demonstrate the solitary wave solutions behavior for $u(x,t)$, $v(x,t)$ at parameters values $\omega = 0.005, l = 0.1, c = 10$ and α =1 for MB and ALW equations. Fig 2 (a) and (b) shows the CVIM's results and the exact solution for $u(x,t)$. Fig. 2 (c) and (d) are responding to CVIM's results and the exact solution for $v(x,t)$.

Fig 3 (a) and (b) shows the behavior solitary wave solution and the closed form solution for $u(x,t)$. Fig 3 (c) and (d) are responding to CVIM's results and the closed form solution for $v(x,t)$. Fig 4 (a) and (b) illustrates behavior of CVIM's results and the closed form solution for solitary wave solution for $u(x,t)$ and Fig. 4 (c) and (d) $v(x,t)$ at $p = 3, q = 2, r = 2, s = 1$ and α =1.

Tables 1-3 numerically explore the comparison of obtaining solutions with the exact solution for α =0.5, α =0.75 and α =1 at $k = 0.1, \beta = 1.5, c_1 = 1.5, c_0 = 1.5$. Table 4-7 numerically present the comparison of obtaining solutions with the exact solution for α =0.5, α =0.75 and α =1 at $\omega = 0.005,\ l = 0.1,\ c = 10$. Tables 8 and 9 numerically present the comparison of obtaining solutions with the exact solution for α =0.5, α =0.75 and α =1 at $p = 3,\ q = 2,\ r = 2,\ s = 1$. In all instances, we achieve identical solitary wave solutions for various values of parameters which absolutely show that the final solution is not always based on these parameters effectively. Therefore, we can consider random values of these parameters as input into our solutions.

4. CONCLUSIONS

In this paper, the fundamental objective of CVIM has been utilized to search the new solitary wave solutions and has been efficaciously applied to study HS coupled KdV, MB equations, ALW equations and DSW equations of time-fractional order. The consequences of comparison of the exact solutions with those acquired by CVIM show that CVIM is influential, compatible and proficient technique for physical phenomena of nonlinear PDEs of fractional order. Some surface plots have been given to demonstrate the dynamical behavior of the obtained solutions when the parameters take some special values.

It is observed that the obtained approximate solutions are very close to the exact solution [40-41] at particularly α =1. And we also explained the physical meaning of each one estimated wave behavior through figures. It is apparent from the analysis that these models give rise to a variety of solitary wave solutions that clarify the complex physical phenomena. The outcomes attained through this new technique are straight forward and very encouraging to determine solitary wave solutions of other NLDEs.

REFERENCES

[1] Podlubny, I., *IEEE Transactions on Automatic Control*, **44**, 208, 1999.

[2] Atangana, A., *Derivative with a new parameter: Theory, methods and applications*, Academic Press, 2015.

[3] Yang, X.J., Baleanu, D., Srivastava, H.M., *Local fractional integral transforms and their applications*, Academic Press, 2015.

[4] Khan, Y., *Neural Computing and Applications*, **23**, 411, 2013.

[5] Nikkar, A., Mighani, M., *International Journal of Mathematical Sciences*, **7**, 94, 2013.

[6] Khan, M., Gondal, M.A., *Journal of Advanced Research in Differential Equations*, **2**, 53, 2010.

