ORIGINAL PAPER

EULER AND TAYLOR POLYNOMIALS METHOD FOR SOLVING VOLTERRA TYPE INTEGRO DIFFERENTIAL EQUATIONS WITH NONLINEAR TERMS

DENİZ ELMACI[1], NURCAN BAYKUŞ SAVAŞANERİL[2], FADİME DAL[3], MEHMET SEZER[4]

Manuscript received: 21.10.2020; Accepted paper: 01.05.2021; Published online: 30.06.2021.

Abstract. *In this study, the first order nonlinear Volterra type integro-differential equations are used in order to identify approximate solutions concerning Euler polynomials of a matrix method based on collocation points. This method converts the mentioned nonlinear integro-differential equation into the matrix equation with the utilization of Euler polynomials along with collocation points. The matrix equation is a system of nonlinear algebraic equations with the unknown Euler coefficients. Additionally, this approach provides analytic solutions, if the exact solutions are polynomials. Furthermore, some illustrative examples are presented with the aid of an error estimation by using the Mean-Value Theorem and residual functions. The obtained results show that the developed method is efficient and simple enough to be applied. And also, convergence of the solutions of the problems were examined. In order to obtain the matrix equations and solutions for the selected problems, code was developed in MATLAB.*

Keywords: *Euler and Taylor polynomials; collocation points; residual error analysis; matrix method; nonlinear terms; Volterra integro differential equation.*

1. INTRODUCTION

Nonlinear Volterra type integro-differential equations are based on many problems such as quantum mechanics, electrodynamics, electronic systems, control problems, number theory, mechanics, astronomy, biology, economics, electrostatics and industry [1-7]. Since it is difficult to obtain analytical solutions of this type of equations, numerical methods are required. In last years, the mentioned problems have been numerically studied by other researchers [9-19]. In this study, Volterra integro-differential equation with nonlinear terms is shown as

$$\sum_{p=0}^{r}\sum_{q=0}^{p}Q_{pq}(t)y^{(p)}(t)y^{(q)}(t)=g(t)+\int_{a}^{t}K(t,s)y(s)ds, \qquad a\le t\le b \tag{1}$$

with conditions

[1] Dokuz Eylul University, Bergama Vocational School, Izmir, Turkey. E-mail: deniz.elmaci@deu.edu.tr.
[2] Dokuz Eylul University, Izmir Vocational School, Izmir, Turkey. E-mail: nurcan.savasaneril@deu.edu.tr.
[3] Ege University, Department of Mathematics, Faculty of Science, Izmir, Turkey. E-mail: fadime.dal@ege.edu.tr.
[4] Celal Bayar University, Department of Mathematics, Faculty of Science, Manisa, Turkey.
E-mail: mehmet.sezer@cbu.edu.tr

$$\sum_{k=0}^{r-1} a_{jk} y^{(k)}(a) + b_{jk} y^{(k)}(b) = \mu_j, \; j = 0,1,\ldots,r-1. \tag{2}$$

where $Q_{pq}(t)$ and $g(t)$ are functions defined on the interval $a \le t \le b$; a_{jk}, b_{jk} and μ_j are appropriate constants; $y(t)$ is an unknown solution function to be determined. For this aim, the Euler polynomials solution of the problem Eq.(1) – Eq.(2) in the finite series form is assumed

$$y(t) \cong y_N(t) = \sum_{n=0}^{N} a_n \, \mathrm{E}_n(t), \qquad a \le t \le b \tag{3}$$

where $E_n(t)$ indicates the Euler-Taylor polynomials which are described as

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad |t| < \pi . \tag{4}$$

Euler polynomials are strictly connected with Bernoulli ones, and are used in the Taylor expansion in a neighborhood of the origin of trigonometric and hyperbolic secant functions. Recursive computation of Euler polynomials can be obtained by using the following formula [8];

$$E_n(t) + \sum_{k=0}^{n} \binom{n}{k} E_k(t) = 2t^n, \quad n = 1,2,\ldots . \tag{5}$$

Also, Euler polynomials $E_n(t)$ can be defined as polynomials of degree $n \ge 0$ satisfying the conditions

$$\begin{aligned} &E_m'(t) = mE_{m-1}(t), \quad m \ge 1 \\ &E_m(t+1) + E_m(t) = 2t^m, m \ge 1. \end{aligned} \tag{6}$$

By means of (4), (5) or (6), the first Euler polynomials are described as

$$E_0(t) = 1, \quad E_1(t) = t - \frac{1}{2}, \quad E_2(t) = t^2 - t, \quad E_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{4},$$
$$E_4(t) = t^4 - 2t^3 + 2, \quad E_5(t) = t^5 - \frac{5}{2}t^4 + \frac{5}{2}t^2 - \frac{1}{2}, \ldots$$

