ORIGINAL PAPER

ON THE HARMONIC EVOLUTE SURFACES OF TUBULAR SURFACES IN EUCLIDEAN 3-SPACE

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Abstract. *The aim of this study is to investigate and interpret the geometric properties of the harmonic evolute surfaces of the tubular surfaces in Euclidean 3-space. For this purpose, the harmonic evolute surface is defined by considering the definitions and theorems for the tubular surface constructed in the Euclidean 3-space. The characterizations of the s and ψ parameter curves of the harmonic evolute surface obtained are examined, and then parameter curves of the tubular surface and harmonic evolute surface are compared. Finally, the harmonic evolute surface of a tubular surface is given an example and the graphics of these surfaces are drawn.*

Keywords: *tubular surfaces; harmonic evolute surface; geodesic curve; asymptotic curve; line of curvature.*

1. INTRODUCTION

The harmonic evolute surface of a tubular surface is called the geometrical location of the points at the inverse distance in terms of multiplication of the mean curvature from the surface in the direction of the normal vector field of the surface. However, for a non-minimal surface, the harmonic evolute surface of a surface can be defined. Let $T(s,\psi)$ be a surface in E^3 which is no minimal, then the parametric equation of the harmonic evolute surface of a surface $T(s,\psi)$ can be written with

$$\hbar(s,\psi)=T(s,\psi)+\frac{1}{H(s,\psi)}N(s,\psi) \tag{1}$$

where $H(s,\psi)$ and $N(s,\psi)$ are the mean curvature and the normal vector field of a surface $T(s,\psi)$, respectively. Many international studies have been conducted on the harmonic evolute surface of a surface, some of which are [1-5]. In this study, harmonic evolute surface of a tubular surface was investigated in order to give direction to the surfaces of differential geometry, which finds application in multiple disciplines. The tubular surface is a special form of canal surface. The canal surfaces, for the first time in 1850, were defined by Gaspard Monge as the envelope of the moving sphere of variable radius [6]. In addition, in 2006, the geometric and analytical properties of these canal surfaces were given by Xu, Feng and Sun. [7]. The tubular surfaces are obtained when the radius of the sphere forming the canal surface is constant. The tubular surface and the characterizations of the parameter curves of this

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surface have been investigated in Euclidean space, see [8-13]; in Minkowski space, see [14-16] and in Galilean space, see [17,18].
Our inspiration from the studies above is to investigate the geometric properties of the harmonic evolute surface of a tubular surface. Therefore, it is aimed to compare and interpret the tubular surface and the harmonic evolute surface obtained from this surface. Moreover, all computations in this study are performed on an intel i7-3630QM@2.40 Ghz/16gb computer using Mathematica 9 software.

2. PRELIMINARIES

In Euclidean 3-space, Euclidean inner product is given by $<\xi,\zeta>=\xi_1\zeta_1+\xi_2\zeta_2+\xi_3\zeta_3$, where $\xi=(\xi_1,\xi_2,\xi_3),\zeta=(\zeta_1,\zeta_2,\zeta_3)\in E^3$. The norm of $\xi\in E^3$ is $\|\xi\|=\sqrt{<\xi,\xi>}$. If $\|\xi'(s)\|=1$, then the curve ξ is unit speed curve in E^3. Let $\boldsymbol{t}$, $\boldsymbol{n}$ and $\boldsymbol{b}$ be tangent, principal normal and binormal unit vectors at point $\xi(s)$ of the curve ξ, then the well-known Frenet formulae are given by

$$\begin{bmatrix}\boldsymbol{t}\\ \boldsymbol{n}\\ \boldsymbol{b}\end{bmatrix}_s=\begin{bmatrix}0&\kappa&0\\ -\kappa&0&\tau\\ 0&-\tau&0\end{bmatrix}\begin{bmatrix}\boldsymbol{t}\\ \boldsymbol{n}\\ \boldsymbol{b}\end{bmatrix}$$

where $\kappa=\|\xi''\|$ and $\tau=\dfrac{\det(\xi'\times\xi'',\xi''')}{\|\xi'\times\xi''\|^2}$ are the curvature and the torsion of the curve ξ with the arc-length s, respectively.

Let T_s and T_ψ be tangent vectors of a surface $T(s,\psi)$, then the normal vector field of the surface is calculated by as

$$N=\frac{T_s\times T_\psi}{\|T_s\times T_\psi\|}. \tag{2}$$

The coefficients of the first fundamental form and the second fundamental form $T(s,\psi)$ are defined by

$$I=Eds^2+2Fdsd\psi+Gd\psi^2,$$
$$II=eds^2+2fdsd\psi+gd\psi^2$$

such that

$$E=\langle T_s,T_s\rangle,\ F=\langle T_s,T_\psi\rangle,\ G=\langle T_\psi,T_\psi\rangle \text{ and } e=\langle T_{ss},N\rangle,\ f=\langle T_{s\psi},N\rangle,\ g=\langle T_{\psi\psi},N\rangle, \tag{3}$$

respectively. The Gaussian curvature and the mean curvature of the surface $T(s,\psi)$ are

$$K=\frac{eg-f^2}{EG-F^2} \quad \text{and} \quad H=\frac{1}{2}\frac{Eg-2Ff+Ge}{EG-F^2}, \tag{4}$$

respectively.

