ORIGINAL PAPER

NEW EXPONENTIAL PROBABILITY INEQUALITY AND COMPLETE CONVERGENCE FOR $\mathcal{F}-LNQD$ RANDOM VARIABLES SEQUENCE WITH APPLICATION TO $AR(1)$MODEL GENERATED BY $\mathcal{F}-LNQD$ ERRORS

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Manuscript received: 30.11.2020; Accepted paper: 28.03.2021;
Published online: 30.06.2021.

Abstract. *The exponential probability inequalities have been important tools in probability and statistics. In this paper, we prove a new tail probability inequality for the distri-butions of sums of conditionally linearly negative quadrant dependent ($\mathcal{F}-LNQD$, in short) random variables, and obtain a result dealing with conditionally complete con-vergence of first-order autoregressive processes with identically distributed ($\mathcal{F}-LNQD$) innovations.*

Keywords: *autoregressive processes; random variables; $\mathcal{F}-LNQD$ sequence; conditionally complete convergence; exponential inequalities.*

1. INTRODUCTION

The exponential inequality plays an important role in various proofs of limit theorems. In particular, it provides a measure of the complete convergence for partial sums.

Firstly, we will recall the de nitions of conditionally negative quadrant dependent, condition-ally negatively associated, and conditionally linearly negative quadrant dependent sequence. Let $(\Omega,\mathcal{A},\mathbb{P})$ be a probability space, and all random variables in this paper are defined on it unless otherwise mentioned. Let $\mathcal{F}$ be a sub-algebra of $\mathcal{A}$, two random variables ζ_1 and ζ_2 are said to be conditionally negative quadrant dependent given $\mathcal{F}(\mathcal{F}-\mathcal{NQD}$, in short if, for all $\epsilon 1,\epsilon 2 \in\mathbb{R}$

$$\mathbb{P}^{\mathcal{F}}(\zeta_1\le\epsilon_1,\zeta_2\le\epsilon_2)\le\mathbb{P}^{\mathcal{F}}(\zeta_1\le\epsilon_1)\mathbb{P}^{\mathcal{F}}(\zeta_2\le\epsilon_2) \tag{1}$$

One of the many possible multivariate generalizations of conditionally negative quadrant dependence is conditionally negatively association introduced by Yuan et al [1]. A finite collection of random variables $\zeta_1,\zeta_2,\dots,\zeta_n$ is said to be conditionally negatively associated ($\mathcal{F}-NA$, in short) if for every pair of disjoint subsets A,B of $\{1,2,\dots,n\}$

$$cov^{\mathcal{F}}\left(f(\zeta_i:i\in A),g\big(\zeta_j:j\in B\big)\right)\le 0,$$

whenever f and g are coordinatewise nondecreasing such that this covariance exists. An infinite sequence $\{\zeta_n,n\ge 1\}$ is $\mathcal{F}-NA$ if every finite subcollection is $\mathcal{F}-NA$.

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We now propose another multivariate generalization of conditionally negative quadrant dependence called conditionally linearly negative quadrant dependence, which is weaker than $\mathcal{F}-NA$ property.

1.1. DEFINITION

A finite sequence of random variables $\{\zeta_n, n \geq 1\}$ is said to be conditionally linearly negative quadrant dependent given ($\mathcal{F}-LNQD$, in short) if for any disjoint subsets $A, B \subset \mathbb{Z}$ and positive $r_j's$,

$$\sum_{k\in A} r_k\zeta_k \ and \ \sum_{j\in B} r_j\zeta_j \ are \ \mathcal{F}-NQD$$

As mentioned earlier, it can be shown that the concepts of linearly negative quadrant dependent and conditional linearly negative quadrant dependent are not equivalent. See, for example, Yuan and Xie [2], where various of counterexamples are given.

A concrete example where conditional limit theorems are useful is the study of statistical inference for non-ergodic models as discussed in Bassawa and Prakasa Rao [3] and Basawa and Scott [4]. For instance, if one wants to estimate the mean off-spring for a Galton-Watson branching process, the asymptotic properties of the maximum likelihood estimator depend on the set of non-extinction.

As it was pointed out earlier, the conditional $LNQD$ property does not imply the $LNQD$ property and the opposite implication is also not true. Hence one does have to derive limit theorems under conditioning if there is a need for such results even through the results and proofs of such results may be analogous to those under the non-conditioning setup. This one of the reasons for developing results for sequences of $\mathcal{F}-LNQD$ random variables in this paper.

As mentioned earlier, large numbers of results for $LNQD$ random variables have been achieved. However, nothing is variable for conditional $LNQD$ random variables. Yuan and Wu [5] extended many results from negative association to asymptotically negative association, Yuan and Yang [6] extended many results from association to conditional association, Yuan et al [1] extended many results from negative association to conditional negative association, and these motivate our original interest in conditional $LNQD$.

On the other hand, the concept of complet convergence of a sequence of random variables was introduced by [7]. Note that complete convergence implies almost sure convergence in view of the Borel-Cantelli lemma. Now we extend this concept a conditionally converge completely given $\mathcal{F}$ to a constant a if $\sum_{i=1}^{\infty} \mathrm{P}(|\mathrm{X_i} - a\,| > \varepsilon/\mathcal{F}) < \infty$ for every $\varepsilon > 0$, and we whrite $\mathrm{X_n} \rightarrow a$ conditionally completely given $\mathcal{F}$.

The main purpose of this paper is to establish a new probability inequality and conditional complete convergence for the $\mathcal{F}-LNQD$ random variables and to extend and improve the results of Wang et al [8].