2. MATERIALS AND METHODS

2.1. MATRIX RELATIONS FOR EULER POLYNOMIALS

The nonlinear Volterra integro-differential Eq.(1) is considered to create the matrices of each term. The desired solution y(t) defined by the truncated Euler series Eq.(3) of Eq.(1) is modified to extract the matrix form, for n = 0, 1, 2, ..., N, as

$$y(t) \cong y_N(t) = \mathbf{E}(t)\mathbf{A} \tag{7}$$

where

$$\mathbf{E}(t) = \begin{bmatrix} E_0(t) & E_1(t) & \cdots & E_N(t) \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} a_0 & a_1 & \cdots & a_N \end{bmatrix}^T.$$

On the other hand, using Euler polynomials and Taylor expansion, and by means of Eq.(5), the matrix relation between standard base matrix and Euler base matrix is constructed as

$$\begin{bmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^N \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{2}\binom{1}{0} & 1 & 0 & \cdots & 0 \\ \frac{1}{2}\binom{2}{0} & \frac{1}{2}\binom{2}{1} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}\binom{N}{0} & \frac{1}{2}\binom{N}{1} & \frac{1}{2}\binom{N}{2} & \cdots & 1 \end{bmatrix} \begin{bmatrix} E_0(t) \\ E_1(t) \\ E_2(t) \\ \vdots \\ E_N(t) \end{bmatrix}$$

$$\mathbf{T}^T(t) = \left(\mathbf{S}^{-1}\right)^T \mathbf{E}^T(t) \Leftrightarrow \mathbf{T}(t) = \mathbf{E}(t)\left(\mathbf{S}^{-1}\right) \Rightarrow \mathbf{E}(t) = \mathbf{T}(t)\mathbf{S} \tag{8}$$

where $\mathbf{T}(t) = \begin{bmatrix} 1 & t & \cdots & t^N \end{bmatrix}$

$$\left(\mathbf{S}^{-1}\right)^T = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \frac{1}{2}\binom{1}{0} & 1 & 0 & \cdots & 0 \\ \frac{1}{2}\binom{2}{0} & \frac{1}{2}\binom{2}{1} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}\binom{N}{0} & \frac{1}{2}\binom{N}{1} & \frac{1}{2}\binom{N}{2} & \cdots & 1 \end{bmatrix}_{(N+1)\times(N+1)}.$$

The relation between the matrix **E**(t) and its derivatives are

$$\mathbf{E}'(t) = \mathbf{T}'(t)\mathbf{S} = \mathbf{T}(t)\mathbf{B}\mathbf{S}$$

where

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & N \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{B}^0 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$y^{(k)}(t)=\mathbf{E}^{(k)}(t)\mathbf{A}=\mathbf{T}(t)\mathbf{B}^{k}\mathbf{SA},\quad k=0,1,2,\ \dots\ . \tag{9}$$

In the similar manner, from Eq.(9), the matrix form of the nonlinear part $y^{(p)}(t)y^{(q)}(t)$ in Eq.(1) can be written as, for $p,q=0,1\ldots,r$

$$y^{(p)}(t)y^{(q)}(t)=\mathbf{T}(t)\mathbf{B}^{p}\mathbf{S}\bar{\mathbf{T}}(t)\overline{\mathbf{B}^{q}}\,\bar{\mathbf{S}}\bar{\mathbf{A}}=\mathbf{R}_{pq}(t)\bar{\mathbf{A}} \tag{10}$$

$$\bar{\mathbf{T}}=\begin{bmatrix}\mathbf{T}(t_0) & 0 & \cdots & 0\\ 0 & \mathbf{T}(t_1) & 0 & \vdots\\ \vdots & \cdots & \ddots & 0\\ 0 & \cdots & 0 & \mathbf{T}(t_N)\end{bmatrix},\quad \overline{\mathbf{B}^{q}}=\begin{bmatrix}\mathbf{B}^{q} & 0 & \cdots & 0\\ 0 & \mathbf{B}^{q} & 0 & \vdots\\ \vdots & \cdots & \ddots & 0\\ 0 & \cdots & 0 & \mathbf{B}^{q}\end{bmatrix}$$

$$\bar{\mathbf{A}}=diag\begin{bmatrix}a_0\mathbf{A} & a_1\mathbf{A} & \cdots & a_N\mathbf{A}\end{bmatrix}^{T}$$