Theorem 2.1. The surface is a minimal surface (developable surface) if and only if the mean curvature (Gaussian curvature) of the surface is vanish [6].

3. PROPERTIES OF THE HARMONIC EVOLUTE SURFACES OF THE TUBULAR SURFACES IN EUCLIDEAN 3-SPACE

In this section, we explore the harmonic evolute surface of the tubular surface whose mean curvature does not vanish in E^3. The canal surface with a fixed radius is called tubular surface. The parametric equation of a tubular surface is given as follows

$$T:T\left(s,\psi\right)=\xi(s)+\rho\left(\cos\psi\boldsymbol{n}(s)+\sin\psi\boldsymbol{b}(s)\right) \tag{5}$$

where $\psi\in[0,2\pi)$, $\xi(s)$ is center curve and ρ is radius of the tubular surface. In addition, the circle $\cos\psi\boldsymbol{n}(s)+\sin\psi\boldsymbol{b}(s)$ is always perpendicular to the center curve ξ at point $\xi(s)$. The tubular surface is formed by moving of these circles around the center curve ξ. The tangent vectors T_s and T_ψ of the tubular surface T are found by

$$\begin{aligned} T_s &= \left(1-\rho\kappa\cos\psi\right)\boldsymbol{t}+\rho\tau\left(-\sin\psi\boldsymbol{n}+\cos\psi\boldsymbol{b}\right),\\ T_\psi &= \rho\left(-\sin\psi\boldsymbol{n}+\cos\psi\boldsymbol{b}\right). \end{aligned} \tag{6}$$

From the equations (2) and (6), the normal vector field of the tubular surface T gets as $N=\pm\rho\left(\cos\psi\boldsymbol{n}+\sin\psi\boldsymbol{b}\right)$. In this study, the normal vector field of the tubular surface T is considered

$$N=-\rho\left(\cos\psi\boldsymbol{n}+\sin\psi\boldsymbol{b}\right). \tag{7}$$

Corollary 3.1. The tubular surface T is a regular surface if and only if $\left(1-\rho\kappa\cos\psi\right)\neq 0.$

Theorem 3.2. Let T be a tubular surface given with the parametrization (5), then the Gaussian curvature and the mean curvature of the tubular surface T are

$$K=\frac{-\kappa\cos\psi}{\rho\left(1-\rho\kappa\cos\psi\right)} \text{ and } H=\frac{\left(1-2\rho\kappa\cos\psi\right)}{2\rho\left(1-\rho\kappa\cos\psi\right)},$$

$\left(1-\rho\kappa\cos\psi\right)\neq 0$, respectively.

Proof: From the equations (3) and (6), the coefficients of the first fundamental form are found as

$$E=\left(1-\rho\kappa\cos\psi\right)^2+\rho^2\tau^2,\ F=\rho^2\tau,\ G=\rho^2. \tag{8}$$

The second order partial derivatives of the tubular surface T are obtained by

$$\begin{aligned} T_{ss} &= \rho\left(-\kappa'\cos\psi + \kappa\tau\sin\psi\right)\boldsymbol{t} + \left(\kappa\left(1-\rho\kappa\cos\psi\right) - \rho\left(\tau'\sin\psi + \tau^2\cos\psi\right)\right)\boldsymbol{n} \\ &\quad + \rho\left(\tau'\cos\psi - \tau^2\sin\psi\right)\boldsymbol{b}, \\ T_{s\psi} &= \rho\left(\kappa\sin\psi\boldsymbol{t} - \tau\cos\psi\boldsymbol{n} - \tau\sin\psi\boldsymbol{b}\right), \\ T_{\psi\psi} &= -\rho\left(\cos\psi\boldsymbol{n} + \sin\psi\boldsymbol{b}\right). \end{aligned} \tag{9}$$

Using the equations (3), (7) and (9), the coefficients of the second fundamental form are

$$e = -\kappa\cos\psi\left(1-\rho\kappa\cos\psi\right) + \rho\tau^2,\ f = \rho\tau,\ \mathrm{g} = \rho. \tag{10}$$

Considering the equations (4), (8) and (10), the Gaussian curvature and the mean curvature of the tubular surface T are obtained as

$$K = \frac{-\kappa\cos\psi}{\rho\left(1-\rho\kappa\cos\psi\right)} \text{ and } H = \frac{\left(1-2\rho\kappa\cos\psi\right)}{2\rho\left(1-\rho\kappa\cos\psi\right)},$$

respectively.

