ORIGINAL PAPER

APPLICATION OF AN EFFECTIVE METHOD ON THE SYSTEM OF NONLINEAR FUZZY INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. *In real world physical applications purpose, it is complicated to acquire an exact solution of fuzzy differential equations due to complexities in fuzzy arithmetic and therefore creating the need for the use of reliable and efficient techniques in the solution of fuzzy differential equations. The purpose of this research paper is to utilize the reliable analytic approach of homotopy perturbation Sumudu transform method for better understanding of systems of non-linear fuzzy integro-differential equations, while using the concept of fuzzy parameter in certain dynamical problems to remove the hurdles faced in numerical approach. These mathematical models are of great interest in engineering and physics. Some numerical examples are also given to demonstrate the efficiency and superiority of the method, followed by graphical representation of the comparison of exact and approximated solution by using Maple 2017.*

Keywords: *homotopy perturbation method; Sumudu transforms; nonlinear fuzzy integro-differential equations; analytical solution.*

1. INTRODUCTION

Modeling unknown problems with fuzzy set theory is a powerful method. As a result, fuzzy concepts have been used to model a wide range of natural phenomena. In many cases, while modeling the real-world phenomena, the data about the behavior of a dynamical system is uncertain and we have to consider these uncertainties to get close to a more realistic model [1-2]. Fuzzy differential equations are used to model a variety of fascinating real-world problems. The fuzzy differential equation, in particular, is a widely used model in a variety of fields. Fuzzy analysis and fuzzy differential equations have recently been proposed as solutions to the ambiguity induced by missing data in certain mathematical or computer models that predict real-world phenomena. While these fuzzy integral equations are abundantly used in dynamical systems whose characteristics based on vagueness and uncertainty, e.g. in engineering and sciences identified as Volterra's population development models, flexible waves electromagnetic issues and artificial intelligence. In recent times, this research on fuzzy integral and fuzzy integro-differential equations is rapidly and abruptly growing [3-11]. Ahmad and Nosher presented the approximate solution of FIDEs by using the numerical approach of Laplace homotopy perturbation [12]. Biswas and Roy executed ADM on the IVPs of fuzzy IDEs [13]. While in 2019, Hamoud and Ghadle approached these fuzzy Volterra-Fredholm integral equations with the modified Adomian decomposition method [14]. Further, Hooshangian gave the solution of non-linear fuzzy VIDE and gave its existence and uniqueness of solution [15], while Kang, Iqbal Habib contributed in presenting

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the solution Fuzzy IDEs by implementing Sumudu decomposition method [16]. Ahmad and Bibi accessed different types of fuzzy IDEs by using the approach of modified VIM [17]. Recently, Padmapriya et al investigated the numerical solutions of fuzzy fractional delay differential equations using a proposed novel technique [18].

2. MATERIALS AND METHODS

In this section, we will introduce some basic notation related to fuzzy numbers, Sumudu transform and fuzzy integrals.

The Sumudu transform on any set Z is defined by

$$Z = \{\mathrm{w} : \mathrm{w}(t) < Me^{\frac{|t|}{k}}, t \epsilon (-1)^j \times [0, \infty); (M, k_1, k_2 > 0)\}. \tag{1}$$

The Sumudu transform method (Watugala, G. K., 1993) is written by the formula

$$S(f(t)) = \int_0^\infty f(t) e^{-ut} dt = F(v), v \epsilon (-k_1, k_2). \tag{2}$$

The system of non-linear integro-differential equations is represented as

$$\begin{aligned} v^{(m)}(t,\beta) &= f_1(t,\beta) + \int_{a(t)}^{b(t)} \left[K_1\left(t, r, N_1\left(v(r,\beta)\right)\right) + K_1\left(t, r, N_1\left(u(r,\beta)\right)\right) \right] dr, \\ u^{(m)}(t,\beta) &= f_2(t,\beta) + \int_{a(t)}^{b(t)} \left[K_2\left(t, r, N_2\left(v(r,\beta)\right)\right) + K_2\left(t, r, N_2\left(u(r,\beta)\right)\right) \right] dr, \end{aligned} \tag{3}$$

where $v^{(m)}(t,\beta)$ and $u^{(m)}(t,\beta)$ are m^{th} order $f_1(t,\beta)$ and $f_2(t,\beta)$ are fuzzy functions given in advance, λ is constant parameter, $\mathrm{N}(v(r,\beta))$ and $\mathrm{N}(u(r,\beta))$ are non-linear terms and $a(t)$ and $b(t)$ are limits these integrals.

Consider the general system of m^{th} order non-linear FIDEs-2

$$\begin{aligned} y^{(m)}(t,\beta) &= g_1(t,\beta) + \lambda \int_{a(t)}^{b(t)} \left[K_1(t,s) N_1\left(y(s,\beta)\right) + K_1(t,s) N_1\left(w(s,\beta)\right) \right] ds, \\ w^{(m)}(t,\beta) &= g_2(t,\beta) + \lambda \int_{a(t)}^{b(t)} \left[K_2(t,s) N_2\left(y(s,\beta)\right) + K_2(t,s) N_2\left(w(s,\beta)\right) \right] ds, \end{aligned} \tag{4}$$

subject to conditions

$$\begin{aligned} & y(0,\beta) = (a_0, b_0), y'(0,\beta) = (a_1, b_1), y''(0,\beta) = (a_2, b_2), \ldots, y^{(m-1)}(0,\beta) = (a_{m-1}, b_{m-1}), \\ & w(0,\beta) = (c_0, d_0), w'(0,\beta) = (c_1, d_1), w''(0,\beta) = (c_2, d_2), \ldots, w^{(m-1)}(0,\beta) = (c_{m-1}, d_{m-1}), \end{aligned} \tag{5}$$

where $y^{(m)}(t,\beta)$ and $w^{(m)}(t,\beta)$ are m^{th} order derivative of fuzzy functions $y(t,\beta)$ and $w(t,\beta)$ also $g_1(t,\beta)$ and $g_2(t,\beta)$ are fuzzy functions given in advance, λ is constant parameter, $K_1(t,s),\tilde{K}_1(t,s)$ and $K_2(t,s),\tilde{K}_2(t,s)$ are kernels and $N(w(s,\beta))$ and $N(y(s,\beta))$ are the non-linear terms and $a(t)$ and $b(t)$ are limits of FIDE-2. Where

$$w^{(m)}(t,\beta)=(\underline{w}^{(m)}(t,\beta),\overline{w}^{(m)}(t,\beta)),\ y^{(m)}(t,\beta)=(\underline{y}^{(m)}(t,\beta),\overline{y}^{(m)}(t,\beta)),\text{ and}$$
$$g_1(t,\beta)=(\underline{g}_1(t,\beta),\overline{g}_1(t,\beta)),\ g_2(t,\beta)=(\underline{g}_2(t,\beta),\overline{g}_2(t,\beta)).$$