Besides, the matrix form of the kernel function **K**(t, s) in Eq.(1) is computed as follows

$$\mathbf{K}(t,s)=\mathbf{T}(t)\mathbf{K}\mathbf{T}(s)^{T} \tag{11}$$

where $\mathbf{K}=\left[k_{mn}\right],\quad m,n=0,1,\dots,N \qquad k_{mn}=\frac{1}{m!n!}.\frac{\partial^{m+n}\mathbf{K}(0,0)}{\partial t^{m}\partial s^{n}}$

$$\int_a^t \mathbf{K}(t,s)y(s)ds=\mathbf{T}(t)\mathbf{K}\mathbf{Q}(t)\mathbf{SA} \tag{12}$$

where

$$\mathbf{Q}(t)=\left[q_{mn}(t)\right]=\int_a^t \mathbf{T}^{T}(s)\mathbf{T}(s)ds,$$

$$q_{mn}(t)=\frac{t^{m+n+1}-a^{m+n+1}}{m+n+1},\quad m,n=0,1,\dots,N\ .$$

By substituting the matrix relations Eq. (10) and Eq. (12) into Eq.(1) and then by using the collocation points

$$t_i=a+\frac{b-a}{N}i,\quad i=0,1,\dots,N.$$

The system of matrix equations are obtained as follows:

$$\sum_{p=0}^{r}\sum_{q=0}^{p}\mathrm{Q}_{pq}(t_i)\mathbf{R}_{pq}(t_i)\bar{\mathbf{A}}=g(t_i)+\bar{\mathbf{T}}(t_i)\bar{\mathbf{K}}\bar{\mathbf{Q}}(t_i)\mathbf{SA} \tag{13}$$

where

$$\mathbf{Q}_{pq} = diag\left[\mathbf{Q}_{pq}(t_0) \quad \mathbf{Q}_{pq}(t_1) \quad \cdots \quad \mathbf{Q}_{pq}(t_N)\right],$$

$$\mathbf{R}_{pq} = \begin{bmatrix} \mathbf{T}(t_0)\mathbf{B}^p\mathbf{S}\bar{\mathbf{T}}(t_0)\overline{\mathbf{B}^q}\bar{\mathbf{S}} \\ \mathbf{T}(t_1)\mathbf{B}^p\mathbf{S}\bar{\mathbf{T}}(t_1)\overline{\mathbf{B}^q}\bar{\mathbf{S}} \\ \vdots \\ \mathbf{T}(t_N)\mathbf{B}^p\mathbf{S}\bar{\mathbf{T}}(t_N)\overline{\mathbf{B}^q}\bar{\mathbf{S}} \end{bmatrix},$$

$$\bar{\mathbf{Q}}(t) = \begin{bmatrix} \mathbf{Q}(t_0) \\ \mathbf{Q}(t_1) \\ \vdots \\ \mathbf{Q}(t_N) \end{bmatrix}, \quad \bar{\mathbf{K}} = \begin{bmatrix} \mathbf{K} & 0 & \cdots & 0 \\ 0 & \mathbf{K} & 0 & \vdots \\ \vdots & \cdots & \ddots & 0 \\ 0 & \cdots & 0 & \mathbf{K} \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} g(t_0) \\ g(t_1) \\ \vdots \\ g(t_N) \end{bmatrix},$$

and the fundamental matrix form

$$\sum_{p=0}^{r}\sum_{q=0}^{p} \mathrm{Q}_{pq}\mathbf{R}_{pq}\bar{\mathbf{A}} - \bar{\mathbf{T}}\bar{\mathbf{K}}\bar{\mathbf{Q}}\mathbf{S}\mathbf{A} = \mathbf{G}$$

or briefly

$$\mathbf{WA} + \mathbf{V}\bar{\mathbf{A}} = \mathbf{G} \ \Leftrightarrow \ \left[\mathbf{W}\ ;\ \mathbf{V}\ :\ \mathbf{G}\right] \tag{14}$$

where

$$\mathbf{W} = -\bar{\mathbf{T}}\bar{\mathbf{K}}\bar{\mathbf{Q}}\mathbf{S}\ , \qquad \mathbf{V} = \sum_{p=0}^{r}\sum_{q=0}^{p} \mathrm{Q}_{pq}\mathbf{R}_{pq}\bar{\mathbf{A}}$$