Corollary 3.3. The tubular surface T is a minimal surface (developable surface) if and only if

$$\rho = \frac{1}{2\kappa\cos\psi}\ (\kappa = 0).$$

Let's give some theorems about geometric interpretation of parametric curves of the tubular surface T.

Theorem 3.4. Let T be a tubular surface. the $\psi-$parameter curves of the tubular surface T are geodesic curve but the $s-$parameter curves of the tubular surface T are not geodesic curves.

Proof: Let T be a tubular surface given by equation (5), then from the equation (7) and (9), we have

$$N \times T_{ss} \neq 0 \text{ and } N \times T_{\psi\psi} = 0$$

where $\times$ denotes the cross product. In that case, the proof is complete.

Theorem 3.5. Let T be a tubular surface, then the s and $\psi-$parameter curves of the surface T are not asymptotic curves.

Proof: Let T be a tubular surface given by equation (5), then from the equation (7) and (9), we have

$$\left\langle N, T_{ss}\right\rangle \neq 0 \text{ and } \left\langle N, T_{\psi\psi}\right\rangle \neq 0.$$

Thus, the proof is complete.

Theorem 3.6. Let's assume that T is the tubular surface. The s and $\psi-$parameter curves of the tubular surface T are lines of curvature if and only if

$$\tau = 0.$$

Proof: Let T be a tubular surface given by equation (5), then from the equation (8) and (10), we have

$$F = \rho^2\tau \text{ and } f = \rho\tau$$

where $\rho \neq 0$. $F = f = 0$ if and only if $\tau = 0$. So, the proof is complete.

From now on, we construct the harmonic evolute surface of a tubular surface which is no minimal surface. Suppose that tubular surface is no minimal surface, then using the equation (1), the harmonic evolute surface of a tubular represents

$$\hbar : \hbar(s,\psi) = \xi(s) + \sigma(s,\psi)\left(\cos\psi \boldsymbol{n}(s) + \sin\psi \boldsymbol{b}(s)\right) \tag{11}$$

where $\sigma(s,\psi) = -\dfrac{\rho}{(1-2\rho\kappa\cos\psi)} \neq 0$ and $\rho \neq \dfrac{1}{2\kappa\cos\psi}$. The tangent vectors $\hbar_s$ and $\hbar_\psi$ of the harmonic evolute surface $\hbar$ are found by

$$\begin{aligned} \hbar_s &= (1-\kappa\sigma Cos\psi)\boldsymbol{t} + (\sigma_s Cos\psi - \tau\sigma Sin\psi)\boldsymbol{n} + (\sigma_s Sin\psi + \tau\sigma Cos\psi)\boldsymbol{b} \\ \hbar_\psi &= (\sigma_\psi Cos\psi - \sigma Sin\psi)\boldsymbol{n} + (\sigma_\psi Sin\psi + \sigma Cos\psi)\boldsymbol{b} \end{aligned} \tag{12}$$

From the equations (2) and (12), the normal vector field of the harmonic evolute surface $\hbar$ gets as

$$u = \frac{(\sigma\sigma_s - \tau\sigma\sigma_\psi)\boldsymbol{t} + (1-\kappa\sigma Cos\psi)\left(-(\sigma_\psi \sin\psi + \sigma\cos\psi)\boldsymbol{n} + (\sigma_\psi\cos\psi - \sigma\sin\psi)\boldsymbol{b}\right)}{(1-\kappa\sigma Cos\psi)\sqrt{\sigma_\psi{}^2 + \sigma^2 + \sigma^2(\sigma_s - \tau\sigma_\psi)^2}} \tag{13}$$

where $(1-\sigma\kappa\cos\psi) > 0$ and $\sigma_\psi{}^2 + \sigma^2 + \sigma^2(\sigma_s - \tau\sigma_\psi)^2 \neq 0$.

From the equations (3) and (12), the coefficients of the first fundamental form of the surface $\hbar$ are found as

$$\bar{E} = (1-\sigma\kappa\cos\psi)^2 + \sigma_s{}^2 + \sigma^2\tau^2,\ \bar{F} = \sigma_s\sigma_\psi + \sigma^2\tau,\ \bar{G} = \sigma_\psi{}^2 + \sigma^2. \tag{14}$$