Throughout the paper, let $S_n = \sum_{i=1}^{n} X_i$ for a sequence $\{X_n, n \geq 1\}$ of random variables defined on a probability space $(\Omega, \mathrm{A}, \mathrm{P})$. Let $\mathcal{F}$ is a sub- σ-algebra of $\mathcal{A}$, $\{X_n, n \geq 1\}$ will be called $\mathcal{F}$-centered if $\mathbb{E}^{\mathcal{F}} X_n = 0$ for every $n \geq 0$. Denote $B_n = \sum_{i=1}^{n} \mathbb{E}^{\mathcal{F}} |X_i|^2$ for each $1 \leq i \leq n$.

2. SOME LEMMAS

Lemma 2.1. [2] Let random variables X and Y be $\mathcal{F}-NQD$. Then

i. $\mathbb{E}^{\mathcal{F}}(XY) \leq \mathbb{E}^{\mathcal{F}}(X)\mathbb{E}^{\mathcal{F}}(Y)$;
ii. $\mathbb{P}^{\mathcal{F}}(X > x, Y > y) \leq \mathbb{P}^{\mathcal{F}}(X > x)\mathbb{P}^{\mathcal{F}}(Y > y)$;
iii. If c and g are both nondecreasing (or both nonincreasing) functions, then $f(X)$ and $g(Y)$ are $\mathcal{F}-NQD$.

Corollary 2.1. [2] Let $\{X_n, n \geq 1\}$ be a sequence of $\mathcal{F}-LNQD$ random variables and $t > 0$, then for each $n \geq 1$,

$$\mathbb{E}^{\mathcal{F}}\left[\sum_{i=1}^{n} \exp(tX_i)\right] \leq \prod_{i=1}^{n} \mathbb{E}^{\mathcal{F}}(\exp(tX_i)) \tag{2}$$

Lemma 2.2. [9] For any $x \in \mathbb{R}$, we have

$$\exp(x) \leq 1 + x + \frac{|x|}{2}\ln(1+|x|)\exp(2|x|).$$

Lemma 2.3. Let $\{X_n, n \geq 1\}$ be a sequence of $\mathcal{F}-LNQD$ random variables with $\mathbb{E}^{\mathcal{F}}(X_n)$=0 for each $n \geq 1$. If there exists a sequence of positive numbers $\{c_n, n \geq 1\}$such that $|X_i| \leq c_i$ for each $i \geq 1$, then for any $t > 0$,

$$\mathrm{E^F}\exp\{t\textstyle\sum_{i=1}^{n} X_i\} \leq \exp\left\{\frac{t^2}{2}\sum_{i=1}^{n} e^{2tc_i}\mathrm{E^F}\,|X_i|^2\right\} \tag{3}$$

Proof: By Lemma 2.2, for all $x \in \mathbb{R}$, $\exp(x) \leq 1 + x + \frac{|x|}{2}\ln(1+|x|)\exp(2|x|)$. Thus, by $\mathbb{E}^{\mathcal{F}}(X_i)$=0 and $|X_i| \leq c_i$ for each $i \geq 1$, we have

$$\begin{aligned}
\mathrm{E^F}\exp(tX_i) &\leq \mathrm{E^F}\left\{1 + tX_i + \frac{t}{2}|X_i|\ln(1+|tX_i|)\exp(2|tX_i|)\right\} \\
&= 1 + t\mathrm{E^F}X_i + \frac{t}{2}\mathrm{E^F}\{|X_i|\ln(1+|tX_i|)\exp(2|tX_i|)\} \\
&= 1 + \frac{t}{2}\mathrm{E^F}\{|X_i|\ln(1+|tX_i|)\exp(2|tX_i|)\} \\
&= 1 + \frac{t}{2}\mathrm{E^F}\{|X_i|\ln(1+|tX_i|)\exp(2tc_i)\} \\
&= 1 + \frac{t}{2}\exp(2tc_i)\,\mathrm{E^F}\{t|X_i|^2\} \\
&= 1 + \frac{t^2}{2}\exp(2tc_i)\,\mathrm{E^F}\{t|X_i|^2\} \\
&\leq \exp\left\{\frac{t^2}{2}\exp(2tc_i)\,\mathrm{E^F}\{|X_i|^2\}\right\} \quad (\text{using } 1+y \leq \exp(y) \text{ for all } y \in \mathrm{R})
\end{aligned} \tag{4}$$

for any $t > 0$. By Lemma 2.1 and (4) we have can see that

$$\mathbb{E}^{\mathcal{F}} exp\left\{t\sum_{i=1}^{n} X_i\right\} \leq \prod_{i=1}^{n} \mathbb{E}^{\mathcal{F}} \exp\{tX_i\} \quad (5)$$

$$\leq exp\left\{\frac{t^2}{2}\sum_{i=1}^{n} e^{2tc_i}\mathbb{E}^{\mathcal{F}} |X_i|^2\right\}. \quad (6)$$

The lemma is thus proved.

Lemma 2.4. Let $\{X_n, n \geq 1\}$ be a sequence of $\mathcal{F} - LNQD$ random variables with $\mathbb{E}^{\mathcal{F}}(X_n)$= 0 for each $n \geq 1$. If there exists a sequence of positive numbers $\{c_n, n \geq 1\}$ such that $|X_i| \leq c_i$ for each $i \geq 1$, then for any $t > 0$ and $\varepsilon > 0$

$$\mathbb{P}^{\mathcal{F}}\left\{\left|\sum_{i=1}^{n} X_i\right| \geq \varepsilon\right\} \leq exp\left\{-t\varepsilon + \frac{t^2}{2}\sum_{i=1}^{n} e^{2tc_i}\mathbb{E}^{\mathcal{F}} |X_i|^2\right\}. \quad (7)$$

Proof: By Markov’s inequality and Lemma 2.3, we can see that

$$\begin{aligned}\mathbb{P}^{\mathcal{F}}(\textstyle\sum_{i=1}^{n} X_i \geq \varepsilon) &\leq \exp(-t\varepsilon)\mathbb{E}^{\mathcal{F}}\, exp\{t\textstyle\sum_{i=1}^{n} X_i\} \\ &\leq \exp(-t\varepsilon)\textstyle\prod_{i=1}^{n} \mathbb{E}^{\mathcal{F}} \exp\{tX_i\} \\ &\leq exp\left\{-t\varepsilon + \tfrac{t^2}{2}\textstyle\sum_{i=1}^{n} e^{2tc_i}\mathbb{E}^{\mathcal{F}} |X_i|^2\right\}. \quad (8)\end{aligned}$$

The desired result follows by remplacing X_i by $-X_i$ in (8). This completes the proof of the lemma.