Besides, we can find for the condition Eq.(2), by using the relation Eq.(5),

$$\sum_{k=0}^{r-1}\left(a_{jk}\mathbf{T}(a)\mathbf{B}^k\mathbf{S} + b_{jk}\mathbf{T}(b)\mathbf{B}^k\mathbf{S}\right)\mathbf{A} = \left[\mu_j\right] \quad \text{or} \quad \left[\mathbf{U};\ \ \bar{\mathbf{0}}\ \ :\ \ \mathbf{A}\right] \tag{15}$$

where

$$\mathbf{U} = \begin{bmatrix} u_{00} & u_{01} & \cdots & u_{0N} \\ u_{10} & u_{11} & \cdots & u_{1N} \\ \vdots & \cdots & \ddots & \vdots \\ u_{(r-1)0} & \cdots & \cdots & u_{(r-1)N} \end{bmatrix}, \quad \bar{\mathbf{0}} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}_{r\times(N+1)^2}.$$

Consequently, any one row of Eq.(14) by the row matrix Eq.(15) is replaced, hence the desired augmented matrix or the resulted matrix equation comes out as

$$\left[\mathbf{W};\ \ \mathbf{V}\ \ :\ \ \tilde{\mathbf{G}}\right] \qquad \text{or} \qquad \mathbf{W\,A} + \mathbf{V}\bar{\mathbf{A}} = \tilde{\mathbf{G}} \tag{16}$$

which suits to the system of nonlinear algebraic equations with the Euler coefficients $a_N = 0,1,...,N$. The solution of this system provides the matrix $\mathbf{A}$ and the solution of Eq.(1) – Eq.(2) is

$$y_N(t) = \mathbf{E}(t)\mathbf{A} = \mathbf{T}(t)\mathbf{SA}$$

2.2. RESIDUAL ERROR ANALYSIS

We define the residual function using both the linear and nonlinear parts of (1) for the present method as [9-12]

$$R_N(t) = L[y_N(t)] + N[y_N(t)] - g(t) \tag{17}$$

where the linear part is

$$L[y_N(t)] = -\int_a^t K(t,s)\,y_N(s)\,ds$$

and the nonlinear part is

$$N[y_N(t)] = \sum_{p=0}^{r}\sum_{q=0}^{p} Q_{pq}(t)\,y_N^{(p)}(t)\,y_N^{(q)}(t)$$

By means of the residual function defined by $R_N(t)$ and the mean value of the function $|R_N(t)|$ on the interval [a, b], the accuracy of the solution can be controlled and the error can be estimated [6-7,10-12]. Thus, we can estimate the upper bound of the mean error $\overline{R_N}$ as follows:

$$\left|\int_a^b R_N(t)\,dt\right| \le \int_a^b \left|R_N(t)\,dt\right|$$

$$\int_a^b |R_N(t)|\,dt = (b-a)|R_N(c)|, \quad a \le c \le b$$

$$\Rightarrow \left|\int_a^b R_N(t)\,dt\right| = (b-a)|R_N(c)|$$

$$\Rightarrow (b-a)|R_N(c)| \le \int_a^b |R_N(t)|\,dt$$

$$|R_N(c)| \le \frac{\int_a^b |R_N(t)|\,dt}{b-a} = \overline{R_N}$$

Kürkçü and Coworkers developed the convergence of Dickson polynomial solution of the nonlinear model problem using the residual function in Banach space [9]. We reveal the following convergence criteria for Euler polynomial solutions. Now, we can use the following theorem for our investigation.

Theorem 1. [9] Let B be a Banach space. The residual function sequence $\{R_N(t)\}_{N=2}^{\infty}$ is convergent in B and the following inequality is satisfied so that $0<\mu_N<1$. Here μ_N is constant in B:

$$\|R_{N+1}(t)\| < \mu_N \|R_N(t)\| \tag{18}$$

μ_N values are obtained by using estimated upper bound of the mean error which is found above. If μ_N values are in (0, 1) as N increases, the residual error will be convergent.

3. NUMERİCAL EXAMPLES

In this section, some numerical examples of the problem Eq. (1) are given to illustrate the accuracy and effectiveness properties of the method.

Example 3.1. Consider the first order Volterra integro-differential equation with nonlinear terms,

$$y'(t)y(t)+t^2y^2(t)=t^4+t-\frac{t^3}{2}+\int_0^t ty(s)ds, \quad y(0)=0, \quad 0\le t\le 1$$

Here $Q_{00}(t)=t^2, \quad Q_{10}(t)=1, \quad g(t)=t^4-\frac{t^3}{2}+t, \quad K(t,s)=t$

The approximate solutions $y_2(t)$ for $N=2$ is given by

$$y(t)\cong y_2(t)=\sum_{n=0}^{2} a_n E_n(t),$$

For $a=0,$ b=1 and $N=2$, we have $\left\{t_0=0,\ t_1=\frac{1}{2},\ t_2=1\right\}$.