The second order partial derivatives of the surface $\hbar$ are obtained by

$$\begin{aligned} \hbar_{ss} &= \left(-(\kappa_s\sigma + 2\kappa\sigma_s)\cos\psi + \kappa\tau\sigma\sin\psi\right)\boldsymbol{t} \\ &\quad + \left(\kappa(1-\sigma\kappa\cos\psi) + (\sigma_{ss} - \tau^2\sigma)\cos\psi - (\tau_s\sigma + 2\tau\sigma_s)\sin\psi\right)\boldsymbol{n} \\ &\quad + \left((\tau_s\sigma + 2\tau\sigma_s)\cos\psi + (\sigma_{ss} - \tau^2\sigma)\sin\psi\right)\boldsymbol{b}, \\ \hbar_{s\psi} &= (\kappa\sigma_s\cos\psi + \kappa\sigma\sin\psi)\boldsymbol{t} + \left((\sigma_{s\psi} - \tau\sigma)\cos\psi - (\sigma_s + \tau\sigma_\psi)\sin\psi\right)\boldsymbol{n} \\ &\quad + \left((\sigma_s + \tau\sigma_\psi)\cos\psi + (\sigma_{s\psi} - \tau\sigma)\sin\psi\right)\boldsymbol{b}, \\ \hbar_{\psi\psi} &= \left((\sigma_{\psi\psi} - \sigma)\cos\psi - 2\sigma_\psi\sin\psi\right)\boldsymbol{n} + \left(2\sigma_\psi\cos\psi + (\sigma_{\psi\psi} - \sigma)\sin\psi\right)\boldsymbol{b}. \end{aligned} \tag{15}$$

Using the equations (3), (13) and (15), the coefficients of the second fundamental form are

$$
\begin{aligned}
\bar{e} &== \frac{\begin{pmatrix} \sigma\left(\sigma_s-\tau\sigma_\psi\right)\left(\tau\kappa\sigma\sin\psi-\left(\sigma\kappa_s+2\kappa\sigma_s\right)\cos\psi\right) \\ +\left(1-\kappa\sigma\cos\psi\right)\left(\sigma_\psi\left(\sigma\tau_s+2\tau\sigma_s\right)-\sigma\left(\sigma_{ss}-\sigma\tau^2\right)-\kappa\left(1-\kappa\sigma\cos\psi\right)\left(\sigma\cos\psi+\sigma_\psi\sin\psi\right)\right)\end{pmatrix}}{\left(1-\kappa\sigma Cos\psi\right)\sqrt{{\sigma_\psi}^2+\sigma^2+\sigma^2\left(\sigma_s-\tau\sigma_\psi\right)^2}}, \\
\bar{f} &= \frac{\left(\kappa\sigma\left(\sigma_s-\tau\sigma_\psi\right)\left(\sigma\sin\psi-\sigma_s\cos\psi\right)+\left(1-\kappa\sigma\cos\psi\right)\left(\sigma_\psi\left(\sigma_s+\tau\sigma_\psi\right)+\sigma(\tau\sigma-\sigma_{s\psi})\right)\right)}{\left(1-\kappa\sigma Cos\psi\right)\sqrt{{\sigma_\psi}^2+\sigma^2+\sigma^2\left(\sigma_s-\tau\sigma_\psi\right)^2}}, \qquad (16) \\
\bar{g} &= \frac{\left(1-\kappa\sigma Cos\psi\right)\left(2\sigma_\psi^2+\sigma\left(\sigma-\sigma_{\psi\psi}\right)\right)}{\left(1-\kappa\sigma Cos\psi\right)\sqrt{{\sigma_\psi}^2+\sigma^2+\sigma^2\left(\sigma_s-\tau\sigma_\psi\right)^2}}.
\end{aligned}
$$

Considering the equations (4), (14) and (16), the Gaussian curvature and the mean curvature of the surface $\hbar$ given by equation (11) are obtained as

$$
\bar{K} = \frac{\begin{array}{c}\left(1-\kappa\sigma\cos\psi\right)\left(2\sigma_\psi^2+\sigma\left(\sigma-\sigma_{\psi\psi}\right)\right)\begin{pmatrix}\sigma\left(\sigma_s-\tau\sigma_\psi\right)\left(\left(2\kappa\sigma_s-\sigma\kappa_s\right)\cos\psi+\tau\kappa\sigma\sin\psi\right) \\ +(1-\kappa\sigma\cos\psi)\begin{pmatrix}\sigma(\sigma\tau^2-\sigma_{ss})+\sigma_\psi(\sigma\tau_s+2\tau\sigma_s) \\ -\kappa\sigma\cos\psi-\kappa\sigma_\psi\sin\psi\end{pmatrix}\end{pmatrix} \\ -\left(\sigma\left(\kappa\sigma\sin\psi-\kappa\sigma_s\cos\psi\right)\left(\sigma_s-\tau\sigma_\psi\right)+\left(1-\kappa\sigma\cos\psi\right)\left(\sigma\left(\tau-\sigma_{s\psi}\right)+\sigma_\psi\left(\sigma_s+\tau\sigma_\psi\right)\right)\right)^2\end{array}}{\left(1-\kappa\sigma Cos\psi\right)^2\left({\sigma_\psi}^2+\sigma^2+\sigma^2\left(\sigma_s-\tau\sigma_\psi\right)^2\right)\begin{pmatrix}-\left(\sigma^2\tau+\sigma_s\sigma_\psi\right)^2 \\ +\left(\sigma^2+\sigma_\psi^2\right)\left(\sigma^2\tau^2+\sigma_s^2+\left(1-\kappa\sigma\cos\psi\right)^2\right)\end{pmatrix}}
$$