[7] Wazwaz, A.M., *Chaos, Solitons & Fractals*, **37**, 1136, 2008.
[8] Atangana, A., Baleanu, D., Alsaedi, A., *Open Mathematics*, **13**, 889, 2015.
[9] Bohner, M., Hatipoğlu, V.F., *Mathematical Methods in the Applied Sciences*, **41**, 9010, 2018.
[10] Bohner, M., Hatipoğlu, V.F., *Nonlinear Analysis: Hybrid Systems*, **32,** 157, 2019.
[11] Rosales, J. J., Godinez, F.A. V., Banda, G. H., *Optik*, **178** 1010, 2019.
[12] Cenesiz, Y., Kurt, A., *Acta Universitatis Sapientiae, Mathematica*, **7**, 130, 2015.
[13] Çenesiz, Y., Baleanu, D., Kurt, A., Tasbozan, O., *Waves in Random and Complex Media*, **27**, 103, 2017.
[14] Tasbozan, O., Cenesiz, Y., Kurt, A., *The European Physical Journal Plus*, **131**, 244, 2016.
[15] Tasbozan, O., Senol, M., Kurt, A., Ozkan, O., *Ocean Engineering*, **161**, 62, 2018.
[16] Kurt, A. et al., Journal *of Ocean Engineering and Science*, **4**, 24, 2019.
[17] Hirota, R., Satsuma, J., *Physics Letters A*, **85**, 407, 1981.
[18] Khater, M. M., Zahran, E. H., Shehata, M. S., *Journal of the Egyptian Mathematical Society*, **25**, 8, 2017.
[19] Sontakke, B.R., Shaikh, A., Nisar, K.S., *Malaysian Journal of Mathematical Sciences*, **12**, 175, 2018.
[20] Ganji, Z.Z., Ganji, D.D., Rostamiyan, Y., *Applied Mathematical Modelling*, **33**, 3107, 2009.
[21] Whitham, G.B., *Proceedings of the Royal Society of London A*, **299,** 6, 1967.
[22] Broer, L.J.F., *Applied Scientific Research*, **31,** 377, 1975.
[23] Kaup, D.J., *Progress of Theoretical Physics*, **54**, 396, 1975.
[24] Ray, S.S., *Mathematical Methods in the Applied Sciences*, **38,** 1352, 2015.
[25] Drinfeld, V.G., Sokolov, V.V., *Russian Academy of Sciences*, **258,** 11, 1981.
[26] Wilson, G., *Physics Letters A*, **89,** 332, 1982.
[27] Jaradat, H.M., Al-Shara, S., Khan, Q.J., Alquran, M., Al-Khaled, K., *International Journal of Applied Mathematics*, **46,** 64, 2016.
[28] Ali, A., Shah, K., Khan, R.A., *Alexandria Engineering Journal*, **57,** 1991, 2018.
[29] Veeresha, P., Prakasha, D.G., Qurashi, M.A., Baleanu, D., arXiv: 1904.01230, 2019.
[30] Mainardi, F., Fractional calculus. In *Fractals and fractional calculus in continuum mechanics*, Springer, Vienna 1997.
[31] Khan, Y., Faraz, N., Yildirim, A., Wu, Q., *Computers & Mathematics with Applications*, **62,** 2273, 2011.
[32] Caputo, M., *Geophysical Journal International*, **13,** 529, 1967.
[33] Bayram, M., Hatipoğlu, S., Alkan, S.E., Das, A., *Thermal Science*, **22**, S7, 2017.
[34] Mozaffari, F.S., Hassanabadi, H., Sobhani, H., Chung, W.S., *Journal of the Korean Physical Society*, **72**, 987, 2018.
[35] Hosseini, K., Bekir, A., Ansari, R., *Optik*, **132**, 203, 2017.
[36] He, J.H., *Communications in Nonlinear Science and Numerical Simulation*, **2,** 230, 1997.
[37] He, J. H., *International Journal of Non-Linear Mechanics*, **34,** 699, 1999.
[38] Das, N., Singh, R., Wazwaz, A.M., Kumar, J., *Journal of Mathematical Chemistry*, **54,** 527, 2016.
[39] Acan, O., Firat, O., Keskin, Y., *Waves in Random and Complex Media*, **30**(2), 250, 2018.
[40] Khalil, R. et al., *Journal of Computational and Applied Mathematics*, **264**, 65, 2014.
[41] Wu, Y., Geng, X., Hu, X., Zhu, S., *Physics Letters A*, **255,** 259, 1999.
[42] Chen, X., Shen, W., Wang, L., Wang, F., *International Journal of Applied Mathematics*, **47,** 156, 2017.