The homotopy is constructed as

$$\begin{aligned}
&H(\underline{x},p,\beta)=(1-p)\left[\underline{x}^{(m)}(t,\beta)-\underline{y}_0(t,\beta)\right]+\\
&p\left[\underline{x}^{(m)}(t,\beta)-\underline{g}_1(t,\beta)-\int_{a(t)}^{b(t)}\left[K_1(t,s)\underline{N_1}(x(s,\beta))+K_1(t,s)\underline{N_1}(z(s,\beta))\right]ds\right]=0,\\
&H(\overline{x},p,\beta)=(1-p)\left[\overline{x}^{(m)}(t,\beta)-\overline{y}_o(t,\beta)\right]+\\
&p\left[\overline{x}^{(m)}(t,\beta)-\overline{g}_1(t,\beta)-\int_{a(t)}^{b(t)}\left[K_1(t,s)\overline{N_1}(x(s,\beta))+K_1(t,s)\overline{N_1}(z(s,\beta))\right]ds\right]=0,\\
&H(\underline{z},p,\beta)=(1-p)\left[\underline{z}^{(m)}(t,\beta)-\underline{w}_0(t,\beta)\right]+\\
&p\left[\underline{z}^{(m)}(t,\beta)-\underline{g}_2(t,\beta)-\int_{a(t)}^{b(t)}\left[K_2(t,s)\underline{N_2}(x(s,\beta))+K_2(t,s)\underline{N_2}(z(s,\beta))\right]ds\right]=0,\\
&H(\overline{z},p,\beta)=(1-p)\left[\overline{z}^{(m)}(t,\beta)-\overline{w}_o(t,\beta)\right]+\\
&p\left[\overline{z}^{(m)}(t,\beta)-\overline{g}_2(t,\beta)-\int_{a(t)}^{b(t)}\left[K_2(t,s)\overline{N_2}(x(s,\beta))+K_2(t,s)\overline{N_2}(z(s,\beta))\right]ds\right]=0.
\end{aligned}\tag{6}$$

As the initial conditions are defined as

$$\begin{cases}\underline{y}_0(t,\beta)=\underline{g}_1(t,\beta)\\ \overline{y}_0(t,\beta)=\overline{g}_1(t,\beta)\\ \underline{w}_0(t,\beta)=\underline{g}_2(t,\beta)\\ \overline{w}_0(t,\beta)=\overline{g}_2(t,\beta)\end{cases}. \tag{7}$$

Then applying Sumudu transform of Eq. (7) as

$$\begin{cases} S\left\{\underline{x}^{(m)}(t,\beta)\right\} = S\left\{\underline{g}_1(t,\beta) + p\int_{a(t)}^{b(t)}\left[K_1(t,s)\underline{N_1}\left(x(s,\beta)\right) + K_1(t,s)\underline{N_1}\left(z(s,\beta)\right)\right]ds\right\} \\ S\left\{\overline{x}^{(m)}(t,\beta)\right\} = S\left\{\overline{g}_1(t,\beta) + p\int_{a(t)}^{b(t)}\left[K_1(t,s)\overline{N_1}\left(x(s,\beta)\right) + K_1(t,s)\overline{N_1}\left(z(s,\beta)\right)\right]ds\right\} \\ S\left\{\underline{z}^{(m)}(t,\beta)\right\} = S\left\{\underline{g}_2(t,\beta) + p\int_{a(t)}^{b(t)}\left[K_2(t,s)\underline{N_2}\left(x(s,\beta)\right) + K_2(t,s)\underline{N_2}\left(z(s,\beta)\right)\right]ds\right\} \\ S\left\{\overline{z}^{(m)}(t,\beta)\right\} = S\left\{\overline{g}_2(t,\beta) + \int_{a(t)}^{b(t)}\left[K_2(t,s)\overline{N_2}\left(x(s,\beta)\right) + \widetilde{K_2(t,s)}\overline{\overline{N_2(z(s,\beta))}}\right]ds\right\} \end{cases}. \tag{8}$$

Now using the differential property of Sumudu transform and inverse Sumudu transform on Eq. (8) respectively

$$\begin{cases} \underline{x}^{(m)}(t,\beta) = S^{-1}\left\{(v^m)\left(\begin{array}{l} \dfrac{\underline{x}(0,\beta)}{v^m} + \dfrac{\underline{x}'(0,\beta)}{v^{m-1}} + \dfrac{\underline{x}''(0,\beta)}{v^{m-2}} + \ldots + \dfrac{\underline{x}^{m-1}(0,\beta)}{v} \\ +S\left\{\underline{g}_1(t,\beta) + p\int_{a(t)}^{b(t)}\left[K_1(t,s)\underline{N_1}\left(x(s,\beta)\right) + K_1(t,s)\underline{N_1}\left(z(s,\beta)\right)\right]ds\right\}\end{array}\right)\right\} \\ \overline{x}^{(m)}(t,\beta) = S^{-1}\left\{(v^m)\left(\begin{array}{l} \left\{\dfrac{\overline{x}(0,\beta)}{v^m} + \dfrac{\overline{x}'(0,\beta)}{v^{m-1}} + \dfrac{\overline{x}''(0,\beta)}{v^{m-2}} + \ldots + \dfrac{\overline{x}^{m-1}(0,\beta)}{v}\right\} \\ +S\left\{\overline{g}_1(t,\beta) + p\int_{a(t)}^{b(t)}\left[K_1(t,s)\overline{N_1}\left(x(s,\beta)\right) + \widetilde{K_1(t,s)}\overline{\overline{N_1(z(s,\beta))}}\right]ds\right\}\end{array}\right)\right\} \\ \underline{z}^{(m)}(t,\beta) = S^{-1}\left[(v^m)\left(\begin{array}{l} \left\{\dfrac{\underline{z}(0,\beta)}{v^m} + \dfrac{\underline{z}'(0,\beta)}{v^{m-1}} + \dfrac{\underline{z}''(0,\beta)}{v^{m-2}} + \ldots + \dfrac{\underline{z}^{m-1}(0,\beta)}{v}\right\} \\ +S\left\{\underline{g}_2(t,\beta) + p\int_{a(t)}^{b(t)}\left[K_2(t,s)\underline{N_2}\left(x(s,\beta)\right) + \widetilde{K_2(t,s)}\underline{\widetilde{N_2(z(s,\beta))}}\right]ds\right\}\end{array}\right)\right] \\ \overline{z}^{(m)}(t,\beta) = S^{-1}\left\{(v^m)\left(\begin{array}{l} \left\{\dfrac{\overline{z}(0,\beta)}{v^m} + \dfrac{\overline{z}'(0,\beta)}{v^{m-1}} + \dfrac{\overline{z}''(0,\beta)}{v^{m-2}} + \ldots + \dfrac{\overline{z}^{m-1}(0,\beta)}{v}\right\} \\ +S\left\{\overline{g}_2(t,\beta) + p\int_{a(t)}^{b(t)}\left[K_2(t,s)\overline{N_2}\left(x(s,\beta)\right) + \widetilde{K_2(t,s)}\overline{\overline{N_2(z(s,\beta))}}\right]ds\right\}\end{array}\right)\right\} \end{cases}. \tag{9}$$