3. MAIN RESULTS AND PROOFS

Theorem 3.1. Let $\{X_n, n \geq 1\}$ be a sequence of $\mathcal{F} - LNQD$ random variables with $\mathbb{E}^{\mathcal{F}}(X_i)$= 0. If there exists a positive number c such that $|X_i| \leq c_i, i \geq 1$ where $B_n = \sum_{i=1}^{n} \mathbb{E}^{\mathcal{F}} {X_i}^2$, then for any $p > 1$, $\varepsilon > 0$ and $n > 0$, then

$$\mathbb{P}^{\mathcal{F}}(S_n/B_n \geq \varepsilon) \leq exp\left\{\frac{1}{q}b^{q/p}e\right\}exp\left\{-\left(\frac{\varepsilon 2^{p-1}bp}{B_n^{p-1}}\right)^{\frac{1}{2p-1}}\varepsilon B_n\left(1-\frac{1}{p-1}\right)\right\} \quad (9)$$

Proof: By Markov's inequality, we have that for any $t > 0$;

$$\begin{aligned}\mathbb{P}^{\mathcal{F}}(S_n/B_n \geq \varepsilon) &= \mathbb{P}^{\mathcal{F}}(e^{tS_n} \geq e^{t\varepsilon B_n}), \\ &\leq e^{t\varepsilon B_n}\,\mathbb{E}^{\mathcal{F}}\left(\prod_{i=1}^{n} e^{tX_i}\right), \\ &\leq \exp\left\{-t\varepsilon B_n + \frac{t^2}{2}e^{2t\max_{1\leq i\leq n} c_i}B_n\right\}. \quad (10)\end{aligned}$$

Let $p > 1$. It is well known that

$$uv = \inf_{b>0}\left\{\frac{1}{pb}u^p + \frac{1}{q}b^{q/p}v^q\right\} \; for \; u > 0, v > 0 \;\; and \;\; \frac{1}{p}+\frac{1}{q} = 1.$$

This yields the inequality

$$\frac{t^2}{2}e^{2t\max\limits_{1\le i\le n} c_i}B_n \le \frac{1}{pb}\frac{t^{2p}}{2^p}B_n^p + \frac{1}{q}b^{q/p}e^{2tq\max\limits_{1\le i\le n} c_i} \tag{11}$$

We can thus conclude that for every $p > 1$, there for all $t > 0$, such that

$$\mathbb{P}^{\mathcal{F}}(S_n/B_n \ge \varepsilon) \le \; exp\left\{-t\varepsilon B_n + \frac{1}{pb}\frac{t^{2p}}{2^p}B_n^p\right\}.$$

$$\times \; exp \; \left\{\frac{1}{q}b^{q/p}e^{2tq\max_{1\le i\le n} c_i}\right\}$$

$$= exp\left\{\frac{1}{q}b^{q/p}e^{2t\,q\max\limits_{1\le i\le n} c_i}\right\}\exp\big(\Phi(t,n)\big) \tag{12}$$

The equation $\frac{\partial\Phi(t,n)}{\partial t} = 0$ has the unique solution .

$$\mathrm{t} = \left(\frac{\varepsilon\, 2^{p-1}bp}{B_n^{p-1}}\right)^{\frac{1}{2p-1}} \tag{13}$$

which minimizes $\Phi(t,n)$. Then from (12), (13)) and taking $2tq\max_{1\le i\le n} c_i \le 1$ we obtain (9).

Theorem 3.2. Let $\{X_n, n \ge 1\}$ be a sequence of $\mathcal{F} - LNQD$ random variables $\mathbb{E}^{\mathcal{F}}(X_i) = 0$**.** If there exists a positive number $\boldsymbol{c}$ such that $|X_i| \le c_i, i \ge 1$, then for any $p > 1$, $\boldsymbol{\varepsilon} > 0$ and $\boldsymbol{n} \ge \boldsymbol{1}$,

$$\mathbb{P}^{\mathcal{F}}(|S_n| \ge \varepsilon) \le 2exp\;\left\{\frac{1}{q}b^{q/p}e\right\}\left\{-\left(\frac{\varepsilon\, 2^{p-1}bp}{B_n^p}\right)^{\frac{1}{2p-1}}\varepsilon\left(1-\frac{1}{P-1}\right)\right\} \tag{14}$$

Proof: From conditions $\mathbb{E}^{\mathcal{F}}(X_i) = 0$ and $|X_i| \le c_i$ for each $i \ge 1$. By Markov's inequality and Lemma 2.4, Corollary 2.1 with the fact that $1 + x \le e^x$, then

$$\mathbb{P}^{\mathcal{F}}(S_n \ge \varepsilon) \le e^{-t\varepsilon}\,\mathbb{E}^{\mathcal{F}}(e^{tS_n}),$$

$$\le e^{-t\varepsilon}\prod_{i=1}^n exp\left(\frac{t^2}{2}e^{2tc_i}\mathbb{E}^{\mathcal{F}}|X_i|^2\right),$$

$$\le \exp\left\{-t\varepsilon + \frac{t^2}{2}e^{2t\max_{1\le i\le n} c_i}B_n\right\} \tag{15}$$