From Eq. (14), the fundamental matrix equation of the problem becomes

$$\sum_{p=0}^{1}\sum_{q=0}^{p} \mathrm{Q}_{pq}\mathbf{R}_{pq}\overline{\mathbf{A}}-\overline{\mathbf{T}}\overline{\mathbf{K}}\overline{\mathbf{Q}}\mathbf{S}\mathbf{A}=\mathbf{G},$$

$$\mathbf{W}=-\overline{\mathbf{T}}\overline{\mathbf{K}}\overline{\mathbf{Q}}\mathbf{S}, \qquad \mathbf{V}=\sum_{p=0}^{1}\sum_{q=0}^{p}\mathbf{Q}_{pq}\mathbf{R}_{pq}$$

The augmented matrix for this fundamental matrix equation is calculated as

$$[\boldsymbol{W} \;\; ; \;\; \boldsymbol{V} \;\; : \;\; \boldsymbol{G}]=\left[\begin{array}{ccccccccccccccccc} 0 & 0 & 0 & & 0 & 0 & 0 & 1 & -\frac{1}{2} & 0 & -1 & \frac{1}{2} & 0 & & 0 \\ -\frac{1}{4} & \frac{1}{16} & \frac{1}{24} & ; & \frac{1}{4} & 0 & -\frac{1}{16} & 1 & 0 & -\frac{1}{4} & -\frac{1}{16} & 0 & \frac{1}{64} & : & \frac{1}{2} \\ -1 & 0 & \frac{1}{6} & & 1 & \frac{1}{2} & 0 & \frac{3}{2} & \frac{3}{4} & 0 & 1 & \frac{1}{2} & 0 & & \frac{3}{2} \end{array}\right]$$

Hence, the new augmented matrix based on condition can be obtained as follows

$$\left[\boldsymbol{W} \;\; ; \;\; \boldsymbol{V} \;\; : \;\; \boldsymbol{G}\right]=\left[\begin{array}{ccc|ccccccccc|c} 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} & 0 & -1 & \frac{1}{2} & 0 & 0 \\ -\frac{1}{4} & \frac{1}{16} & \frac{1}{24} & \frac{1}{4} & 0 & -\frac{1}{16} & 1 & 0 & -\frac{1}{4} & -\frac{1}{16} & 0 & \frac{1}{64} & \frac{1}{2} \\ 1 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

By solving this system, substituting the resulting unknown Euler coefficients matrix into Eq. (3) we obtain the exact solution for N = 2 as $y(t)=t$.

Example 3.2. Consider the first order Volterra integro-differential equation with nonlinear terms,

$$y'(t)\,y(t)+t^2e^{-t}y^2(t)=e^{2t}-t^2+\int_0^t t^2 y(s)\,ds, \qquad y(0)=1, \qquad 0\le t\le 1$$

The augmented matrix based on condition can be obtained as previous example

$$\left[\boldsymbol{W} \;\; ; \;\; \boldsymbol{V} \;\; : \;\; \boldsymbol{G}\right]=\left[\begin{array}{ccc|ccccccccc|c} 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} & 0 & -1 & \frac{1}{2} & 0 & 1 \\ -\frac{1}{8} & \frac{1}{32} & \frac{1}{48} & \frac{e^{-1/2}}{4} & 0 & -\frac{e^{-1/2}}{16} & 1 & 0 & 0 & -\frac{e^{-1/2}}{16} & 0 & \frac{e^{-1/2}}{64} & e-\frac{1}{4} \\ 1 & \frac{-1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

While the exact solution is $y(t)=e^t$, the proposed method is applied and the approximate solutions are obtained as

$$y_2(t)=1+t+0.71016t^2,$$
$$y_3(t)=1+t+0.43955t^2+0.39765t^3,$$
$$y_4(t)=1+t+0.47958t^2+0.27545t^3+0.0918t^4,$$
$$y_5(t)=1+t+0.51818t^2+0.03377t^3+0.16057t^4+0.08974t^5,$$
$$y_6(t)=1+t+0.50166t^2+0.15126t^3-0.15088t^4+0.44772t^5-0.14976t^6$$

for N = 2, 3, 4, 5, 6 respectively. The comparison of the exact solution and the approximate solutions is given in Fig.1, also the comparative solutions are given in Table 1. The upper error bounds are obtained as

$$\overline{R_2}=0.012281,\ \overline{R_3}=0.365716,\ \overline{R_4}=0.394521,\ \overline{R_5}=0.248215,\ \overline{R_6}=0.241687.$$