Assuming the solution of Eq. (3) is stated in series as power of parameter p

$$\begin{cases} \underline{x}(t,\beta) = \sum_{j=0}^{\infty} p^j \underline{x}_j \\ \overline{x}(t,\beta) = \sum_{j=0}^{\infty} p^j \overline{x}_j \\ \underline{z}(t,\beta) = \sum_{j=0}^{\infty} p^j \underline{z}_j \\ \overline{z}(t,\beta) = \sum_{j=0}^{\infty} p^j \overline{z}_j \end{cases}. \tag{10}$$

As a result, we have

$$p^0:\begin{cases}\underline{x}_0(t,\beta)=S^{-1}\left\{(v^m)\left(\begin{array}{l}\left\{\dfrac{\underline{x}(0,\beta)}{v^m}+\dfrac{\underline{x}'(0,\beta)}{v^{m-1}}+\dfrac{\underline{x}''(0,\beta)}{v^{m-2}}+\ldots+\dfrac{\underline{x}^{m-1}(0,\beta)}{v}\right\}\\+S\left\{\underline{g}_1(t,\beta)\right\}\end{array}\right)\right\}\\ \overline{x}_0(t,\beta)=S^{-1}\left\{(v^m)\left(\begin{array}{l}\left\{\dfrac{\overline{x}(0,\beta)}{v^m}+\dfrac{\underline{x}'(0,\beta)}{v^{m-1}}+\dfrac{\underline{x}''(0,\beta)}{v^{m-2}}+\ldots+\dfrac{\overline{x}^{m-1}(0,\beta)}{v}\right\}\\+S\left\{\overline{g}_1(t,\beta)\right\}\end{array}\right)\right\}\\ \underline{z}_0(t,\beta)=S^{-1}\left\{(v^m)\left(\begin{array}{l}\left\{\dfrac{\underline{z}(0,\beta)}{v^m}+\dfrac{\underline{z}'(0,\beta)}{v^{m-1}}+\dfrac{\underline{z}''(0,\beta)}{v^{m-2}}+\ldots+\dfrac{\underline{z}^{m-1}(0,\beta)}{v}\right\}\\+S\left\{\underline{g}_2(t,\beta)\right\}\end{array}\right)\right\}\\ \overline{z}_0(t,\beta)=S^{-1}\left\{(v^m)\left(\begin{array}{l}\left\{\dfrac{\overline{z}(0,\beta)}{v^m}+\dfrac{\overline{z}'(0,\beta)}{v^{m-1}}+\dfrac{\overline{z}''(0,\beta)}{v^{m-2}}+\ldots+\dfrac{\overline{z}^{m-1}(0,\beta)}{v}\right\}\\+S\left\{\overline{g}_2(t,\beta)\right\}\end{array}\right)\right\}\end{cases},$$

$$p^1:\begin{cases}\underline{x}_1(t,\beta)=S^{-1}\left\{(v^m)\left(\int_{a(t)}^{b(t)}\left[K_1(t,s)\underline{N_1}\left(x_0(s,\beta)\right)+K_1(t,s)\underline{N_1}\left(z_0(s,\beta)\right)\right]ds\right)\right\}\\ \overline{x}_1(t,\beta)=S^{-1}\left\{(v^m)\left(\int_{a(t)}^{b(t)}\left[K_1(t,s)\overline{N_1}\left(x_0(s,\beta)\right)+K_1(t,s)\overline{N_1}\left(z_0(s,\beta)\right)\right]ds\right)\right\}\\ \underline{z}_1(t,\beta)=S^{-1}\left\{(v^m)\left(\int_{a(t)}^{b(t)}\left[K_2(t,s)\overline{N_2}\left(x_0(s,\beta)\right)+K_2(t,s)\underline{N_2}\left(z_0(s,\beta)\right)\right]ds\right)\right\}\\ \overline{z}_1(t,\beta)=S^{-1}\left\{(v^m)\left(\int_{a(t)}^{b(t)}\left[K_2(t,s)\overline{N_2}\left(x_0(s,\beta)\right)+\widetilde{K_2(t,s)}\overline{N_2(z_0}(s,\beta))\right]ds\right)\right\}\end{cases},$$

$\vdots,$

Finally, the series solution is

$$\begin{cases}\underline{y}(t,\beta)=\lim_{p\to1}\underline{x}(t,\beta)=\sum_{j=0}^{\infty}\underline{x}_j(t,\beta)\\ \overline{y}(t,\beta)=\lim_{p\to1}\overline{x}(t,\beta)=\sum_{j=0}^{\infty}\overline{x}_j(t,\beta)\\ \underline{w}(t,\beta)=\lim_{p\to1}\underline{z}(t,\beta)=\sum_{j=0}^{\infty}\underline{z}_j(t,\beta)\\ \overline{w}(t,\beta)=\lim_{p\to1}\overline{z}(t,\beta)=\sum_{j=0}^{\infty}\overline{z}_j(t,\beta)\end{cases}. \tag{11}$$

3. RESULTS AND DISCUSSION

In this section, we will discuss the different types of non-linear FVIDEs and find their solution by using HPSTM.