and

$$
\bar{H} = \frac{\begin{array}{c}\left(1-\kappa\sigma\cos\psi\right)\left(\sigma^2\tau^2+\sigma_s^2+\left(1-\kappa\sigma\cos\psi\right)^2\right)\left(2\sigma_\psi^2+\sigma\left(\sigma-\sigma_{\psi\psi}\right)\right) \\ +\left(\sigma^2+\sigma_\psi^2\right)\begin{pmatrix}\sigma\left(\sigma_s-\tau\sigma_\psi\right)\left(\left(2\kappa\sigma_s-\sigma\kappa_s\right)\cos\psi+\tau\kappa\sigma\sin\psi\right) \\ +\left(1-\kappa\sigma\cos\psi\right)\begin{pmatrix}\sigma\left(\sigma\tau^2-\sigma_{ss}\right)+\sigma_\psi(\sigma\tau_s+2\tau\sigma_s) \\ -\kappa\left(1-\kappa\sigma\cos\psi\right)\left(\sigma\cos\psi+\sigma_\psi\sin\psi\right)\end{pmatrix}\end{pmatrix} \\ -2\left(\sigma^2\tau+\sigma_s\sigma_\psi\right)\begin{pmatrix}\sigma\left(\sigma_s-\tau\sigma_\psi\right)\left(\kappa\sigma\sin\psi-\kappa\sigma_s\cos\psi\right) \\ +\left(1-\kappa\sigma\cos\psi\right)\left(\sigma\left(\tau-\sigma_{s\psi}\right)+\sigma_\psi\left(\sigma_s+\tau\sigma_\psi\right)\right)\end{pmatrix}\end{array}}{2\left(1-\kappa\psi Cos\psi\right)\sqrt{{\sigma_\psi}^2+\sigma^2+\sigma^2\left(\sigma_s-\tau\sigma_\psi\right)^2}\begin{pmatrix}-\left(\sigma^2\tau+\sigma_s\sigma_\psi\right)^2 \\ +\left(\sigma^2+\sigma_\psi^2\right)\left(\sigma^2\tau^2+\sigma_s^2+\left(1-\kappa\sigma\cos\psi\right)^2\right)\end{pmatrix}},
$$

respectively.

Corollary 3.7. The harmonic evolute surface $\hbar$ of the tubular surface T is not both a minimal surface and a developed surface.

Let's give some theorems about geometric interpretation of parametric curves of the harmonic evolute surface $\hbar$ of the tubular surface T.

Theorem 3.8. Let $\hbar$ be a harmonic evolute surface of the tubular surface T, then the following statements are satisfied:

i. The $s-$parameter curves of the harmonic evolute surface $\hbar$ of the tubular surface T are no geodesic curves.

ii. The $\psi-$parameter curves of harmonic evolute surface $\hbar$ of the tubular surface T are geodesic curves if and only if σ is constant.

Proof: The parameter curve on the surface is called geodesic curve, if the acceleration vector of the parameter curve on the surface is parallel to the normal vector of the surface. In that case,

