ORIGINAL PAPER

SOLUTION OF NEWEL-WHITEHEAD-SEGEL EQUATION USING CONFORMABLE FRACTIONAL SUMUDU DECOMPOSITION METHOD

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Abstract. *In the present paper, a numerical method is proposed to solve the time fractional Newel-Whitead-Segel equation subject to initial condition. This method is based d on the unification of conformable Sumudu transform (CST) and Adomian decomposition method (ADM), and then it is used to find the analytical solutions of linear-nonlinear fractional PDE's. The test examples are given for illustration.*

Keywords: *Newell-Whitehead-Segel equation; Sumudu transform; conformable fractional derivative.*

1. INTRODUCTION

In recent years, fractional calculus used in many areas such as electrical networks, control theory of dynamical systems, probability and statistics, electrochemistry of corrosion, chemical physics, optics, engineering, acoustics, viscoelasticity, material science and signal processing can be successfully modelled by linear or nonlinear fractional order differential equations [1].

As it is well known, The Newell-Whitehead-Segel (NWS) equations have wide applicability in mechanical and chemical engineering, ecology, biology and bio-engineering [2].

In the present work, the conformable Sumudu decomposition method (CSDM) is applied to obtain the approximate solution of the linear and nonlinear fractional Newell-Whitehead-Segel of form given below

$$\frac{\partial^\alpha f}{\partial x^\alpha} = k\frac{\partial^2 f}{\partial x^2} + cf - df^r, ...x \in \mathbb{R} \tag{1}$$

where $t \geq 0, 0 < \alpha \leq 1$, r is a intger, k , d and c are reals numbers with $k>0$. $f(x;t)$ is a function of temporal variable t and spatial variable x.

Here f can be considered as the nonlinear temperature distribution in a thin as well as infinitely long rod. It may also be seen as the fluid flow velocity in a pipe of infinite length having small diameter. Recently, Kumar and Sharma provided of the numerical approximation (NWS) of fractional order using homotopy analysis Sumudu transform method (HASTM) and found that homotopy perturbation method (HPM) [3], Adomian decomposition

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method (ADM) and differential transform method (DTM) are particular cases of the solution obtained by (HASTM). The fractional model of Newell–Whitehead–Segel equation has not yet been studied by conformable Sumudu decomposition method [4].

2. PRELIMINARY

Some necessary definitions and theorems of the fractional calculus are listed here for our subsequent development.

2.1. FUNDAMENTAL PROPERTIES OF FRACTIONAL CALCULUS

In this section we give definitions and some basic results.

Definition 1. Given a function $f:[0;\infty)\to\mathbb{R}$, $t\geq 0$ and $\alpha\in(0;1)$.

Conformable derivative of f with respect to t of order α is defined by:

$$D^{\alpha}f(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}$$

Definition 2. Given a function $f:[t_0;\infty)\to\mathbb{R}$, be a real value function with $t_0\in\mathbb{R}$ and $0<\beta\leq 1$. Then, conformable Sumudu transform of the function f of order β is defined by:

$$S_{\beta}^{t_0}\left[f(t)\right](u)=\frac{1}{u}\int_{t_0}^{\infty}\exp\left(-\frac{(t-t_0)^{\beta}}{\beta u}\right)f(t)(t-t_0)^{\beta-1}dt \tag{2}$$

Theorem 1. Let $a\in\mathbb{R}$ and $0<\beta\leq 1$, $f:[a;\infty)\to\mathbb{R}$ be a differentiable function.

$$S_{\alpha}\left[D^{\alpha}f(t)\right]=\frac{1}{u}S_{\alpha}\left[f(t)\right]-\frac{f(a)}{u} \tag{3}$$

Proof: See [5].

3. CONFORMABLE SUMUDU DECOMPOSITION METHOD (CSDM)

We consider the following fractional partial differential equation (FPDEs) of the form:

$$D_t^{\alpha}f(x;t)+D_x^{n}f(x;t)+P\left[f(x;t)\right]+Q\left[f(x;t)\right]=g(x;t).,t>0,.x>0..,0<\alpha\leq 1. \tag{4}$$

with initial condition

$$f(x;0)=k(x) \tag{5}$$

where $D_t^{\alpha} f(x;t)$ is the linear derivative operator in conformable sense of order α in t, P is the linear differential term lower derivatives and, Q is a nonlinear differential operator and $g(x;t)$ is the non-homogenous term. $D_x^n f(x;t)$ is the highest order linear classical derivative operator in x.