Let $p > 1$. It is well known that

$$uv = \inf_{b>0}\left\{\frac{1}{pb}u^p + \frac{1}{q}b^{q/p}v^q\right\}\ for\ u > 0, v > 0\ and\ {}^1/_p + {}^1/_q = 1.$$

This yields the inequality

$$\frac{t^2}{2}e^{2t\max_{1\le i\le n} c_i}B_n \le \frac{1}{pb}\frac{t^{2p}}{2^p}B_n^p + \frac{1}{q}b^{q/p}e^{2tq\max_{1\le i\le n} c_i} \tag{16}$$

We can thus conclude that for every $p > 1$, there for all $t > 0$, such that

$$\begin{aligned}\mathbb{p}^{\mathcal{F}}(|S_n| \ge \varepsilon) &\le 2\ exp\left\{-t\varepsilon + \frac{1}{pb}\frac{t^{2p}}{2^p}B_n^p\right\}.\\ &\times\ exp\ \left\{\frac{1}{q}b^{q/p}e^{2tq\max_{1\le i\le n} c_i}\right\}\\ &= 2exp\left\{\frac{1}{q}b^{q/p}e^{2t\,q\max_{1\le i\le n} c_i}\right\}\exp(\Phi(t,n))\end{aligned} \tag{17}$$

The equation $\frac{\partial\Phi(t,n)}{\partial t} = 0$ has the unique solution

$$t = \left(\frac{\varepsilon\, 2^{p-1}bp}{B_n^p}\right)^{\frac{1}{2p-1}} \tag{18}$$

which minimizes $\Phi(t,n)$. Then from (17),(18)) and taking $2tq\max_{1\le i\le n} c_i \le 1$ we obtain upper bound for the tail probability as

$$\mathrm{p^F}(|\mathrm{S_n}| \ge \varepsilon) \le 2\exp\left\{\frac{1}{\mathrm{q}}\mathrm{b}^{\mathrm{q}/\mathrm{p}}\mathrm{e}\right\}\exp\left\{-\left(\frac{\varepsilon\, 2^{\mathrm{p}-1}\mathrm{bp}}{\mathrm{B_n^p}}\right)^{\frac{1}{2\mathrm{p}-1}}\varepsilon\left(1-\frac{1}{\mathrm{P}-1}\right)\right\} \tag{19}$$

Since $\{-X_n, n \ge 1\}$ is also a sequense of $\mathcal{F} - LNQD$ random variables it follows from (19) that

$$\begin{aligned}\mathrm{p^F}(\mathrm{S_n} \le -\varepsilon) = \mathrm{p^F}(-\mathrm{S_n} \ge \varepsilon) &\le \exp\left\{\frac{1}{\mathrm{q}}\mathrm{b}^{\mathrm{q}/\mathrm{p}}\mathrm{e}\right\}\\ \times\ \exp&\left\{-\left(\frac{\varepsilon\, 2^{\mathrm{p}-1}\mathrm{bp}}{\mathrm{B_n^p}}\right)^{\frac{1}{2\mathrm{p}-1}}\varepsilon\left(1-\frac{1}{\mathrm{P}-1}\right)\right\}\end{aligned} \tag{20}$$

From (19) and (20) we obtain

$$\begin{aligned}\mathbb{p}^{\mathcal{F}}(|S_n| \ge \varepsilon) = \mathbb{p}^{\mathcal{F}}(S_n \ge -\varepsilon) + \mathbb{p}^{\mathcal{F}}(S_n \le \varepsilon) &\le 2\ exp\left\{\frac{1}{q}b^{q/p}e\right\}\\ \times\ exp&\left\{-\left(\frac{\varepsilon\, 2^{p-1}bp}{B_n^p}\right)^{\frac{1}{2p-1}}\varepsilon\left(1-\frac{1}{P-1}\right)\right\}\end{aligned} \tag{21}$$

Theorem 3.3. Let $\{X_n, n \ge 1\}$ be a sequence of $\mathcal{F} - LNQD$ random variables with mean zero and finite variances. If there exists a positive number c such that $|X_i| \le c_i, i \ge 1$, where $\{c_n, n \ge 1\}$ is a sequence of positive numbers. Then for any $p > 1, \varepsilon > 0$ and $n \ge 1$,

$$\mathrm{p^F}(|\mathrm{S_n} - \mathrm{E^F S_n}| \geq \varepsilon) \leq 2\exp\left\{\frac{1}{\mathrm{q}}\mathrm{b}^{\mathrm{q}/\mathrm{p}}\mathrm{e}\right\}\exp\left\{-\left(\frac{\varepsilon\, 2^{\mathrm{p}-1}\mathrm{bp}}{\mathrm{B_n^p}}\right)^{\frac{1}{2\mathrm{p}-1}}\varepsilon\left(1-\frac{1}{\mathrm{P}-1}\right)\right\} \tag{22}$$

Proof: By Markov's inequality and Lemma (2.2), we have that for any $t > 0$,

$$\begin{aligned}\mathrm{p^F}(\mathrm{S_n} - \mathrm{E^F S_n} \geq \varepsilon) &\leq \mathrm{e}^{-\mathrm{t}\varepsilon}\,\mathrm{E^F}[\exp(\mathrm{t}\textstyle\sum_{\mathrm{i}=1}^{\mathrm{n}}(\mathrm{X_i} - \mathrm{E^F X_i}))],\\ &\leq \mathrm{e}^{-\mathrm{t}\varepsilon}\,\mathrm{E^F}\textstyle\prod_{\mathrm{i}=1}^{\mathrm{n}}\left[\mathrm{e}^{-\mathrm{t}(\mathrm{X_i}-\mathrm{E^F X_i})}\right],\\ &\leq \exp\left\{-\mathrm{t}\varepsilon + \frac{\mathrm{t}^2}{2}\mathrm{e}^{2\mathrm{t}\max_{1\leq \mathrm{i}\leq \mathrm{n}}\mathrm{c_i}}\mathrm{B_n}\right\}.\end{aligned} \tag{23}$$