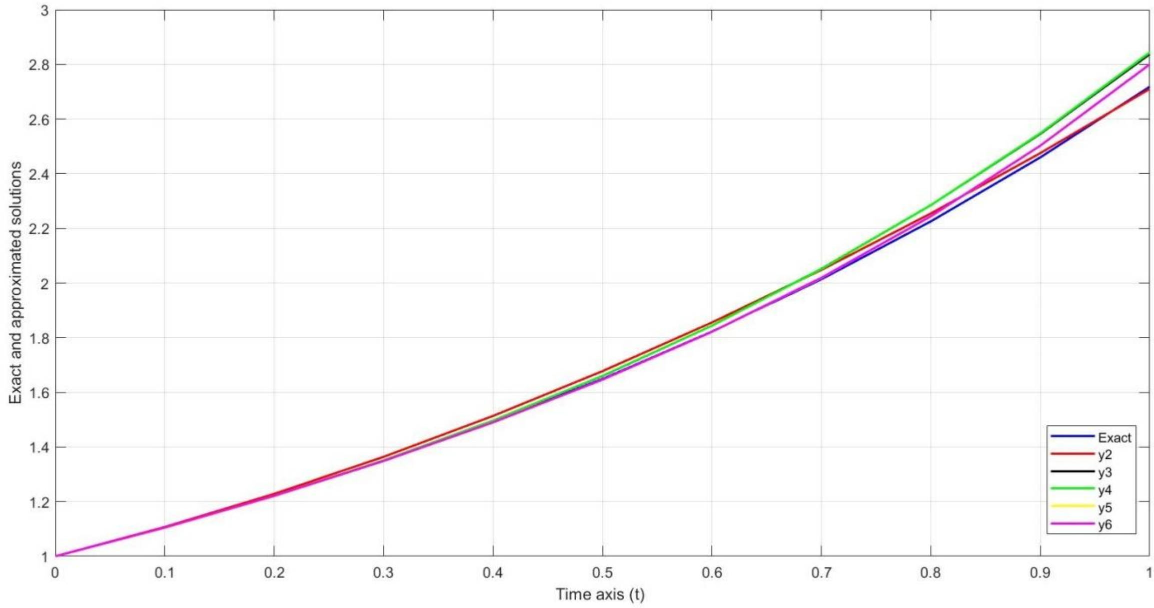

Figure 1. The comparison of the exact solution y(t) = e^t, approximate solutions $y_N(t)$ for N = 2, 3, 4, 5, 6.

Table 1. Exact and approximated solutions of Example 2 for the value of t.

t	Exact Solution	y_2	y_3	y_4	y_5	y_6	R_2	R_3	R_4	R_5	R_6
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	1.1052	1.1071	1.1048	1.1051	1.1052	1.1052	4.30e-02	-6.32e-03	-7.91e-04	2.51e-04	-7.77e-04
0.2	1.2214	1.2284	1.2208	1.2215	1.2213	1.2212	8.61e-02	1.77e-03	8.01e-03	-5.31e-03	-5.34e-03
0.3	1.3499	1.3639	1.3503	1.3513	1.3491	1.3490	1.25e-01	2.93e-02	3.22e-02	-1.32e-02	-1.26e-02
0.4	1.4918	1.5136	1.4958	1.4967	1.4901	1.4901	1.55e-01	8.30e-02	8.00e-02	-1.57e-02	-1.57e-02
0.5	1.6487	1.6775	1.6596	1.6601	1.6466	1.6465	1.64e-01	1.71e-01	1.62e-01	6.79e-04	-1.48e-09
0.6	1.8221	1.8557	1.8441	1.8440	1.8216	1.8215	1.38e-01	3.04e-01	2.95e-01	5.92e-02	5.91e-02
0.7	2.0138	2.0480	2.0518	2.0515	2.0191	2.0191	5.67e-02	4.93e-01	4.97e-01	1.97e-01	1.99e-01
0.8	2.2255	2.2545	2.2849	2.2856	2.2441	2.2442	-1.10e-01	7.54e-01	7.96e-01	4.71e-01	4.71e-01
0.9	2.4596	2.4752	2.5459	2.5495	2.5027	2.5024	-4.00e-01	1.0988	1.2234	9.69e-01	9.46e-01
1.0	2.7183	2.7102	2.8372	2.8468	2.8023	2.8000	-8.64e-01	1.5422	1.8205	1.8192	1.7093

Example 3.3. Consider the first order Volterra integro-differential equation with nonlinear terms,

$$\left(y'(t)\right)^2 + y'(t)y(t) = 1 - t - e^{-t} + \int_0^t (t-s)y(s)ds, \quad y(0)=1, \quad 0 \le t \le 1$$

