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NEW MODIFIED SSOR METHOD FOR NUMERICAL SOLUTION OF LINEAR AUGMENTED SYSTEM

NAJMUDDIN AHMAD[1], FAUZIA SHAHEEN[1]

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Abstract. *In this paper, we have proposed a new modified symmetric SOR method for numerical solutions of linear augmented system. We have also discussed the convergence criteria of this method along with optimal parameters. We will see that the new modified method has a higher rate of convergence with the help of an example. We are also presenting a comparison table with the other existing methods. All the computational work has been performed by MATLAB 2020R.*

Keywords: *SOR method; SSOR method; radius of convergence; error vector.*

1. INTRODUCTION

A linear system of the type (1) is called a linear augmented system, where $A \in \mathbb{R}^{m\times m}$ is

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ q \end{pmatrix} \tag{1}$$

a symmetric positive definite matrix$B \in \mathbb{R}^{m\times n}$ is matrix of full column rank and B^T is a transpose of B. $b \in \mathbb{R}^m$ and $q \in \mathbb{R}^n$ are the column vectors,where m ≥ n.This type of linear augmented system arises in many mathematical problems such as constrained optimization [1], the finite element method for solving Stock's equation [2-4], constrained least square problems and generalized least square problems [5-8].

SOR method is first given by theDavid M.Youngin 1930. The idea of SOR method is the implementation of ω in Gauss Seidal method so that we can get the better results in comparison to Jacobi and Guass Seidal methods. Further after few years a modification in SOR method is made by Russell.Russell just introduced a new relaxation parameter to each block row of a block coefficient matrix.

2. MODIFIED SSOR METHOD

For the sake of simplicity we write above linear system (1) as

$$\begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b \\ -q \end{pmatrix}. \tag{2}$$

Because of zero blocks in the coefficient matrix, we cannot solve the system by the SOR method. Hence, Golub et al. [9] presented several SOR-like algorithms to solve such an

[1]Integral University, Department of Mathematics & Statistics, 226026 Lucknow, India.
E-mail:najmuddinahmad33@gmail.com; shaheenfauzia97@gmail.com.

augmented system. In this paper, for the coefficient matrix of augmented system (2), we consider the following splitting,

$$\mathfrak{D} = \begin{pmatrix} \frac{A}{\alpha} & 0 \\ 0 & Q \end{pmatrix}, \qquad \mathfrak{L} = \begin{pmatrix} 0 & 0 \\ B^T & \frac{Q}{2} \end{pmatrix}, \qquad \mathfrak{U} = \begin{pmatrix} \left(\frac{1}{\alpha} - 1\right)A & -B \\ 0 & \frac{Q}{2} \end{pmatrix}$$

where $\mathfrak{D}$is a block diagonal matrix, $\mathfrak{L}$ is a strictly lower triangular and $\mathfrak{U}$ is strictly upper triangular matrixand $\alpha$is a nonzero and real number, $Q \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix.

The iteration scheme of the MSSOR-like method is,

$$(\mathfrak{D} - \omega\mathfrak{L})Z^{\left(i+\frac{1}{2}\right)} = [(1-\omega)\mathfrak{D} + \omega\,\mathfrak{U}]Z^{(i)} + \omega c$$

That is,

$$Z^{\left(i+\frac{1}{2}\right)} = L_{\alpha,\beta,\omega}Z^{(i)} + \omega(\mathfrak{D} - \omega\mathfrak{L})^{-1}c \tag{3}$$

$$L_{\alpha,\beta,\omega} = (\mathfrak{D} - \omega\mathfrak{L})^{-1}[(1-\omega)\,\mathfrak{D} + \omega\mathfrak{U}]$$

$$(\mathfrak{D} - \omega\mathfrak{L}) = \begin{pmatrix} \frac{A}{\alpha} & 0 \\ 0 & Q \end{pmatrix} - \omega\begin{pmatrix} 0 & 0 \\ B^T & \frac{Q}{2} \end{pmatrix} = \begin{pmatrix} \frac{A}{\alpha} & 0 \\ -\omega B^T & (1-\frac{\omega}{2})Q \end{pmatrix}$$