Let $p > 1$. It is well known that

$$uv = \inf_{b>0}\left\{\frac{1}{pb}u^p + \frac{1}{q}b^{q/p}v^q\right\} \; for\; u > 0, v > 0 \; and \; {}^1/_p + {}^1/_q = 1.$$

This yields the inequality

$$\frac{t^2}{2}e^{2t\max_{1\leq i\leq n}c_i}B_n \leq \frac{1}{pb}\frac{t^{2p}}{2^p}B_n^p + \frac{1}{q}b^{q/p}e^{2t\,q\max_{1\leq i\leq n}c_i} \tag{24}$$

We can thus conclude that for every $p > 1$, there for all $t > 0$, such that

$$\begin{aligned}\mathbb{p}^{\mathcal{F}}(|S_n - \mathbb{E}^{\mathcal{F}}S_n| \geq \varepsilon) &\leq 2\,exp\left\{-t\varepsilon + \frac{1}{pb}\frac{t^{2p}}{2^p}B_n^p\right\}.\\ &\times\; exp\;\left\{\frac{1}{q}b^{q/p}e^{2tq\max_{1\leq i\leq n}c_i}\right\}\\ &= 2\exp\left\{\frac{1}{\mathrm{q}}\mathrm{b}^{\mathrm{q}/\mathrm{p}}\mathrm{e}^{2\mathrm{tq}\max_{1\leq \mathrm{i}\leq \mathrm{n}}\mathrm{c_i}}\right\}\exp(\Phi(\mathrm{t},\mathrm{n}))\end{aligned} \tag{25}$$

The equation $\frac{\partial\Phi(t,n)}{\partial t} = 0$ has the unique solution.

Taking $t = \left(\frac{\varepsilon\, 2^{p-1}bp}{B_n^p}\right)^{\frac{1}{2p-1}}$. Hence it follows from (23) that

$$\mathrm{p^F}(\mathrm{S_n} - \mathrm{E^F S_n} \geq \varepsilon) \leq \exp\left\{\frac{1}{\mathrm{q}}\mathrm{b}^{\mathrm{q}/\mathrm{p}}\mathrm{e}\right\}\exp\left\{-\left(\frac{\varepsilon\, 2^{\mathrm{p}-1}\mathrm{bp}}{\mathrm{B_n^p}}\right)^{\frac{1}{2\mathrm{p}-1}}\varepsilon\left(1-\frac{1}{\mathrm{P}-1}\right)\right\} \tag{26}$$

Let $-S_n = T_n = \sum_{i=1}^{n}(-X_n)$. Since $\{-X_n, n \geq 1\}$ is also a sequence of $\mathcal{F} - LNQD$ random variables we also have

$$\begin{aligned}\mathbb{p}^{\mathcal{F}}(S_n - \mathbb{E}^{\mathcal{F}}S_n \leq -\varepsilon) = \mathbb{p}^{\mathcal{F}}(T_n - \mathbb{E}^{\mathcal{F}}T_n \geq \varepsilon) &\leq exp\left\{\frac{1}{q}b^{q/p}e\right\}\\ \times\; \exp\left\{-\left(\frac{\varepsilon\, 2^{\mathrm{p}-1}\mathrm{bp}}{\mathrm{B_n^p}}\right)^{\frac{1}{2\mathrm{p}-1}}\varepsilon\left(1-\frac{1}{\mathrm{P}-1}\right)\right\}\end{aligned} \tag{27}$$

by Combining (26) and(27) we get (22)

Corollary 3.1. Let $\{X_n, n \geq 1\}$ be a sequence of $\mathcal{F}-LNQD$ random variables. Assume that there exists a positive integer n_0 such that $|X_i| \leq c_n$, for each $1 \leq i \leq n, n \geq n_0$, where $\{c_n, n \geq 1\}$ is a sequence of positive numbers. Then for any $\varepsilon > 0$ and $p > 1$

$$\mathrm{p^F}\left(\left|\mathrm{S_n}-\mathrm{E^F S_n}\right| \geq \mathrm{n}\varepsilon\right) \leq 2\exp\left\{\frac{1}{\mathrm{q}}\mathrm{b}^{\mathrm{q}/\mathrm{p}}\mathrm{e}\right\}\exp\left\{-\left(\frac{\mathrm{n}\varepsilon\, 2^{\mathrm{p}-1}\mathrm{bp}}{\mathrm{B_n^p}}\right)^{\frac{1}{2\mathrm{p}-1}}\mathrm{n}\varepsilon\left(1-\frac{1}{\mathrm{P}-1}\right)\right\} \tag{28}$$

Theorem 3.4. Let $\{X_n, n \geq 1\}$ be a sequence of $\mathcal{F}-LNQD$ random variables with $\mathbb{E}^{\mathcal{F}}(X_i)=0$. If there exists a positive numbers c such that $|X_i| \leq c_i, i \geq 1$. Then for any $r > 0$

$$\mathrm{n^{-r}\ S_n} \to 0 \quad \text{completely}, \quad \mathrm{n} \to \infty. \tag{29}$$