i. Using the equations (13) and (15), we get

$$u\times\hbar_{ss}=\frac{1}{w}\left\{\begin{array}{l}
\left(-1+\kappa\sigma\cos\psi\right)\left(\begin{array}{l}
(-2\tau\sigma_s\cos\psi\left(\sigma\cos\psi+\sigma_\psi\sin\psi\right)\\
+\sigma_{ss}\left(\begin{array}{l}\rho\left(-1+\kappa^3\rho^2\tau^2\right)\cos\psi\sin\psi\\ -\sigma_\psi\left(\kappa^3\sigma^2\tau^2\cos\psi^2+\sin\psi^2\right)\end{array}\right)\\
+\sigma\left(\left(\tau^2\sin\psi(\sigma\cos\psi+\sigma_\psi\sin\psi\right)-\sigma\tau_s\right)\\
+2\tau\sigma_s\sin\psi\left(-\sigma\sin\psi+\sigma_\psi\cos\psi\right)
\end{array}\right)\boldsymbol{t},\\
\left(\begin{array}{l}
\cos\psi\left(2\sigma\tau\sigma_s^2-\sigma\tau_s\right)+\sigma\tau\sigma_\psi\left(\left(\sigma\tau^2-\sigma_{ss}\right)\sin\psi\right)\\
+\sigma\sigma_s\left(\begin{array}{l}\sigma_{ss}\sin\psi+2\kappa\kappa_s\cos\psi\left(-1+\kappa\sigma\cos\psi\right)\left(-\sigma\sin\psi+\sigma_\psi\cos\psi\right)\\ -\tau^2\left(\sigma\sin\psi+2\sigma_\psi\cos\psi\right)+\sigma\tau_s\cos\psi\end{array}\right)\\
+\tau\kappa\sigma\sin\psi(1-\kappa\sigma\cos\psi)\left(\sigma\sin\psi-\sigma_\psi\cos\psi\right)
\end{array}\right)\boldsymbol{n},\\
\left(\begin{array}{l}
-2\kappa\sigma\kappa_s\sigma_s\cos\psi\left(1-\kappa\sigma\cos\psi\right)\left(\sigma\cos\psi+\sigma_\psi\sin\psi\right)\\
+\sigma\left(-\sigma_s+\tau\sigma_\psi\right)\left(\kappa^3\sigma^2\tau^2\sigma_{ss}\cos\psi-\left(\sigma\tau_s+2\tau\sigma_s\right)\sin\psi\right)\\
-\tau\kappa\sigma\sin\psi\left(1-\kappa\sigma\cos\psi\right)(\sigma\cos\psi+\sigma_\psi\sin\psi)
\end{array}\right)\boldsymbol{b}
\end{array}\right\}$$

where $w=\left(1-\kappa\sigma Cos\psi\right)\sqrt{\sigma_\psi{}^2+\sigma^2+\sigma^2\left(\sigma_s-\tau\sigma_\psi\right)^2}$. Since $u\times\hbar_{ss}\neq 0$, the $s-$parameter curves of the harmonic evolute surface $\hbar$ of the tubular surface T are no geodesic curves.

ii. Using the equations (13) and (15), we have

$$u\times\hbar_{\psi\psi}=\frac{1}{w}\left\{\begin{array}{l}\sigma_\psi\left(\sigma+\sigma_{\psi\psi}\right)\left(1-\kappa\sigma\cos\psi\right)\boldsymbol{t}+\sigma\left(\sigma_s-\tau\sigma_\sigma\right)(2\sigma_\psi\cos\psi-\left(\sigma-\sigma_{\psi\psi}\right)\sin\psi)\boldsymbol{n}\\ +\sigma\left(\sigma_s-\tau\sigma_\psi\right)(2\sigma_\psi\sin\psi+(\sigma-\sigma_{\psi\psi})\cos\psi)\boldsymbol{b}\end{array}\right\}$$

where $w=\left(1-\kappa\sigma Cos\psi\right)\sqrt{\sigma_\psi{}^2+\sigma^2+\sigma^2\left(\sigma_s-\tau\sigma_\psi\right)^2}$.

If σ is nonzero constant, considering $\boldsymbol{t}$, $\boldsymbol{n}$ and $\boldsymbol{b}$ are linearly independent, $u \times \hbar_{\psi\psi} = 0$. So the $\psi -$parameter curves the harmonic evolute surface $\hbar$ of the tubular surface T are geodesic curves if and only if σ is nonzero constant.

Corollary 3.9. Let $\hbar$ be a harmonic evolute surface of the tubular surface T, then the following statements are satisfied:

i. The $s -$parameter curves of both the harmonic evolute surface $\hbar$ and the tubular surface T are not geodesic curves.

ii. While the $\psi -$parameter curves of the tubular surface T are geodesic curves, the $\psi -$parameter curves of the harmonic evolute surface $\hbar$ are not geodesic curves.

Theorem 3.10. Let $\hbar$ be a harmonic evolute surface of the tubular surface T, then the following statements are satisfied;

i. The $s -$parameter curves of the harmonic evolute surface $\hbar$ of the tubular surface T are asymptotic curves if and only if

$$\sigma = \frac{\kappa \cos\psi}{\tau^2 + \kappa^2 \cos\psi^2}$$

is nonzero constant.

ii. The $\psi -$parameter curves of the harmonic evolute surface $\hbar$ of the tubular surface T cannot be asymptotic curves.