$$(\mathfrak{D} - \omega\mathfrak{U}) = \begin{pmatrix} \frac{A}{\alpha} & 0 \\ 0 & Q \end{pmatrix} - \omega\begin{pmatrix} \left(\frac{1}{\alpha} - 1\right)A & -B \\ 0 & \frac{Q}{2} \end{pmatrix} = \begin{pmatrix} \frac{1-\omega+\omega\alpha}{\alpha}A & \omega B \\ 0 & Q(1-\frac{\omega}{2}) \end{pmatrix}$$

Since the matrix A is SPD and Q is nonsingular, we obtain that

$$\boldsymbol{det(\mathfrak{D} - \omega\mathfrak{L}) = \frac{\left(1-\frac{\omega}{2}\right)^n}{\alpha^m} det(A)\, det\,(Q) \neq 0}$$

$$det(\mathfrak{D}\text{-}\,\omega\mathfrak{U}) = \frac{\left(1-\frac{\omega}{2}\right)^n(1-\omega+\omega\alpha)^m}{\alpha^m} det(A)\, det(Q) \neq 0$$

If and only if $\frac{1}{\alpha}(1-\frac{\omega}{2}) \neq 0 \& (1-\omega+\omega\alpha)) \neq 0$ i.e. $\omega \neq \left\{2, \frac{1}{(1-\alpha)}\right\}$ and $\alpha \neq 0$

$$\boldsymbol{L_{\alpha,\beta,\omega} = \begin{pmatrix} \frac{A}{\alpha} & 0 \\ -\omega B^T & \left(1-\frac{\omega}{2}\right)Q \end{pmatrix}^{-1} \begin{pmatrix} \frac{(1-\omega\alpha)}{\alpha}A & -\omega B \\ 0 & \left(1-\frac{\omega}{2}\right)Q \end{pmatrix}}$$
$$= \begin{pmatrix} (1-\omega\alpha)I_m & -\omega\alpha A^{-1}B \\ \frac{2\omega(1-\omega\alpha)}{2-\omega}Q^{-1}B^T & I_n - \frac{2\omega^2\alpha}{2-\omega}Q^{-1}B^T A^{-1}B \end{pmatrix}$$

$\omega$is a positive real number with $\omega \neq 0$ MSOR-like method.Given initial vectors $x^{(0)} \in \mathbb{R}^m$and $y^{(0)} \in \mathbb{R}^n$, two relaxation factors α and ω. For k = 0, 1, 2,... until the iteration sequence $(x^{(k)}, y^{(k)})^t$is convergent, compute$z^{(i+\frac{1}{2})}$and so by the backward SOR, we compute$z^{(i+1)}$ from $z^{(i+\frac{1}{2})}$.

$$\mathrm{z}^{(\mathrm{i}+1)} = \mathfrak{U}_{\alpha,\beta,\omega}\, \mathrm{z}^{(\mathrm{i}+\frac{1}{2})} + \omega(\mathfrak{D}-\omega\mathfrak{U})^{-1}\mathrm{c} \tag{4}$$

$$\mathfrak{U}_{\alpha,\beta,\omega} = (\mathfrak{D}-\omega\mathfrak{U})^{-1}[(1-\omega)\,\mathfrak{D} + \omega\,\mathfrak{L}]$$

$$= \begin{pmatrix} \frac{1-\omega+\omega\alpha}{\alpha}A & \omega B \\ 0 & Q(1-\frac{\omega}{2}) \end{pmatrix}^{-1} \begin{pmatrix} \frac{A}{\alpha}(1-\omega) & 0 \\ \omega B^T & \left(1-\frac{\omega}{2}\right)Q \end{pmatrix}$$