Now applying the conformable Sumudu transform S_α to Eq(4), we have:

$$S_\alpha\left[D_t^\alpha f\right]+S_\alpha\left[D_x^n f\right]+S_\alpha\left[P(f)+Q(f)\right]=S_\alpha\left[g\right]$$

From the property Eq(3), we obtain:

$$\frac{1}{u}S_\alpha\left[f\right]-\frac{f(x;0)}{u}+S_\alpha\left[D_x^n f\right]+S_\alpha\left[P(f)+Q(f)\right]=S_\alpha\left[g\right]$$

which implies that:

$$S_\beta\left[f\right]=f(x;0)-uS_\alpha\left[D_x^n f\right]-uS_\alpha\left[P(f)+Q(f)\right]+uS_\alpha\left[g\right] \tag{6}$$

we applied the inverse Sumudu transform conformable sense to Eq.(6), we get:

$$f(x;t)=S_\alpha^{-1}\left[f(x;0)+uS_\alpha(g)\right]-S_\alpha^{-1}\left[uS_\alpha\left[D_x^n f\right]\right]-S_\alpha^{-1}\left[uS_\alpha\left[P(f)+Q(f)\right]\right] \tag{7}$$

So, according to Sumudu decomposition method (SDM), we can obtain the solution $f(x;t)$ as the form:

$$f(x;t)=\sum_{m=0}^{\infty} f_m(x;t) \tag{8}$$

the nonlinear operator is decomposed as:

$$Q\left[f(x;t)\right]=\sum_{m=0}^{\infty} A_m(f_0;f_1;f_2;.......;f_m) \tag{9}$$

where

$$A_m(f_0;f_1;f_2;.......;f_m)=\frac{1}{m!}\frac{\partial^m}{\partial\lambda^m}Q\left[\sum_{i=0}^{\infty}\lambda^i f_i\right] \tag{10}$$

Now, substituting Ed(8) and Eq(10) into Eq(7), we obtain:

$$\begin{cases}\displaystyle\sum_{m=0}^{\infty} f_m(x;t)=S_\alpha^{-1}\left[f(x;0)+uS_\alpha(g)\right]-S_\alpha^{-1}\left[uS_\alpha\left[D_x^n\sum_{m=0}^{\infty} f_m(x;t)\right]\right]\\ \displaystyle -S_\alpha^{-1}\left[uS_\alpha\left[P(\sum_{m=0}^{\infty} f_m(x;t))+Q(A_m)\right]\right]\end{cases} \tag{11}$$

Comparing both side of Eq-11), we get

$$\begin{cases} f_0(x;t) = S_\alpha^{-1}\left[f(x;0) + uS_\alpha(g)\right] \\ f_{m+1}(x;t) = -S_\alpha^{-1}\left[uS_\alpha\left[D_x^n f_m(x;t)\right]\right] - S_\alpha^{-1}\left[uS_\alpha\left[P(f_m(x;t)) + Q(A_m)\right]\right], m \geq 0. \end{cases} \tag{12}$$

4. ILLUSTRATIVE EXAMPLES

In this section we shall test examples using the CSDM technique.

4.1. EXAMPLE 1

Consider a linear time-fractional Newell- Whitehehead-Segel equation:

$$D_t^\alpha f(x;t) = D_x^2 f(x;t) - 2f(x;t).,t \geq 0,.,0 < \alpha \leq 1. \tag{13}$$

with the initial condition

$$f(x;0) = \exp(x)$$

we applying the conformable Sumudu transform to both sides of the Eq(12), we get :

$$\frac{1}{u}S_\alpha\left[f(x;t)\right] - \frac{f(x;0)}{u} = S_\alpha\left[D_x^2 f(x;t)\right] - 2S_\alpha\left[f(x;t)\right] \tag{14}$$

Then, we have

$$\left(\frac{1}{u} + 2\right)S_\alpha\left[f(x;t)\right] = S_\alpha\left[D_x^2 f(x;t)\right] + \frac{e^x}{u}$$

$$\Rightarrow S_\alpha\left[f(x;t)\right] = \frac{u}{1+2u}S_\alpha\left[D_x^2 f(x;t)\right] + \frac{e^x}{1+2u} \tag{15}$$