3.1. CONSIDER A SYSTEM OF NON-LINEAR FUZZY VIDE-2 AS

$$\begin{aligned} y'(t,\beta) &= g_1(t,\beta)+\int_0^t (y^2(s,\beta)+w^2(s,\beta))ds \\ w'(t,\beta) &= g_2(t,\beta)+\int_0^t y(s,\beta)w(s,\beta))ds, \end{aligned} \tag{12}$$

subject to the initial conditions $y(0,\beta)=w(0,\beta)=(0,0)$ and $\lambda=1, 0\le s\le t, 0\le\beta\le 1$ where

$$g_1(t,\beta)=(\beta,(3-\beta)), g_2(t,\beta)=(2t\beta^2,\ 2t(3-\beta)^2). \tag{13}$$

According to the above described procedure, we have

$$\begin{cases} H(\underline{x},p,\beta)=\underline{x}'(t,\beta)-\beta-p\int_0^t\left(\underline{x}^2(s,\beta)+\underline{u}^2(s,\beta)\right)ds=0 \\ H(\overline{x},p,\beta)=\overline{x}'(t,\beta)-(3-\beta)-p\int_0^t\left(\overline{x}^2(s,\beta)+\overline{u}^2(s,\beta)\right)ds=0 \\ H(\underline{u},p,\beta)=\underline{u}'(t,\beta)-2t\beta^2-p\int_0^t\left(\underline{x}(s,\beta)\underline{u}(s,\beta)\right)ds=0 \\ H(\overline{u},p,\beta)=\overline{u}'(t,\beta)-2t(3-\beta)^2-p\int_0^t\left(\overline{x}(s,\beta)\overline{u}(s,\beta)\right)ds=0 \end{cases}. \tag{14}$$

By using Sumudu transform, differential property and inverse transform on Eq. (14), we have

$$\begin{cases} \underline{x}(t,\beta)=S^{-1}\left\{v\left(S\left\{\beta+p\int_0^t\left(\underline{x}^2(s,\beta)+\underline{u}^2(s,\beta)\right)ds\right\}\right)\right\} \\ \overline{x}(t,\beta)=S^{-1}\left\{v\left(S\left\{(3-\beta)+p\int_0^t\left(\overline{x}^2(s,\beta)+\overline{u}^2(s,\beta)\right)ds\right\}\right)\right\} \\ \underline{u}(t,\beta)=S^{-1}\left\{v\left(S\left\{2t\beta^2+p\int_0^t\left(\underline{x}(s,\beta)\underline{u}(s,\beta)\right)ds\right\}\right)\right\} \\ \overline{u}(t,\beta)=S^{-1}\left\{v\left(S\left\{2t(3-\beta)^2+p\int_0^t\left(\overline{x}(s,\beta)\overline{u}(s,\beta)\right)ds\right\}\right)\right\} \end{cases}. \tag{15}$$

Consequently, we get

$$p^0:\begin{cases}\underline{x}_0(t,\beta)=t\beta\\ \overline{x}_0(t,\beta)=t(3-\beta)\\ \underline{u}_0(t,\beta)=t^2\beta^2\\ \overline{u}_0(t,\beta)=t^2(3-\beta)^2\end{cases},$$

$$p^1:\begin{cases}\underline{x}_1(t,\beta)=\dfrac{t^4}{6}\beta^2\\ \overline{x}_1(t,\beta)=\dfrac{t^4}{6}(3-\beta)^2\\ \underline{u}_1(t,\beta)=\dfrac{t^5}{20}\beta^3\\ \overline{u}_1(t,\beta)=\dfrac{t^5}{20}(3-\beta)^3\end{cases},$$

$\vdots,$

The approximated solution is given as

$$\begin{cases}\underline{y}(t,\beta)=t\beta+\dfrac{t^4}{6}\beta^2+\ldots\\ \overline{y}(t,\beta)=t(3-\beta)+\dfrac{t^4}{6}(3-\beta)^2+\ldots\\ \underline{w}(t,\beta)=t^2\beta^2+\dfrac{t^5}{20}\beta^3+\ldots\\ \overline{w}(t,\beta)=t^2(3-\beta)^2+\dfrac{t^5}{20}(3-\beta)^3+\ldots\end{cases}. \tag{16}$$

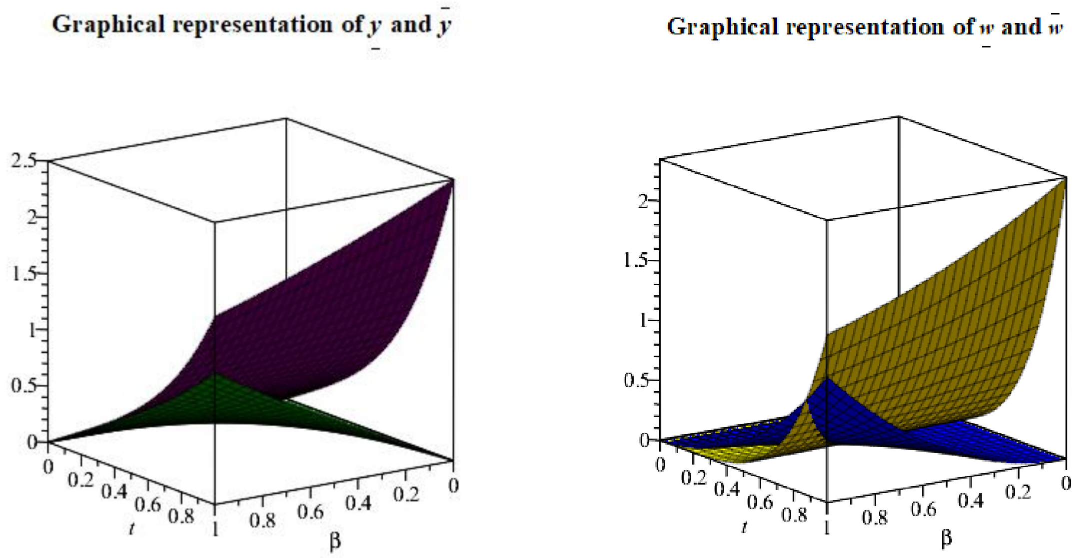

Figure 1. Graphical representation of *y* and *w* in 3D.