Proof: Let $B=\sum_{n=1}^{\infty}\mathbb{E}^{\mathcal{F}}(X_n)^2 \leq \infty$. For any $\varepsilon > 0$, it follows from Theorem 3.2 we have

$$\sum_{\mathrm{n}=1}^{\infty}\mathrm{p^F}(|\mathrm{S_n}| \geq \mathrm{n^r}\varepsilon) \leq 2\sum_{\mathrm{n}=1}^{\infty}\exp\left\{\frac{1}{\mathrm{q}}\mathrm{b}^{\mathrm{q}/\mathrm{p}}\mathrm{e}\right\}\exp\left\{-\left(\frac{\mathrm{n^r}\varepsilon\, 2^{\mathrm{p}-1}\mathrm{bp}}{\mathrm{B_n^p}}\right)^{\frac{1}{2\mathrm{p}-1}}\varepsilon\mathrm{n^r}\left(1-\right.\right.$$

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$$\leq 2\sum_{n=1}^{\infty} exp\left\{\frac{1}{q}b^{q/p}e\right\} exp\left\{-\left(\frac{\varepsilon\, 2^{p-1}bp}{B_n^p}\right)^{\frac{1}{2p-1}}\varepsilon\left(1-\frac{1}{P-1}\right)\right\}^{\frac{2rp}{n^{2p-1}}}$$

$$\leq 2exp\left\{\frac{1}{q}b^{q/p}e\right\}\sum_{n=1}^{\infty}[\exp(-c)]^{\frac{2rp}{n^{2p-1}}} \tag{30}$$

where C is positive number not depending on n. (by the inequality $e^{-y} \leq \left(\frac{a}{ey}\right)^a$), choosing $a=\frac{2p-1}{rp}$, since $a > 0, y > 0$. Then the right-hand side of (30) become,

$$\begin{aligned}
\sum_{\mathrm{n}=1}^{\infty}\mathrm{p^F}(|\mathrm{S_n}| \geq \mathrm{n^r}\varepsilon) &\leq 2\exp\left\{\frac{1}{\mathrm{q}}\mathrm{b}^{\mathrm{q}/\mathrm{p}}\mathrm{e}\right\}\sum_{\mathrm{n}=1}^{\infty}\left(\frac{\mathrm{a}}{\mathrm{ec}}\right)^{\mathrm{a}}\left(\frac{1}{\mathrm{n}}\right)^{\left(\frac{2\mathrm{rp}}{\mathrm{n}^{2\mathrm{p}-1}}\right)^{\mathrm{a}}} \\
&\leq 2exp\left\{\frac{1}{q}b^{q/p}e\right\}\frac{a^a}{(ec)^a}\sum_{n=1}^{\infty}\frac{1}{\frac{2rpa}{n^{2p-1}}} \\
&\leq 2exp\left\{\frac{1}{q}b^{q/p}e\right\}\frac{a^a}{(ec)^a}\sum_{n=1}^{\infty}\frac{1}{n^2}, \\
&\leq 2exp\left\{\frac{1}{q}b^{q/p}e\right\}\frac{a^a}{(ec)^a}\frac{\pi^2}{6}, \\
&< \infty
\end{aligned} \tag{31}$$

Theorem 3.5. Let $\{X_n, n \geq 1\}$ be a sequence of $\mathcal{F}-LNQD$ random variables. Assume that there exists a positive integer n_0 such that $|X_i| \leq c_n$, for each $1 \leq i \leq n, n \geq n_0$, where $\{c_n, n \geq 1\}$ is a sequence of positive numbers. Then,

$$\sum_{\mathrm{n}=1}^{\infty}\mathrm{p^F}\left(\frac{1}{\mathrm{n}}|\mathrm{S_n}-\mathrm{E^F S_n}| \geq \varepsilon_\mathrm{n}\right) < \infty \tag{32}$$

Theorem 3.6. Let $\{X_n, n \geq 1\}$ be a sequence of $\mathcal{F}-LNQD$ random variables with $\mathbb{E}^{\mathcal{F}}(X_i)$=0. If there exists a positive number c such that $|X_i| \leq c_{n\,i}, i \geq 1$. Then for any $r > 0$

$$n^{-r}\ (S_n - E^F S_n) \to 0 \ \text{ completely}, \quad n \to \infty. \tag{33}$$

Proof: For any $\varepsilon > 0$, it follows from Corollary 3.1 that

$$\sum_{n=1}^{\infty} \mathbb{P}^{\mathcal{F}}(|S_n - \mathbb{E}^{\mathcal{F}} S_n| \geq n^r \varepsilon) \leq$$
$$2\sum_{n=1}^{\infty} \exp\left\{\frac{1}{q} b^{q/p} e\right\} \exp\left\{-\left(\frac{n^r \varepsilon\, 2^{p-1} bp}{B_n^p}\right)^{\frac{1}{2p-1}} \varepsilon n^r \left(1 - \frac{1}{P-1}\right)\right\}$$
$$\leq 2\sum_{n=1}^{\infty}\left[exp\left\{\frac{1}{q} b^{q/p} e\right\}\right]$$
$$\times\left[\exp\left\{-\left(\frac{\varepsilon\, 2^{p-1} bp}{B_n^p}\right)^{\frac{1}{2p-1}} \varepsilon\left(1 - \frac{1}{P-1}\right)\right\}\right]^{\frac{2rp}{n^{2p-1}}} \tag{34}$$

By this result we get (33).

4. APPLICATIONS TO THE RESULTS TO AR(1) MODEL

The basic object of this section is applying the results to first-order autoregressive processes $(AR(1))$.