$$\mathfrak{U}_{\alpha,\beta,\omega} = \begin{pmatrix} \frac{\alpha}{1-\omega+\omega\alpha}\mathrm{A}^{-1} & \frac{-2\omega\alpha\mathrm{A}^{-1}\mathrm{B}Q^{-1}}{(1-\omega+\omega\alpha)(2-\omega)} \\ 0 & \frac{2}{2-\omega}Q^{-1} \end{pmatrix} \begin{pmatrix} \frac{A}{\alpha}(1-\omega) & 0 \\ \omega B^T & \left(1-\frac{\omega}{2}\right)Q \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\alpha(1-\omega+\omega\alpha)(2-\omega)}\left((1-\omega)(2-\omega)I_m - 2\omega^2\alpha^2\mathrm{A}^{-1}\mathrm{B}Q^{-1}B^T\right) & \left(\frac{-\omega\alpha}{(1-\omega+\omega\alpha)}\right)\mathrm{A}^{-1}\mathrm{B} \\ \left(\frac{2\omega}{2-\omega}\right)Q^{-1}B^T & I_n \end{pmatrix}$$

$$\mathrm{z}^{(\mathrm{i}+1)} = \mathfrak{U}_{\alpha,\beta,\omega}\left[L_{\alpha,\beta,\omega}Z^{(i)} + \omega(\mathfrak{D}-\omega\mathfrak{L})^{-1}c\right] + \omega(\mathfrak{D}-\omega\mathfrak{U})^{-1}\mathrm{c}$$

$$\mathrm{z}^{(\mathrm{i}+1)} = \mathfrak{U}_{\alpha,\beta,\omega}L_{\alpha,\beta,\omega}Z^{(i)} + \mathfrak{U}_{\alpha,\beta,\omega}\omega(\mathfrak{D}-\omega\mathfrak{L})^{-1}c + \omega(\mathfrak{D}-\omega\mathfrak{U})^{-1}\mathrm{c} \tag{5}$$

$$\mathrm{z}^{(\mathrm{i}+1)} = v_\omega Z^{(i)} + \mathfrak{C} \tag{6}$$

This is the iterative formula for SSOR method .The iteration matrix for modified SSOR system is given as,

$$v_\omega = \mathfrak{U}_{\alpha,\beta,\omega}\mathfrak{L}_{\alpha,\beta,\omega} \tag{7}$$

$$= \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \tag{8}$$

$$v_{11} = \left(\frac{1}{\alpha(1-\omega+\omega\alpha)(2-\omega)}\left((1-\omega)(2-\omega)I_m - 2\omega^2\alpha^2\mathrm{A}^{-1}\mathrm{B}Q^{-1}B^T\right)\right)((1-\omega\alpha)\mathrm{I_m}) + \left(\left(\frac{-\omega\alpha}{(1-\omega+\omega\alpha)}\right)\mathrm{A}^{-1}\mathrm{B}\right)\left(\frac{2\omega(1-\omega\alpha)}{2-\omega}Q^{-1}B^T\right)$$

$$= \left(\frac{(1-\omega\alpha)}{\alpha(1-\omega+\omega\alpha)(2-\omega)}\right)\left((1-\omega)(2-\omega)I_m - 2\omega^2\alpha^2\mathrm{A}^{-1}\mathrm{B}Q^{-1}B^T - 2\omega^2\alpha^2\mathrm{A}^{-1}\mathrm{B}Q^{-1}B^T\right)$$

$$=\left(\frac{(1-\omega\alpha)}{\alpha(1-\omega+\omega\alpha)(2-\omega)}\right)\left((1-\omega)(2-\omega)I_m - 4\omega^2\alpha^2\mathrm{A}^{-1}\mathrm{B}Q^{-1}B^T\right)$$