3.2. CONSIDER THE SYSTEM OF NON-LINEAR FFIDE-2

$$\begin{aligned} y'(t,\beta) &= g_1(t,\beta) + \int_{-1}^{1} (y^2(s,\beta) + w^2(s,\beta))\, ds, \\ w'(t,\beta) &= g_2(t,\beta) + \int_{-1}^{1} (y^2(s,\beta) - w^2(s,\beta))\, ds \ , \end{aligned} \tag{17}$$

with initial conditions $y(0,\beta) = w(0,\beta) = (0,0)$ with $\lambda = 1,\ -1 \le s \le 1,\ 0 \le \beta \le 1,$

$$g_1(t,\beta) = (\beta,\ (5-\beta))$$
$$g_2(t,\beta) = (2t\beta,\ 2t(5-\beta))\ .$$

According to the above described procedure, we have

$$\begin{cases} H(\underline{x}, p, \beta) = \underline{x}'(t,\beta) - \beta - p\int_{-1}^{1}\left(\underline{x}^2(s,\beta) + \underline{u}^2(s,\beta)\right) ds = 0 \\ H(\overline{x}, p, \beta) = \overline{x}'(t,\beta) - (5-\beta) - p\int_{-1}^{1}\left(\overline{x}^2(s,\beta) + \overline{u}^2(s,\beta)\right) ds = 0 \\ H(\underline{u}, p, \beta) = \underline{u}'(t,\beta) - 2t\beta - p\int_{-1}^{1}\left(\underline{x}^2(s,\beta) - \underline{u}^2(s,\beta)\right) ds = 0 \\ H(\overline{u}, p, \beta) = \overline{u}'(t,\beta) - 2t(5-\beta) - p\int_{-1}^{1}\left(\overline{x}^2(s,\beta) - \overline{u}^2(s,\beta)\right) ds = 0 \end{cases} . \tag{18}$$

Applying the Sumudu transform and inverse transform as

$$\begin{cases} \underline{x}(t,\beta) = S^{-1}\left\{ v\left(S\left\{ \beta + p\int_{-1}^{1}\left(\underline{x}^2(s,\beta) + \underline{u}^2(s,\beta)\right) ds \right\}\right)\right\} \\ \overline{x}(t,\beta) = S^{-1}\left\{ v\left(S\left\{ (5-\beta) + p\int_{-1}^{1}\left(\overline{x}^2(s,\beta) + \overline{u}^2(s,\beta)\right) ds \right\}\right)\right\} \\ \underline{u}(t,\beta) = S^{-1}\left\{ v\left(S\left\{ 2t\beta + p\int_{-1}^{1}\left(\underline{x}^2(s,\beta) - \underline{u}^2(s,\beta)\right) ds \right\}\right)\right\} \\ \overline{u}(t,\beta) = S^{-1}\left\{ v\left(S\left\{ 2t(5-\beta) - p\int_{-1}^{1}\left(\overline{x}^2(s,\beta) - \overline{u}^2(s,\beta)\right) ds \right\}\right)\right\} \end{cases} . \tag{19}$$

Equating the powers of parameter p

$$p^0:\begin{cases}\underline{x}_0(t,\beta)=t\,\beta\\ \overline{x}_0(t,\beta)=t(5-\beta)\\ \underline{u}_0(t,\beta)=t^2\beta\\ \overline{u}_0(t,\beta)=t^2(5-\beta)\end{cases},$$

$$p^1:\begin{cases}\underline{x}_1(t,\beta)=\frac{16}{15}t\beta^2\\ \overline{x}_1(t,\beta)=\frac{16}{15}\mathrm{t}(5-\beta)^2\\ \underline{u}_1(t,\beta)=\frac{4}{15}t\,\beta^2\\ \overline{u}_1(t,\beta)=\frac{4}{15}\mathrm{t}(5-\beta)^2\end{cases},$$

$\vdots,$

Consequently, the approximate solution is

$$\begin{cases}\underline{y}(t,\beta)=t\,\beta+\frac{16}{15}t\beta^2+\dots\\ \overline{y}(t,\beta)=t(5-\beta)+\frac{16}{15}\mathrm{t}(5-\beta)^2+\dots\\ \underline{w}(t,\beta)=t^2\beta^2+\frac{4}{15}t\beta^2+\dots\\ \overline{w}(t,\beta)=t^2(5-\beta)+\frac{4}{15}\mathrm{t}(5-\beta)^2+\dots\end{cases}. \tag{20}$$

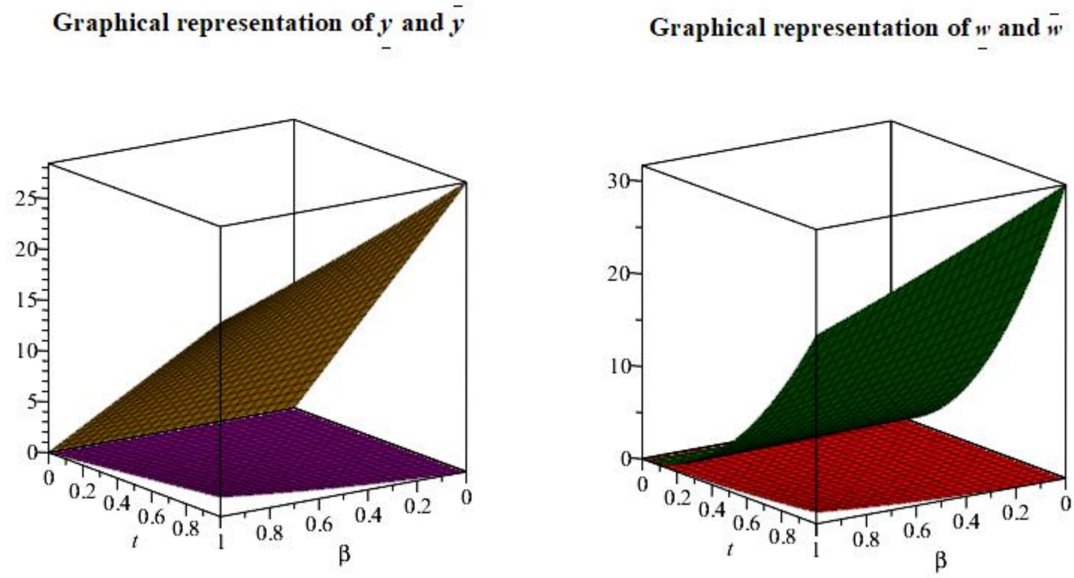

Figure 2. Graphical demonstration of y and w in 3D.

3.3. CONSIDER THE SYSTEM OF NON-LINEAR FFIDE-2

$$\begin{aligned} y'(t,\beta) &= g_1(t,\beta) + \int_{-1}^{1} (w^2(s,\beta) + z^2(s,\beta))ds, \\ w'(t,\beta) &= g_2(t,\beta) + \int_{-1}^{1} (y^2(s,\beta) + z^2(s,\beta))ds, \\ z'(t,\beta) &= g_3(t,\alpha) + \int_{-1}^{1} (y^2(s,\beta) + w^2(s,\beta))ds, \end{aligned} \tag{21}$$

with $y(0,\beta) = w(0,\beta) = z(0,\beta) = (0,0)$, $\lambda = 1,\ -1 \le s \le 1,\ 0 \le \beta \le 1,$

$$\begin{aligned} g_1(t,\beta) &= (\beta^2 + 2\beta,\ 3 - 2\beta) \\ g_2(t,\beta) &= (2t(\beta^2 + 2\beta),\ 2t(3 - 2\beta))\ . \\ g_3(t,\beta) &= (3t^2(\beta^2 + 2\beta)^2, 3t^2(3 - 2\beta)^2) \end{aligned} \tag{22}$$