The augmented matrix based on condition can be obtained as previous examples

$$\left[\boldsymbol{W} \;;\; \boldsymbol{V} \;:\; \boldsymbol{G}\right] = \left[\begin{array}{ccc|ccccccccc|c} 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2} & -1 & -1 & -\frac{1}{2} & 1 & 0 \\ \frac{1}{8} & -\frac{1}{24} & -\frac{1}{64} & 0 & 0 & 0 & 1 & 1 & -\frac{1}{4} & 0 & 0 & 0 & 1-e^{-1/2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

While the exact solution is $y(t) = e^{-t}$, the proposed method is applied and the approximate solutions are obtained as

$$y_2(t) = 1 - t + 0.50741t^2,$$
$$y_3(t) = 1 - t + 0.48647t^2 - 0.12156t^3,$$
$$y_4(t) = 1 - t + 0.49864t^2 - 0.15956t^3 + 0.02901t^4,$$
$$y_5(t) = 1 - t + 0.49989t^2 - 0.16588t^3 + 0.03950t^4 - 0.00564t^5$$
$$y_6(t) = 1 - t + 0.49999t^2 - 0.16660t^3 + 0.04141t^4 - 0.00785t^5 + 0.00092t^6$$

for N = 2, 3, 4, 5, 6 respectively. The comparison of the exact and the approximate solutions is given in Fig.2. In addition, the comparative solutions are given in Table 2.

The upper error bounds are obtained as $\overline{R_2} = 0.04414$, $\overline{R_3} = 0.00152$, $\overline{R_4} = 7.468e-05$, $\overline{R_5} = 5.849e-06$, $\overline{R_6} = 2.495e-07$.

By using the Theorem 1,

$$\mu_N = \left\{ \frac{\overline{R_3}}{\overline{R_2}}, \frac{\overline{R_4}}{\overline{R_3}}, \frac{\overline{R_5}}{\overline{R_4}}, \frac{\overline{R_6}}{\overline{R_5}}, \ldots \right\} = \{0.034436,\ 0.049131,\ 0.078315,\ 0.042660,\ \ldots\}$$

so,

$$\frac{\overline{|R_{N+1}|}}{\overline{|R_N|}} < 1$$

It can be concluded that the residual error decreases as N values are increasing.

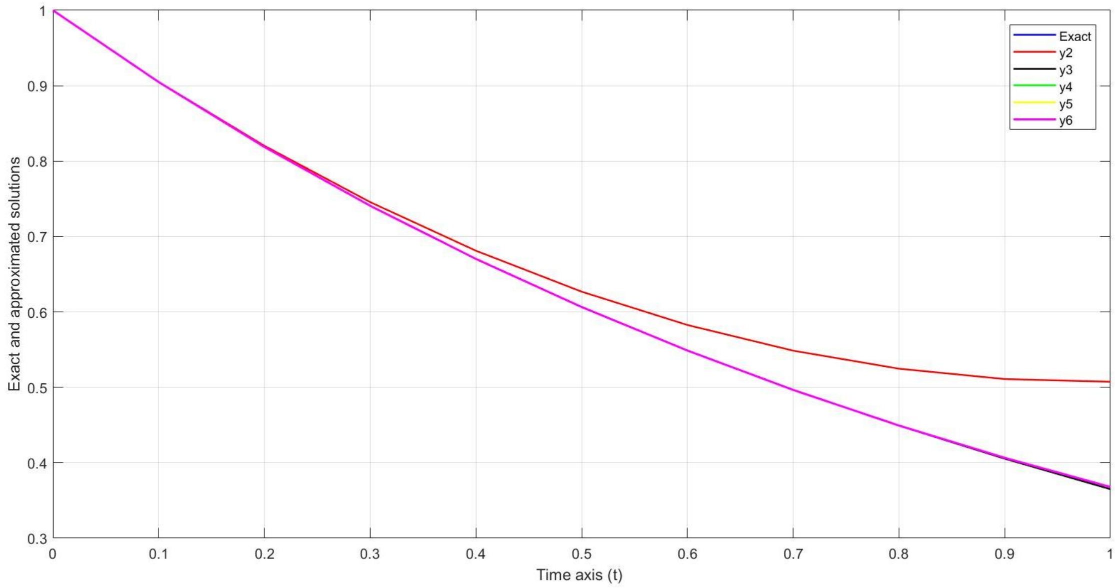

Figure 2. The comparison of the exact solution y(t) = e^{-t}, approximate solutions $y_N(t)$ for N = 2, 3, 4, 5, 6.