The inverse Sumudu transform S_α^{-1} , the Eq (15) can be explicitly given as:

$$f(x;t) = S_\alpha^{-1}\left[\frac{u}{1+2u}S_\alpha\left[D_x^2 f(x;t)\right]\right] + e^x S_\alpha^{-1}\left[\frac{1}{1+2u}\right] \tag{16}$$

Then,

$$f(x;t) = e^x e^{-2\frac{t^\alpha}{\alpha}} + S_\alpha^{-1}\left[\frac{u}{1+2u}S_\alpha\left[D_x^2 f(x;t)\right]\right] \tag{17}$$

Base on the serial solution in Eq(8) and Eq(17) turns into:

$$\sum_{m=0}^{\infty} f_m(x;t) = e^x e^{-2\frac{t^\alpha}{\alpha}} + S_\alpha^{-1}\left[\frac{u}{1+2u}S_\alpha\left[D_x^2 \sum_{m=0}^{\infty} f_m(x;t)\right]\right] \tag{18}$$

we use Eq(12), we obtain the following iterations:

$$f_0 = e^{x-2\frac{t^\alpha}{\alpha}}$$

$$f_1 = \frac{t^\alpha}{\alpha} e^{x-2\frac{t^\alpha}{\alpha}}$$

$$f_2 = \frac{t^{2\alpha}}{\alpha^2 2!} e^{x-2\frac{t^\alpha}{\alpha}}$$

$$f_3 = \frac{t^{3\alpha}}{\alpha^3 3!} e^{x-2\frac{t^\alpha}{\alpha}}$$

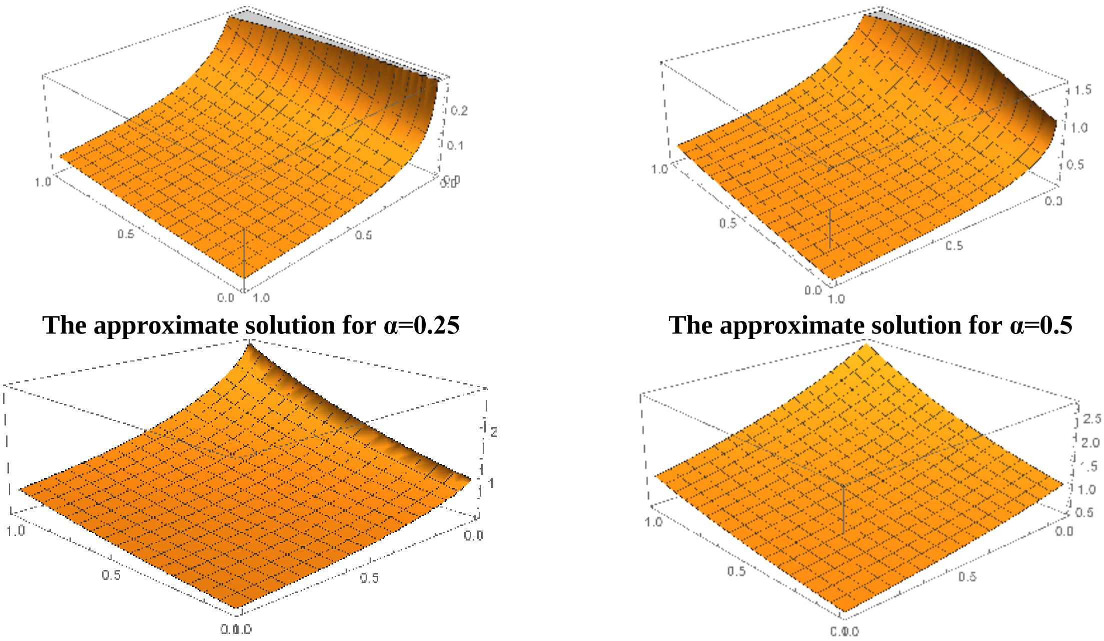
$$\vdots$$

Finally, we obtain the approximate solution:

$$\begin{aligned}
& f(x;t) = e^{x-2\frac{t^\alpha}{\alpha}} + \frac{t^\alpha}{\alpha} e^{x-2\frac{t^\alpha}{\alpha}} + \frac{t^{2\alpha}}{\alpha^2 2!} e^{x-2\frac{t^\alpha}{\alpha}} + \frac{t^{3\alpha}}{\alpha^3 3!} e^{x-2\frac{t^\alpha}{\alpha}} \ldots\ldots\ldots \\
& \Leftrightarrow f(x;t) = e^{x-2\frac{t^\alpha}{\alpha}} \left(1 + \frac{t^\alpha}{\alpha} + \frac{t^{2\alpha}}{\alpha^2 2!} + \frac{t^{3\alpha}}{\alpha^3 3!} + \ldots\ldots\ldots \right) \\
& \Leftrightarrow f(x;t) = e^{x-2\frac{t^\alpha}{\alpha}} \left(1 + \frac{t^\alpha}{\alpha} + \frac{1}{2!}\left(\frac{t^\alpha}{\alpha}\right)^2 + \frac{1}{3!}\left(\frac{t^\alpha}{\alpha}\right)^3 + \ldots\ldots\ldots \right) \\
& \Rightarrow f(x;t) = e^{x-\frac{t^\alpha}{\alpha}}
\end{aligned} \tag{18}$$

If we put $\alpha = 1$ in Eq(18), we can conclude the exact solution:

$$f(x;t) = e^{x-t}$$



The approximate solution for α=0.25 **The approximate solution for α=0.5**

The approximate solution for α=0.75 **The exacte solution for α=1**

Figure 1. 4-step approximate solution graphs of *f* (*x*; *t*) are given for different α values

4.2. EXAMPLE 2

Consider the nonlinear time–fractional Newell-Whitehead-Segel equation:

$$D_t^{\alpha} f(x;t) = D_x^2 f(x;t) + 2f(x;t) - 3f^2(x;t).,t \geq 0,.,0<\alpha \leq 1. \tag{19}$$

with the initial conditions:

$$f(x;0) = \lambda \tag{20}$$

we applying the conformable Sumudu transform to both sides of the Eq(19), we get :

$$\frac{1}{u} S_{\alpha}\left[f(x;t)\right] - \frac{f(x;0)}{u} = S_{\alpha}\left[D_x^2 f(x;t)\right] + 2S_{\alpha}\left[f(x;t)\right] - 3S_{\alpha}\left[f^2(x;t)\right] \tag{21}$$

Then, we have:

$$\begin{aligned} &\left(\frac{1}{u} - 2\right) S_{\alpha}\left[f(x;t)\right] = S_{\alpha}\left[D_x^2 f(x;t)\right] + \frac{\lambda}{u} - 3S_{\alpha}\left[f^2(x;t)\right] \\ &\Rightarrow S_{\alpha}\left[f(x;t)\right] = \frac{\lambda}{1-2u} + \frac{u}{1-2u} S_{\alpha}\left[D_x^2 f(x;t)\right] - \frac{3u}{1-2u} S_{\alpha}\left[f^2(x;t)\right] \end{aligned} \tag{22}$$

The inverse Sumudu transform S_{α}^{-1}, the Eq(22) can be explicitly given as:

$$f(x;t) = S_{\alpha}^{-1}\left[\frac{\lambda}{1-2u}\right] + S_{\alpha}^{-1}\left[\frac{u}{1-2u} S_{\alpha}\left[D_x^2 f(x;t)\right]\right] - 3S_{\alpha}^{-1}\left[\frac{u}{1-2u} S_{\alpha}\left[f^2(x;t)\right]\right] \tag{23}$$

With Adomain polynomials are given as in Eq(10), taking the solution form in Eq(8), the Eq(23) becomes:

$$\sum_{m=0}^{\infty} f_m = \lambda e^{2\frac{t^{\alpha}}{\alpha}} + S_{\alpha}^{-1}\left[\frac{u}{1-2u} S_{\alpha}\left[D_x^2 \sum_{m=0}^{\infty} f_m)\right]\right] - 3S_{\alpha}^{-1}\left[\frac{u}{1-2u} S_{\alpha}\left[\sum_{m=0}^{\infty} A_m\right]\right] \tag{23}$$

A few components of Adomian polynomials above are as follows:

$$\begin{aligned} A_0 &= f_0^2 \\ A_1 &= 2f_0 f_1 \\ A_2 &= 2f_0 f_2 + f_1^2 \\ &\vdots \end{aligned}$$