With the help of homotopy perturbation method, we construct the homotopy as

$$\begin{cases} H(\underline{x}, p, \beta) = \underline{x}'(t,\beta) - (\beta^2 + 2\beta) - p\int_{-1}^{1} \left(\underline{u}^2(s,\beta) + \underline{v}^2(s,\beta)\right) ds = 0 \\ H(\overline{x}, p, \beta) = \overline{x}'(t,\beta) - (3 - 2\beta) - p\int_{-1}^{1} \left(\overline{u}^2(s,\beta) + \overline{v}^2(s,\beta)\right) ds = 0, \\ H(\underline{u}, p, \beta) = \underline{u}'(t,\beta) - 2t(\beta^2 + 2\beta) - p\int_{-1}^{1} \left(\underline{x}^2(s,\beta) + \underline{v}^2(s,\beta)\right) ds = 0, \\ H(\overline{u}, p, \beta) = \overline{u}'(t,\beta) - 2t(3 - 2\beta) - p\int_{-1}^{1} \left(\overline{x}^2(s,\beta) + \overline{v}^2(s,\beta)\right) ds = 0, \\ H(\underline{v}, p, \beta) = \underline{v}'(t,\beta) - 3t^2(\beta^2 + 2\beta)^2 - p\int_{-1}^{1} \left(\underline{x}^2(s,\beta) + \underline{u}^2(s,\beta)\right) ds = 0, \\ H(\overline{v}, p, \beta) = \overline{v}'(t,\beta) - 3t^2(3 - 2\beta)^2 - p\int_{-1}^{1} \left(\overline{x}^2(s,\beta) + \overline{u}^2(s,\beta)\right) ds = 0, \end{cases} . \tag{23}$$

Using Sumudu transform and its differential property and taking the inverse transform, we have

$$\begin{cases} \underline{x}(t,\beta)=S^{-1}\left\{v\left(S\left\{\underline{g}_1(t,\beta)+p\int_{-1}^{1}(\underline{u}^2(s,\beta)+\underline{v}^2(s,\beta))ds\right\}\right)\right\} \\ \overline{x}(t,\beta)=S^{-1}\left\{v\left(S\left\{\overline{g}_1(t,\beta)+p\int_{-1}^{1}(\overline{u}^2(s,\beta)+\overline{v}^2(s,\beta))ds\right\}\right)\right\} \\ \underline{u}(t,\beta)=S^{-1}\left\{v\left(S\left\{\underline{g}_2(t,\beta)+p\int_{-1}^{1}(\underline{x}^2(s,\beta)+\underline{v}^2(s,\beta))ds\right\}\right)\right\} \\ \overline{u}(t,\beta)=S^{-1}\left\{v\left(S\left\{\overline{g}_2(t,\beta)+p\int_{-1}^{1}(\overline{x}^2(s,\beta)+\overline{v}^2(s,\beta))ds\right\}\right)\right\} \\ \underline{v}(t,\beta)=S^{-1}\left\{v\left(S\left\{\underline{g}_3(t,\beta)+p\int_{-1}^{1}(\underline{x}^2(s,\beta)+\underline{u}^2(s,\beta))ds\right\}\right)\right\} \\ \overline{v}(t,\beta)=S^{-1}\left\{v\left(S\left\{\overline{g}_3(t,\beta)+p\int_{-1}^{1}(\overline{x}^2(s,\beta)+\overline{u}^2(s,\beta))ds\right\}\right)\right\} \end{cases}. \tag{24}$$

Consequently, we have

$$p^0:\begin{cases} \underline{x}_0(t,\beta)=t(\beta^2+2\beta) \\ \overline{x}_0(t,\beta)=t(3-2\beta) \\ \underline{u}_0(t,\beta)=t^2(\beta^2+2\beta) \\ \overline{u}_0(t,\beta)=t^2(3-2\beta) \\ \underline{v}_0(t,\beta)=t^3(\beta^2+2\beta)^2 \\ \overline{v}_0(t,\beta)=t^3(3-2\beta)^2 \end{cases},$$

$$p^1=\begin{cases} \underline{x}_1(t,\beta)=t\frac{2}{5}(\beta^2+2\beta)^2+t\frac{2}{7}(\beta^2+2\beta)^4 \\ \overline{x}_1(t,\beta)=t\frac{2}{5}(3-2\beta)^2+t\frac{2}{7}(3-2\beta)^4 \\ \underline{u}_1(t,\beta)=t\frac{2}{3}(\beta^2+2\beta)^2+t\frac{2}{7}(\beta^2+2\beta)^4 \\ \overline{u}_1(t,\beta)=t\frac{2}{3}(3-2\beta)^2+t\frac{2}{7}(3-2\beta)^4 \\ \underline{v}_1(t,\beta)=t\frac{2}{3}(\beta^2+2\beta)^2+t\frac{2}{5}(\beta^2+2\beta)^2 \\ \overline{v}_1(t,\beta)=t\frac{2}{3}(3-2\beta)^2+t\frac{2}{5}(3-2\beta)^2 \end{cases},$$

$\vdots$,

We get the solution in the form of series as

$$\begin{cases} \underline{y}(t,\beta) = t(\beta^2+2\beta) + \frac{2}{5}t(\beta^2+2\beta)^2 + \frac{2}{7}t(\beta^2+2\beta)^4 + \ldots \\ \overline{y}(t,\beta) = t(3-2\beta) + \frac{2}{5}t(3-2\beta)^2 + \frac{2}{7}t(3-2\beta)^4 + \ldots \\ \underline{w}(t,\beta) = t^2(\beta^2+2\beta) + \frac{2}{3}t(\beta^2+2\beta)^2 + \frac{2}{7}t(\beta^2+2\beta)^4 + \ldots \\ \overline{w}(t,\beta) = t^2(3-2\beta) + \frac{2}{3}t(3-2\beta)^2 + \frac{2}{7}t(3-2\beta)^4 + \ldots \\ \underline{v}(t,\beta) = t^3(\beta^2+2\beta)^2 + \frac{2}{3}t(\beta^2+2\beta)^2 + \frac{2}{5}t(\beta^2+2\beta)^2 + \ldots \\ \overline{v}(t,\beta) = t^3(3-2\beta)^2 + \frac{2}{3}t(3-2\beta)^2 + \frac{2}{5}t(3-2\beta)^2 + \ldots \end{cases} \quad (25)$$