$$v_{12} = \left(\frac{1}{\alpha(1-\omega+\omega\alpha)(2-\omega)}\left((1-\omega)(2-\omega)I_m - 2\omega^2\alpha^2\mathrm{A}^{-1}\mathrm{B}Q^{-1}B^T\right)\right)(-\omega\alpha\mathrm{A}^{-1}\mathrm{B}) + \left(\left(\frac{-\omega\alpha}{(1-\omega+\omega\alpha)}\right)\mathrm{A}^{-1}\mathrm{B}\right)\left(I_n - \frac{2\omega^2\alpha}{2-\omega}Q^{-1}B^T\mathrm{A}^{-1}\mathrm{B}\right)$$

$$= \left(\frac{-\omega\alpha}{\alpha(1-\omega+\omega\alpha)(2-\omega)}\right)[(\omega^2-\omega(\alpha+3)+2(\alpha+1))A^{-1}B - 4\omega^2\alpha^2A^{-1}BQ^{-1}B^TA^{-1}B]$$

$$v_{21} = \left(\left(\frac{2\omega}{2-\omega}\right)Q^{-1}B^T\right)(-\omega\alpha A^{-1}B) = \left(\frac{-2\omega^2\alpha)}{(2-\omega)}\right)Q^{-1}B^TA^{-1}B$$

$$v_{22} = \left(I_n - \frac{2\omega^2\alpha}{2-\omega}Q^{-1}B^TA^{-1}B\right)$$

and,

$$\mathfrak{C} = \mathfrak{U}_{\alpha,\beta,\omega}\omega(\mathfrak{D}-\omega\mathfrak{L})^{-1}c + \omega(\mathfrak{D}-\omega\mathfrak{U})^{-1}c \quad (9)$$

where

$$c = \begin{pmatrix} b \\ -q \end{pmatrix} \quad (10)$$

3. CONVERGENCE CRITERIA

Theorem 3.1. Suppose that μ is an eigenvalue of $Q^{-1}B^TA^{-1}B$. If λ satisfies,

$$\boldsymbol{(1-\lambda)[\lambda(1-\omega+\omega\alpha)-(1-\omega\alpha)(1-\omega)] = 4\lambda\alpha\omega^2\mu} \quad \textbf{(11)}$$

then λ is an eigenvalue of v_ω. Conversely, if λ is an eigenvalue of v_ω such that $\lambda \neq (\omega-1)^2$, λ ≠1 and μ satisfies, then μ is a nonzero eigenvalue of$Q^{-1}B^TA^{-1}B$.

Proof:

$$v_\omega\, x = \lambda\, x$$

$$\mathfrak{U}_{\alpha,\beta,\omega}\,\mathfrak{L}_{\alpha,\beta,\omega} x = \lambda\, x$$

$$(\mathfrak{D}-\omega\mathfrak{L})^{-1}[(1-\omega)\mathfrak{D}+\omega\mathfrak{U}]X = [(1-\omega)\mathfrak{D}+\omega\mathfrak{L}]^{-1}(\mathfrak{D}-\omega\mathfrak{U})\lambda\, x$$

$$\begin{pmatrix} (1-\omega\alpha)I_m & -\omega\alpha A^{-1}B \\ \frac{2\omega(1-\omega\alpha)}{2-\omega}Q^{-1}B^T & I_n - \frac{2\omega^2\alpha}{2-\omega}Q^{-1}B^TA^{-1}B \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} =$$

$$\lambda\begin{pmatrix} \frac{(1-\omega+\omega\alpha)}{(1-\omega)}I_m & \frac{\omega\alpha}{(1-\omega)}A^{-1}B \\ \frac{-2\omega(1-\omega+\omega\alpha)}{(1-\omega)(2-\omega)}Q^{-1}B^T & I_n - \frac{2\omega^2\alpha}{(1-\omega)(2-\omega)}Q^{-1}B^TA^{-1}B \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

$$(1-\omega\alpha)I_mX_1 - \omega\alpha A^{-1}BX_2 = \lambda\frac{(1-\omega+\omega\alpha)}{(1-\omega)}I_mX_1 + \lambda\frac{\omega\alpha}{(1-\omega)}A^{-1}BX_2 \quad (12)$$

and

$$\frac{2\omega(1-\omega\alpha)}{2-\omega}Q^{-1}B^TX_1 + \left(I_n - \frac{2\omega^2\alpha}{2-\omega}Q^{-1}B^TA^{-1}B\right)X_2 = \lambda\frac{-2\omega(1-\omega+\omega\alpha)}{(1-\omega)(2-\omega)}Q^{-1}B^TX_1 + \lambda\Big(I_n -$$
$$2\omega 2\alpha 1-\omega(2-\omega)Q-1B^TA-1BX2 \quad (13)$$