Proof: The parameter curves on the surface are called asymptotic curve, if the normal curvature of the parameter curves is zero everywhere. In that case;

i. From the equation (16), we know

$$\left\langle \hbar_{ss}, u \right\rangle = \frac{\begin{pmatrix} \sigma\left(\sigma_s - \tau\sigma_\psi\right)\left(\tau\kappa\sigma \sin\psi - \left(\sigma\kappa_s + 2\kappa\sigma_s\right)\cos\psi\right) \\ +\left(1-\kappa\sigma\cos\psi\right)\left(\sigma_\psi\left(\sigma\tau_s + 2\tau\sigma_s\right) - \sigma\left(\sigma_{ss} - \sigma\tau^2\right) - \kappa\left(1-\kappa\sigma\cos\psi\right)\left(\sigma\cos\psi + \sigma_\psi \sin\psi\right)\right) \end{pmatrix}}{\left(1-\kappa\sigma Cos\psi\right)\sqrt{\sigma_\psi{}^2 + \sigma^2 + \sigma^2\left(\sigma_s - \tau\sigma_\psi\right)^2}}$$

where $\left(1-\kappa\sigma Cos\psi\right) > 0$ and $\sqrt{\sigma_\psi{}^2 + \sigma^2 + \sigma^2\left(\sigma_s - \tau\sigma_\psi\right)^2} \neq 0$. From here, $\left\langle \hbar_{ss}, u \right\rangle = 0$ if and only if $\sigma = \dfrac{\kappa \cos\psi}{\tau^2 + \kappa^2 \cos\psi^2}$ is nonzero constant. Thus, the proof is completed.

ii. From the equation (16), we know that

$$\left\langle \hbar_{\psi\psi}, u \right\rangle = \frac{\left(2\sigma_\psi^2 + \sigma\left(\sigma - \sigma_{\psi\psi}\right)\right)}{\sqrt{\sigma_\psi{}^2 + \sigma^2 + \sigma^2\left(\sigma_s - \tau\sigma_\psi\right)^2}}$$

where $\sqrt{\sigma_\psi{}^2 + \sigma^2 + \sigma^2\left(\sigma_s - \tau\sigma_\psi\right)^2} \neq 0$. Since $\sigma \neq 0$, $\left\langle \hbar_{\psi\psi}, u \right\rangle \neq 0$. Thus $\psi -$parameter curves of the harmonic evolute surface $\hbar$ cannot be asymptotic curves.

Corollary 3.11. The s and $\psi-$parameter curves of both the harmonic evolute surface $\hbar$ and the tubular surface T are not asymptotic curves.

Theorem 3.12. Let $\hbar$ be a harmonic evolute surface of T. The s and $\psi-$parameter curves of harmonic evolute surface $\hbar$ are line of curvature if and only if

$$\tau=0 \text{ and } \sigma$$

is nonzero constant.

Proof: The parameter curves of the harmonic evolute surface $\hbar$ are lines of curvature if and only if $\bar{F}$ and $\bar{f}$ the coefficients of the first and second fundamental form, respectively, are vanish.From the equations (14) and (16), $\bar{F}=\bar{f}=0$ if $\tau=0$ and σ is nonzero constant. Thus the proof is completed.

4. APPLICATIONS

Example 4.1. Let's show the graphics the harmonic evolute surface of a tubular surface. Let us consider the unit speed curve $\xi(s)$ given by the parametric equation

$$\xi(s)=\left(cos\left(\frac{s}{\sqrt{2}}\right),sin\left(\frac{s}{\sqrt{2}}\right),\frac{s}{\sqrt{2}}\right).$$

The Frenet apparatuses of the unit speed curve $\xi(s)$ are obtained as

$$\boldsymbol{t}(s)=\frac{1}{\sqrt{2}}\left(-sin\left(\frac{s}{\sqrt{2}}\right),cos\left(\frac{s}{\sqrt{2}}\right),1\right),\ \boldsymbol{n}(s)=\left(-cos\left(\frac{s}{\sqrt{2}}\right),-sin\left(\frac{s}{\sqrt{2}}\right),0\right),$$
$$\boldsymbol{b}(s)=\frac{1}{\sqrt{2}}\left(sin\left(\frac{s}{\sqrt{2}}\right),-cos\left(\frac{s}{\sqrt{2}}\right),1\right),\ \kappa=\frac{1}{2} \text{ and } \tau=\frac{1}{2}.$$

The equation of the tubular surface T are found as

$$T(s,\psi)=\begin{pmatrix} cos\left(\frac{s}{\sqrt{2}}\right)(1-\cos\psi)+\frac{1}{\sqrt{2}}sin\left(\frac{s}{\sqrt{2}}\right)\sin\psi, \\ (1-\cos\psi)sin\left(\frac{s}{\sqrt{2}}\right)-\frac{1}{\sqrt{2}}cos\left(\frac{s}{\sqrt{2}}\right)\sin\psi,\frac{1}{\sqrt{2}}(s+\sin\psi) \end{pmatrix}$$

where $\rho=1$. Besides, the normal vector field and the mean curvature of the tubular surface T are determined by