If we put the Adomain polynomials into Eq(23), and using the iterative formula in Eq(10), we get:

$$
\begin{aligned}
f_0 &= \lambda e^{2\frac{t^\alpha}{\alpha}} \\
f_1 &= \frac{3}{2}\lambda^2 e^{2\frac{t^\alpha}{\alpha}}\left(1-e^{2\frac{t^\alpha}{\alpha}}\right) \\
f_2 &= \left(\frac{3}{2}\right)^2 \lambda^3 e^{2\frac{t^\alpha}{\alpha}}\left(1-e^{2\frac{t^\alpha}{\alpha}}\right)^2 \\
f_3 &= \left(\frac{3}{2}\right)^3 \lambda^4 e^{2\frac{t^\alpha}{\alpha}}\left(1-e^{2\frac{t^\alpha}{\alpha}}\right)^3 \\
&\vdots
\end{aligned}
$$

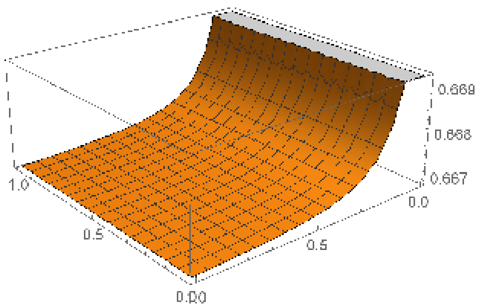
Hence, the solution of Eq(19) is given by:

$$
f(x;t) = \lambda e^{2\frac{t^\alpha}{\alpha}} + \frac{3}{2}\lambda^2 e^{2\frac{t^\alpha}{\alpha}}\left(1-e^{2\frac{t^\alpha}{\alpha}}\right) + \left(\frac{3}{2}\right)^2 \lambda^3 e^{2\frac{t^\alpha}{\alpha}}\left(1-e^{2\frac{t^\alpha}{\alpha}}\right)^2 + \ldots\ldots
$$

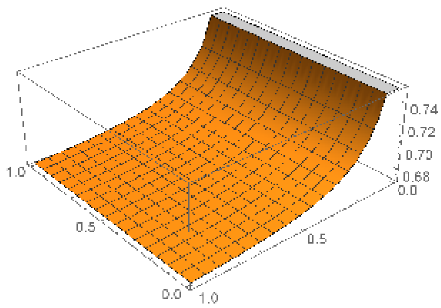
$$
\Leftrightarrow f(x;t) = \lambda e^{2\frac{t^\alpha}{\alpha}}\left(1+\frac{3}{2}\lambda\left(1-e^{2\frac{t^\alpha}{\alpha}}\right)+\left(\frac{3}{2}\right)^2\lambda^2\left(1-e^{2\frac{t^\alpha}{\alpha}}\right)^2+\ldots\ldots\right) \tag{24}
$$

$$
\Leftrightarrow f(x;t) = \lambda e^{2\frac{t^\alpha}{\alpha}}\left(1+\frac{3}{2}\lambda\left(1-e^{2\frac{t^\alpha}{\alpha}}\right)+\left(\frac{3}{2}\lambda\left(1-e^{2\frac{t^\alpha}{\alpha}}\right)\right)^2+\ldots\ldots\right)
$$

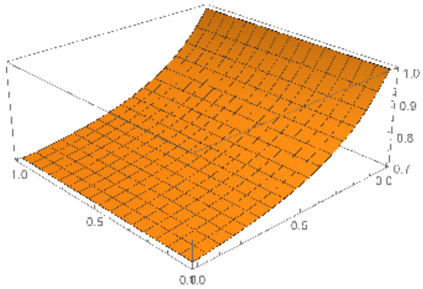
$$
\Rightarrow f(x;t) = \lambda e^{2\frac{t^\alpha}{\alpha}}\left(\frac{1}{1-\frac{3}{2}\lambda\left(1-e^{2\frac{t^\alpha}{\alpha}}\right)}\right) = \frac{2\lambda e^{2\frac{t^\alpha}{\alpha}}}{2-3\lambda\left(1-e^{2\frac{t^\alpha}{\alpha}}\right)} \tag{25}
$$



The approximate solution for α=0.25



The approximate solution for α=0.75



Plot3D of the exact solution for α=1

Figure 2. 3-step approximate solution graphs of $f(x;t)$ are given for different α values and λ=1.

5. CONCLUSION

The application of SDM was extended successfully for solving the FPDEs. The SDM was clearly very efficient and powerful technique in finding the approximate solution of the proposed equations. In order to check the effectiveness of the introduced procedure, the numerical example is tested, by comparing the (AS) approximate solution with the exact solution.

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