We consider an autoregressive time series of first order AR(1) defined by this

$$X_{n+1} = \theta X_n + \zeta_{n+1}, \qquad n = 1, 2, \ldots, \tag{35}$$

where $\{\zeta_n, n \geq 0\}$ is a sequence of identically distributed $\mathcal{F} - LNQD$ random variables with $\zeta_0 = X_0 = 0$, $0 < \mathbb{E}^{\mathcal{F}} \zeta_k^4 < \infty, k = 1,2, \ldots,$ and where θ is a parameter with $|\theta| < 1$. Here, we can rewrite X_{n+1} in (35) as follows:

$$X_{n+1} = \theta^{n+1} X_0 + \theta^n \zeta_1 + \theta^{n-1} \zeta_2 + \cdots + \zeta_{n+1}. \tag{36}$$

The coefficient θ is fitted least squares, giving the estimator

$$\widehat{\theta_n} = \frac{\sum_{j=1}^n X_j X_{j-1}}{\sum_{j=1}^n X_{j\_1}^2} \tag{37}$$

It immediately follows from (35) and (37) that

$$\widehat{\theta_n} - \theta = \frac{\sum_{j=1}^n \zeta_j X_{j-1}}{\sum_{j=1}^n X_{j\_1}^2} \tag{38}$$

Theorem 4.1. Let the conditions of Theorem 3.3 be satisfied then for any $\frac{(\mathbb{E}^{\mathcal{F}} \zeta_1^2)^{1/2}}{\rho^2} < \xi$ positive, we have

$$\mathbb{P}^{\mathcal{F}}\left(\sqrt{n}\left|\widehat{\theta_n} - \theta\right| > \rho\right) \leq 2exp\left\{-\left(\frac{(\rho^2\xi^2 - E^F\zeta_1^2) n 2^{p-1} bp}{B_n^p}\right)^{\frac{1}{2p-1}} (\rho^2\xi^2 - E^F\zeta_1^2) n \left(1 - \right.\right. \tag{39}$$

$$\left.\left.\frac{1}{p-1}\right)\right\}$$

$$\times\, exp\,\left\{\frac{1}{q}b^{q/p}e\right\}+\left\{-\frac{1}{2}n\frac{\left(K_1-n\xi^2\right)^2}{K_2}\right\}$$

where $K_1=\mathbb{E}^{\mathcal{F}}(X_i^2)<\infty\;,K_2=\mathbb{E}^{\mathcal{F}}(X_i^4)<\infty$.

Proof: Firstly, we notice that:

$$\widehat{\theta_n}-\theta=\frac{\sum_{j=1}^{n}\zeta_j X_{j-1}}{\sum_{j=1}^{n}X_{j\_1}^2}\;.$$

It follows that

$$\mathbb{P}^{\mathcal{F}}\left(\sqrt{n}\left|\widehat{\theta_n}-\theta\right|>\rho\right)\leq\mathbb{P}^{\mathcal{F}}\left(\left|\frac{1/\sqrt{n}\sum_{j=1}^{n}\zeta_j X_{j-1}}{1/n\sum_{j=1}^{n}X_{j\_1}^2}\right|>\rho\right)$$

By virtue of the probability properties and Hölder's inequality, we have for any ξ positive

$$\begin{aligned}\mathbb{P}^{\mathcal{F}}\left(\sqrt{n}\left|\widehat{\theta_n}-\theta\right|>\rho\right)&\leq\mathbb{P}^{\mathcal{F}}\left(1/n\sum_{j=1}^{n}\zeta_j^2\geq\rho^2\xi^2\right)+\;\mathbb{P}^{\mathcal{F}}\left(1/n^2\sum_{j=1}^{n}X_{j\_1}^2\leq\xi^2\right)\\&=\mathbb{P}^{\mathcal{F}}\left(\sum_{j=1}^{n}\zeta_j^2\geq(\rho^2\xi^2)n\right)+\;\mathbb{P}^{\mathcal{F}}\left(\sum_{j=1}^{n}X_{j\_1}^2\leq n^2\xi^2\right)\\&=\;\;I_{\,1n}\;+\;I_{\,2n}\,.\end{aligned}$$

Next we estimate $I_{\,1n}$ and $I_{\,2n}$.

$$\begin{aligned}I_{\,1n}&=\mathbb{P}^{\mathcal{F}}\left(\sum_{j=1}^{n}\zeta_j^2\geq(\rho^2\xi^2)n\right)\\&=\mathbb{P}^{\mathcal{F}}\left(\sum_{j=1}^{n}(\zeta_j^2-\mathbb{E}^{\mathcal{F}}\zeta_j^2+\mathbb{E}^{\mathcal{F}}\zeta_j^2)\geq(\rho^2\xi^2)n\right)\\&=\mathbb{P}^{\mathcal{F}}\left(\sum_{j=1}^{n}(\zeta_j^2-\mathbb{E}^{\mathcal{F}}\zeta_j^2)\geq(\rho^2\xi^2-\mathbb{E}^{\mathcal{F}}\zeta_j^2)n\right)\\&\leq\mathbb{P}^{\mathcal{F}}\left(\left|\sum_{j=1}^{n}(\zeta_j^2-\mathbb{E}^{\mathcal{F}}\zeta_j^2)\right|\geq(\rho^2\xi^2-\mathbb{E}^{\mathcal{F}}\zeta_j^2)n\right)\end{aligned}\qquad(40)$$

By using the Theorem 3.3 the right hand side of (40) become

$$\begin{aligned}I_{\,1n}&=\mathbb{P}^{\mathcal{F}}\left(\sum_{j=1}^{n}\zeta_j^2\geq(\rho^2\xi^2)n\right)\\&\leq 2exp\left\{-\left(\frac{(\rho^2\xi^2-\mathbb{E}^{\mathcal{F}}\zeta_1^2)n2^{p-1}bp}{B_n^p}\right)^{\frac{1}{2p-1}}(\rho^2\xi^2-\mathbb{E}^{\mathcal{F}}\zeta_1^2)n\left(1-\frac{1}{p-1}\right)\right\}\\&\times\,exp\,\left\{\frac{1}{q}b^{q/p}e\right\}\end{aligned}\qquad(41)$$

We will bound now, the second probability of the right-hand side of the expression $I_{\,2n}$. According to the *Markov's* inequality, it follows for any positive t.