$$N=\left(\frac{\left(2-\cos\psi\right)\left(2\cos\left(\frac{s}{\sqrt{2}}\right)\cos\psi-\sqrt{2}\sin\left(\frac{s}{\sqrt{2}}\right)\sin\psi\right)}{2\sqrt{\left(-2+\cos\psi\right)^{2}}},\frac{\left(2-\cos\psi\right)\left(2\cos\psi\sin\left(\frac{s}{\sqrt{2}}\right)+\sqrt{2}\cos\left(\frac{s}{\sqrt{2}}\right)\sin\psi\right)}{2\sqrt{\left(-2+\cos\psi\right)^{2}}},\frac{-4\sin\psi+\sin 2\psi}{2\sqrt{9-8\cos\psi+\cos 2\psi}}\right)$$

and

$$H=\frac{1-\cos\psi}{\sqrt{\left(-2+\cos\psi\right)^{2}}}$$

where $\cos\psi\neq 2$ and $8\cos\psi-\cos 2\psi\neq 9$. From here, we express the harmonic evolute surface of the tubular surface T the following as

$$\hbar\left(s,\psi\right)=\left(\frac{2\cos\left(\frac{s}{\sqrt{2}}\right)-\sqrt{2}\sin\left(\frac{s}{\sqrt{2}}\right)\sin\psi}{2-2\cos\psi},\frac{2\sin\left(\frac{s}{\sqrt{2}}\right)+\sqrt{2}\sin\left(\frac{s}{\sqrt{2}}\right)\sin\psi}{2-2\cos\psi},\frac{s-cot[\left(\frac{\psi}{2}\right)}{\sqrt{2}}\right).$$

Now let's draw the graphs of surfaces

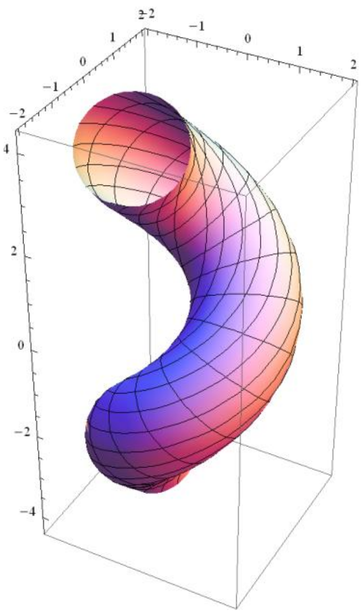

Figure 1.The graph of the tubular surface T with $s\in\left(-5,5\right)$ and $\psi\in\left(0,2\pi\right)$.

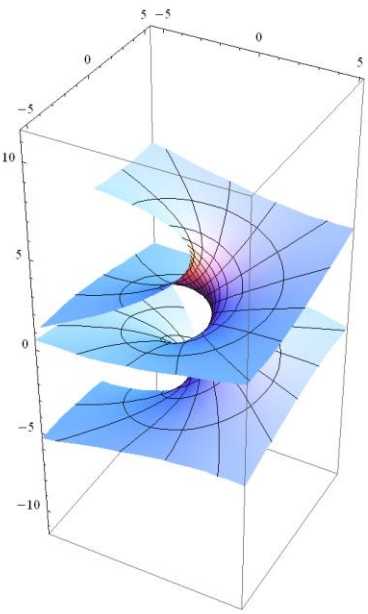

Figure 2. The graph of the harmonic evolute surface $\hbar$ with $s\in\left(-5,5\right)$ and $\psi\in\left(0,2\pi\right)$.

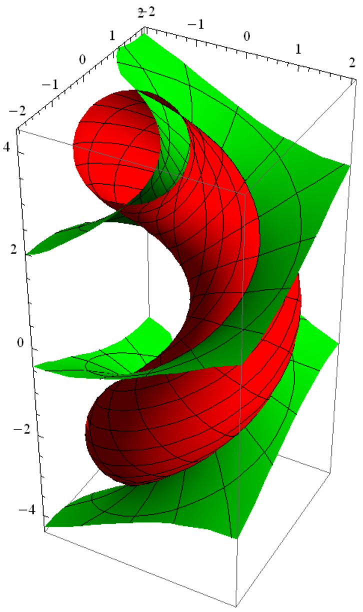

Figure 3. The harmonic evolute surface $\hbar$ given with in green and tubular surface T given with in red for $s \in (-5,5)$ and $\psi \in (0,2\pi)$.

5. CONCLUSION

This article is about the harmonic evolute surface of a tubular surface in E^3. Firstly, the geometric properties of the tubular surface are given, and then the first and second fundamental form, the mean curvature and the Gaussian curvature of the harmonic evolute surface of this tubular surface are calculated. Besides, we give the conditions to be geodesic curve, asymptotic curve and curvature line of parameter curves of these new surfaces and we compare the parameter curves of these surfaces. Finally, the graphics are drawn with examples.

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