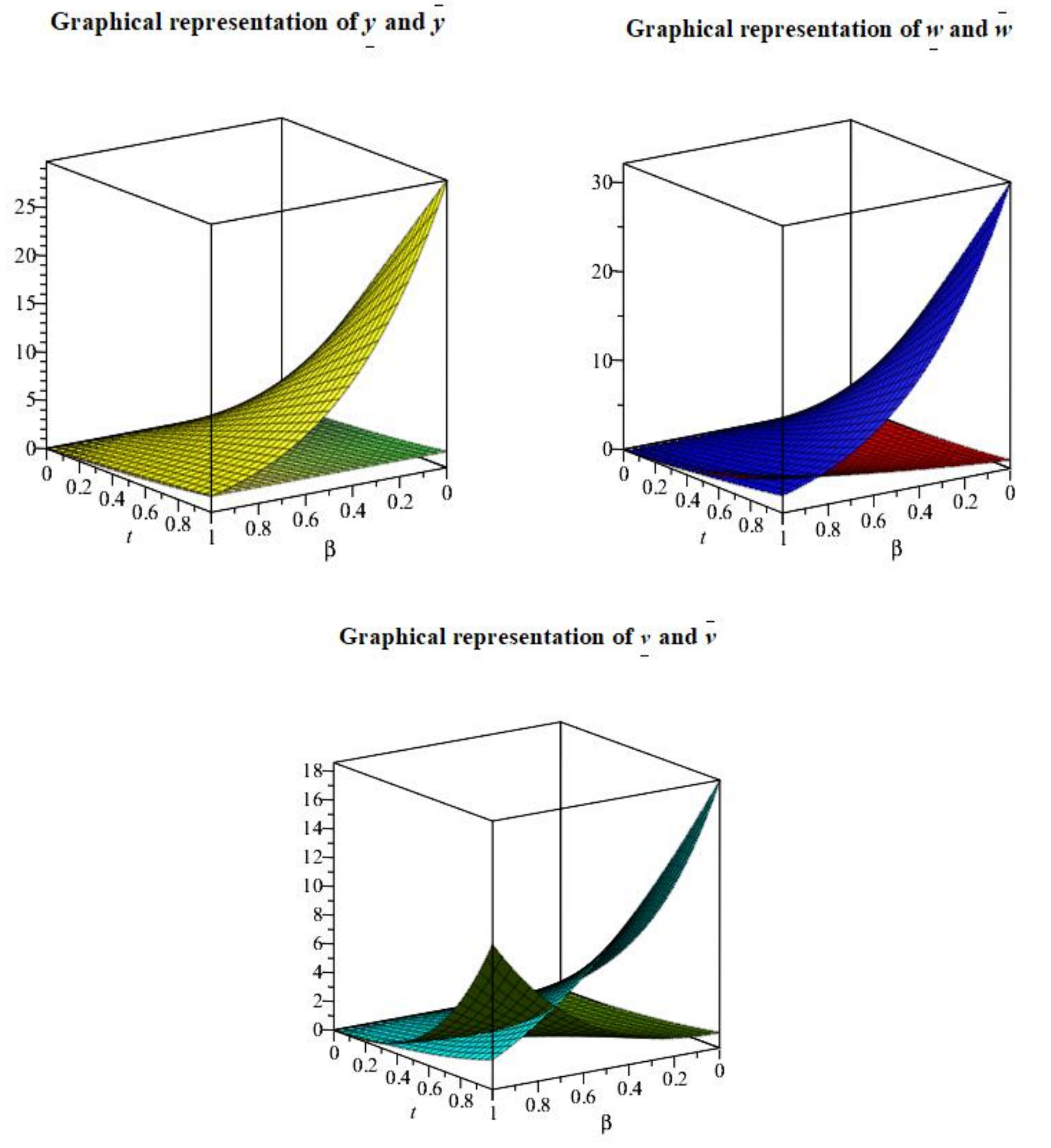

Figure 3. Graphical representation of y, w and z in 3D.

3.4. CONSIDER THE SYSTEM OF NON-LINEAR FVIDE OF 2ND KIND AS

$$
\begin{aligned}
y'(t,\beta) &= g_1(t,\beta) + \int_0^t \left[-y(s,\beta) + w(s,\beta) \right] ds, \\
w'(t,\beta) &= g_2(t,\beta) + \int_0^t \left[y(s,\beta) - w(s,\beta) \right] ds,
\end{aligned} \tag{26}
$$

with initial conditions as

$$y(0,\beta)=0, w(0,\beta)=1 \text{ with } \lambda=1,\ -1\le s\le 1,\ 0\le\beta\le 1.$$

Along fuzzy function

$$
\begin{cases}
\underline{g}_1(t,\beta) = \beta t + \dfrac{3}{26} - \dfrac{3\beta}{26} - \dfrac{t^2}{13} - \dfrac{t^2\beta}{13} \\
\overline{g}_1(t,\beta) = 2t - \beta t + \dfrac{3\beta}{26} - \dfrac{3t^2}{13} + \dfrac{t^2\beta}{13} - \dfrac{3}{26} \\
\underline{g}_2(t,\beta) = \left(\beta + \beta^2\right) \\
\overline{g}_2(t,\beta) = \left(4 - \beta - \beta^3\right)
\end{cases}. \tag{27}
$$

According to the above described procedure, we have

$$
\begin{cases}
\underline{x}(t,\beta) = S^{-1}\left\{ v\left\{ S\left\{ \underline{g}_1(t,\beta) - p\int_0^t \left[-\underline{x}(s,\beta) + \underline{u}(s,\beta) \right] ds \right\}\right\}\right\} \\
\overline{x}(t,\beta) = S^{-1}\left\{ v\left\{ S\left\{ \overline{g}_1(t,\beta) + p\int_0^t \left[-\overline{x}(s,\beta) + \overline{u}(s,\beta) \right] ds \right\}\right\}\right\} \\
\underline{u}(t,\beta) = S^{-1}\left\{ v\left\{ S\left\{ \underline{g}_2(t,\beta) + p\int_0^t \left[\underline{x}(s,\beta) - \underline{u}(s,\beta) \right] ds \right\}\right\}\right\} \\
\overline{u}(t,\beta) = S^{-1}\left\{ v\left\{ S\left\{ \overline{g}_2(t,\beta) + p\int_0^t \left[\overline{x}(s,\beta) - \overline{u}(s,\beta) \right] ds \right\}\right\}\right\}
\end{cases}. \tag{29}
$$