$$\begin{aligned}I_{\,2n}&=\mathbb{P}^{\mathcal{F}}\left(\frac{1}{n^2}\sum_{i=1}^{n}X_{i\_1}^2\leq\xi^2\right)\\&=\mathbb{P}^{\mathcal{F}}\left(\xi^2n^2-\sum_{i=1}^{n}X_{i\_1}^2\geq 0\right)\end{aligned}$$

$$= \mathbb{E}^{\mathcal{F}}\left(\mathbb{I}_{\{n^2 - \sum_{i=1}^{n} X_{i\_1}^2 \geq 0\}}\right)$$
$$\leq \mathbb{E}^{\mathcal{F}}\left(expt\left(n^2\xi^2 - \sum_{i=1}^{n} X_{i\_1}^2\right)\right) \qquad (t > 0)$$
$$\leq e^{tn^2\xi^2}\mathbb{E}^{\mathcal{F}}\left(exp - t\sum_{i=1}^{n} X_{i\_1}^2\right)$$
$$\leq e^{tn^2\xi^2}\prod_{i=1}^{n}\mathbb{E}^{\mathcal{F}}\left(exp - tX_{i_1}^2\right).$$

Since

$$I_{\ 2n} \leq e^{tn^2\xi^2}\prod_{i=1}^{n}\mathbb{E}^{\mathcal{F}}\left(exp - tX_{i_1}^2\right).$$

we first claim that for $x \geq 0$

$$e^{-x} \leq 1 - x + \frac{1}{2}x^2 \,. \tag{42}$$

To see this let $\psi(x) = e^{-x}$ and $\phi(x) = 1 - x + \frac{1}{2}x^2$, $(\psi(x)' = -e^{-x})$ and recall that for every x

$$e^x \geq 1 + x \qquad \forall\, x, \tag{43}$$

so that $\psi(x)' = -e^{-x} \leq -1 + x = \phi(x)'$. Since $\psi(0) = 1 = \phi(0)$ this implies $\psi(x) \leq \phi(x)$ for all $x \geq 0$ and (42) is claimed.

From (42) and (43) it follows that for $t > 0$

$$\begin{aligned} e^{tn^2\xi^2}\prod_{i=1}^{n}\mathbb{E}^{\mathcal{F}}\left(\exp\left(-tX_{i_1}^2\right)\right) &\leq e^{tn^2\xi^2}\left(1 - tK_1 + \frac{t^2}{2}K_2\right)^n \\ &\leq e^{tn^2\xi^2}\left(\exp\left(-tK_1 + \frac{t^2}{2}K_2\right)\right)^n \\ &\leq e^{tn^2\xi^2}\exp\left(-ntK_1 + \frac{t^2}{2}nK_2\right) \end{aligned} \tag{44}$$

where $K_1 = \mathbb{E}^{\mathcal{F}}(X_i^2) < \infty$, $K_2 = \mathbb{E}^{\mathcal{F}}(X_i^4) < \infty$.

Hence

$$I_{\ 2n} = \mathbb{P}^{\mathcal{F}}\left(\sum_{i=1}^{n} X_{i\_1}^2 \leq n^2\xi^2\right) \leq \left[t(n^2\xi^2 - nK_1) + \frac{nt^2K_2}{2}\right]$$

with $h(t) = n^2\xi^2 - nK_1 + \frac{nt^2K_2}{2}$ and $t > 0$, the equation $h'(t) = 0$ has the unique solution $t = \frac{K_1 - n\xi^2}{K_2}$ which minimize $h(t)$. Hence

$$\mathbb{P}^{\mathcal{F}}\left(\sum_{i=1}^{n} X_{i-1}^2 \leq n^2\xi^2\right) \leq \exp\left\{-\frac{1}{2}n\frac{(K_1 - n\xi^2)^2}{K_2}\right\} \tag{45}$$

Then for every $\rho > 0$, $K_1 < \infty$, $K_2 < \infty$, and by the assumption

$$\mathbb{P}^{\mathcal{F}}\left(\sqrt{n}\left|\widehat{\theta_n} - \theta\right| > \rho\right) \leq 2exp\left\{-\left(\frac{(\rho^2\xi^2 - \mathrm{E}^{\mathrm{F}}\zeta_1^2)\mathrm{n}2^{\mathrm{p}-1}\mathrm{bp}}{\mathrm{B}_\mathrm{n}^\mathrm{p}}\right)^{\frac{1}{2\mathrm{p}-1}}(\rho^2\xi^2 - \mathrm{E}^{\mathrm{F}}\zeta_1^2)\mathrm{n}\left(1 - \frac{1}{\mathrm{p}-1}\right)\right\} \tag{46}$$

$$\times exp\left\{\frac{1}{q}b^{q/p}e\right\}+\left\{-\frac{1}{2}n\frac{(K_1-n\xi^2)^2}{K_2}\right\}$$

This completes the proof.

Corollary 4.1. The sequence$(\widehat{\theta_n})_{n\in\mathbb{N}}$ is completels converges to the parameter θ of autoregressive process $AR(1)$ model. Then we have

$$\sum_{n=1}^{\infty} p^F(\sqrt{n}|\widehat{\theta_n}-\theta|>\rho\,)<\infty. \quad (47)$$

Proof: By using Theorem 3.6 and $\mathbb{E}^{\mathcal{F}}(X_i^2)<\infty$,$\mathbb{E}^{\mathcal{F}}(X_i^4)<\infty$ we get the result of (47) immediately.

5. CONCLUSION

Our work consists in establishing some new exponential inequalities for the distribution of sums of $\mathcal{F}-LNQD$ random variables. Using these inequalities, we proved the conditionally complete convergence of first-order autoregressive processes $AR(1)$ with identically distributed ($\mathcal{F}-$ LNQD) errors.

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