Table 2. Exact and approximated solutions of Example 3 for the value of t.

t	Exact Solution	y_2	y_3	y_4	y_5	y_6	R_2	R_3	R_4	R_5	R_6
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	0.9048	0.9051	0.9047	0.9048	0.9048	0.9048	-5.89e-03	1.46e-03	1.02e-04	5.36e-06	2.16e-07
0.2	0.8187	0.8203	0.8185	0.8187	0.8187	0.8187	-1.85e-02	1.24e-03	3.92e-05	-3.00e-09	-5.96e-08
0.3	0.7408	0.7457	0.7405	0.7408	0.7408	0.7408	-3.49e-02	3.31e-04	-2.90e-05	-1.89e-06	-3.57e-08
0.4	0.6703	0.6812	0.6701	0.6703	0.6703	0.6703	-5.19e-02	-5.79e-04	-4.08e-05	-3.16e-08	4.45e-08
0.5	0.6065	0.6269	0.6064	0.6065	0.6065	0.6065	-6.64e-02	-1.02e-03	4.75e-08	1.49e-06	0.0000
0.6	0.5488	0.5827	0.5489	0.5488	0.5488	0.5488	-7.56e-02	-7.10e-04	5.01e-05	-6.41e-08	-5.46e-08
0.7	0.4966	0.5486	0.4967	0.4966	0.4966	0.4966	-7.64e-02	5.17e-04	4.53e-05	-2.99e-06	5.58e-08
0.8	0.4493	0.5247	0.4491	0.4493	0.4493	0.4493	-6.60e-02	2.75e-03	-8.62e-05	4.98e-09	1.31e-07
0.9	0.4066	0.5110	0.4054	0.4066	0.4066	0.4066	-4.15e-02	6.07e-03	-4.13e-04	2.18e-05	-8.73e-07
1.0	0.3679	0.5074	0.3649	0.3681	0.3679	0.3679	2.37e-07	1.06e-02	-9.94e-04	8.01e-05	-5.24e-06

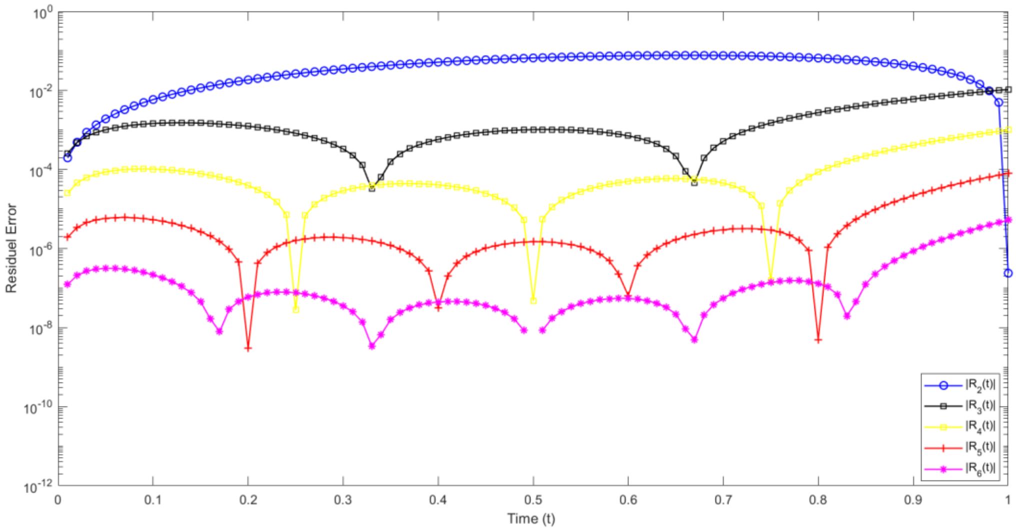

Figure 3. The residual errors of Example 3 for N = 2, 3, 4, 5, 6.

As shown in Table 2 and Fig.3, even if the N value slightly increases the exact can be reached quickly.

4. CONCLUSIONS

In this research, a collocation calculation model based on Euler polynomial for solutions of the Volterra type integro-differential equations with nonlinear terms is presented. Furthermore, the control of the solutions is performed with the utilization of defined techniques. If the exact solution of the problem is a polynomial for instance, then the exact solution can be calculated by applying this technique. The results of presented method are found very close to exact solution's results. Another advantage of the proposed technique is the utilization for testing reliability of the solutions of different problems. The Euler polynomial solutions approach to the exact solution, as N is increased. This situation reflects on the residual functions of problems. Euler matrix method provides two main advantages: first, it is simple to construct the main matrix equations and increasing the efficiency and easiness in the programming. Second considerable advantage is the duration of computation of this method.

Acknowledgements: *The authors would like to acknowledge to TUBITAK for their financial support and would like to thank the reviewers for their thoughtful comments and efforts towards improving our manuscript.*

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