The component form of parameter p is

$$
p^0 : \begin{cases}
\underline{x}_0(t,\beta) = \dfrac{t^2}{2} + \dfrac{3t}{26} - \dfrac{3\beta}{26} - \dfrac{t^3}{39} - \dfrac{2t^3\beta}{39} \\
\overline{x}_0(t,\beta) = t^2 + \dfrac{\beta t}{2} - \dfrac{3\beta t^2}{2} - \dfrac{\beta t^3}{3} - \dfrac{t}{18} - \dfrac{t^3}{52} \\
\underline{u}_0(t,\beta) = \left(\beta+\beta^2\right) + t\left(\beta+\beta^2\right) \\
\overline{u}_0(t,\beta) = \left(4-\beta-\beta^2\right) + t\left(4-\beta-\beta^2\right)
\end{cases},
$$

$$p^1:\begin{cases}\underline{x}_1(t,\beta)=\frac{t^2}{2}\left(\beta^2+\beta+\frac{3\beta}{26}\right)+\frac{t^3}{6}\left(\beta^2+\beta+\frac{2\beta}{39}\right)-\frac{t^4}{24}-\frac{t^6}{120}\\ \overline{x}_1(t,\beta)=\frac{t^2}{2}\left(4-\beta^3-\beta\right)+\frac{t^3}{3}\left(-\beta^3-\beta+\frac{71}{18}\right)+\frac{t^4}{12}\left(1-\frac{\beta}{2}\right)+\frac{t^5}{20}\left(\frac{\beta}{3}-\frac{1}{52}\right)\\ \underline{u}_1(t,\beta)=t\left(\beta^2+\frac{29\beta}{26}\right)+t^2\left(\beta^2+\frac{29\beta}{26}\right)+\frac{t^3}{6}\left(-2+\beta^2+\frac{29\beta}{26}\right)-\frac{t^4}{20}\left(\frac{2}{3}\beta-1\right),\\ \overline{u}_1(t,\beta)=-t\left(4-\beta^3-\beta\right)-\frac{t^2}{2}\left(-\frac{5}{2}\beta+\frac{145}{18}\right)+\frac{t^3}{6}\left(-\beta^2-\frac{5}{2}\beta+\frac{53}{18}\right)+\\ \frac{t^4}{24}\left(2+\beta-\frac{3}{26}\right)+\frac{t^5}{20}\left(\frac{\beta}{3}-\frac{1}{52}\right)\end{cases}$$

$\vdots,$

Finally, we get the solution in the form of series as

$$\begin{cases}\underline{y}(t,\beta)=-\frac{3\beta}{26}+\frac{3t}{26}-+\frac{t^2}{2}\left(1+\beta^2+\frac{29\beta}{26}\right)+\frac{t^3}{6}\left(\beta^2-\frac{2}{13}-\frac{4\beta}{13}+\frac{29\beta}{39}\right)-\frac{t^4}{24}-\frac{t^6}{120}\\ \overline{y}(t,\beta)=t\left(\frac{17}{18}+\frac{\beta}{2}\right)+\frac{t^2}{2}\left(6-\beta^3-4\beta\right)+\frac{t^3}{3}\left(-\beta^3-2\beta+\frac{71}{18}-\frac{1}{52}\right)\\ +\frac{t^4}{12}\left(1-\frac{\beta}{2}\right)+\frac{t^5}{20}\left(\frac{\beta}{3}-\frac{1}{52}\right)\\ \underline{w}(t,\beta)=\left(\beta+\beta^2\right)+t\left(2\beta+2\beta^2+\frac{29\beta}{26}\right)+t^2\left(\beta^2+\frac{29\beta}{26}\right)\\ +\frac{t^3}{6}\left(-2+\beta^2+\frac{29\beta}{26}\right)-\frac{t^4}{20}\left(\frac{2}{3}\beta-1\right)\\ \overline{w}(t,\beta)=\left(4-\beta-\beta^2\right)-\frac{t^2}{2}\left(-\frac{5}{2}\beta+\frac{145}{18}\right)+\frac{t^3}{6}\left(-\beta^2-\frac{5}{2}\beta+\frac{53}{18}\right)+\\ \frac{t^4}{24}\left(2+\beta-\frac{3}{26}\right)+\frac{t^5}{20}\left(\frac{\beta}{3}-\frac{1}{52}\right)\end{cases}\tag{30}$$

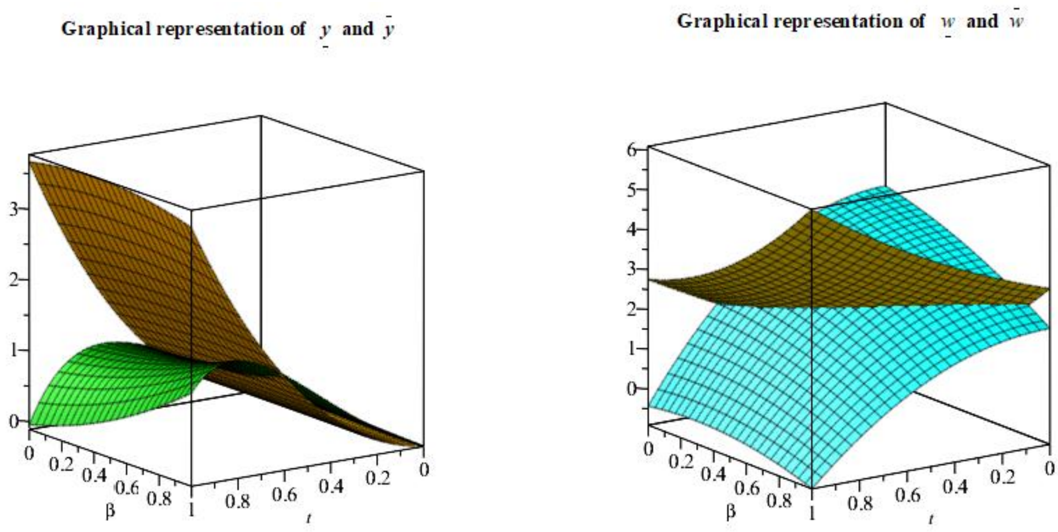

Figure 4. Graphical representation of y, w and z in 3D.

4. CONCLUSIONS

In this paper, we implemented the technique according to the proposed homotopy perturbation Sumudu transform method for attaining the approximate solution of systems of fuzzy integro-differential along with a fuzzy parametric form with suitable initial conditions and source functions. Finally, we combined all the solutions of the above mentioned equations to obtain a solution of the system of fuzzy integro-differential equations. Graphs were plotted for the obtained approximate fuzzy solution. Numerical results and their graphical plots demonstrated that the proposed coupling HPSTM is efficient and accurate for the application to these fuzzy problems.

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