From equation (12) we have,

$$\left((1-\omega\alpha)-\lambda\frac{(1-\omega+\omega\alpha)}{(1-\omega)}\right)\mathrm{I_m}X_1=\left(\omega\alpha+\lambda\frac{\omega\alpha}{(1-\omega)}\right)\mathrm{A}^{-1}\mathrm{B}X_2$$

$$X_1=\frac{\left(\omega\alpha+\lambda\frac{\omega\alpha}{(1-\omega)}\right)}{\left((1-\omega\alpha)-\lambda\frac{(1-\omega+\omega\alpha)}{(1-\omega)}\right)}\mathrm{A}^{-1}\mathrm{B}X_2$$

$$X_1=\frac{\omega\alpha(1-\omega+\lambda)}{[(1-\omega\alpha)(1-\omega)-\lambda(1-\omega+\omega\alpha)]}\mathrm{A}^{-1}\mathrm{B}X_2$$

From equation (13) we have,

$$(I_n-\lambda I_n)X_2=\left(\frac{2\omega^2\alpha}{2-\omega}-\frac{2\omega(1-\omega\alpha)}{2-\omega}\frac{\omega\alpha(1-\omega+\lambda)}{[(1-\omega\alpha)(1-\omega)-\lambda(1-\omega+\omega\alpha)]}+\frac{-2\omega(1-\omega+\omega\alpha)\lambda}{(1-\omega)(2-\omega)}\frac{\omega\alpha(1-\omega+\lambda)}{[(1-\omega\alpha)(1-\omega)-\lambda(1-\omega+\omega\alpha)]}-\frac{2\omega^2\alpha\lambda}{(1-\omega)(2-\omega)}\right)Q^{-1}\mathrm{B}^{\mathrm{T}}A^{-1}BX_2$$

$$(1-\lambda)I_n=\frac{(-4\lambda\alpha\omega^4+12\lambda\alpha\omega^3-8\lambda\alpha\omega^2)\mu}{(2-\omega)(1-\omega)[(1-\omega\alpha)(1-\omega)-\lambda(1-\omega+\omega\alpha)]}$$

$$(1-\lambda)I_n=\frac{-4\lambda\alpha\omega^2\mu}{[(1-\omega\alpha)(1-\omega)-\lambda(1-\omega+\omega\alpha)]}$$

$$(1-\lambda)[\lambda(1-\omega+\omega\alpha)-(1-\omega\alpha)(1-\omega)]=4\lambda\alpha\omega^2\mu.$$

Similarly we can prove converse part of this theorem.This proves the theorem.

Lemma: ([10]). Both roots of the real quadratic equation $x^2-\mathrm{b}x+\mathrm{c}=0$ are less than one in modulus if and only if

$$|c|<1 \quad\text{and}\quad |b|<1+\mathrm{c}. \tag{14}$$

Theorem 3.2. Let A and Q be symmetric positive definite, and B be of full rank. Assume that all eigens values μ of $Q^{-1}B^T\mathrm{A}^{-1}\mathrm{B}$ are real and positive. Then we have the following conditions for the relaxation parameter, α,μ and ω.

$$0<\omega<2 \;\;\&\;\; 0<\alpha<\frac{2(1-\omega)}{\omega^2(2\mu-1)}$$

Proof: Let λ,ω,α and μ be an eigen value of iterative matrix, a relaxation parameter, scaler and eigen value of $Q^{-1}B^T\mathrm{A}^{-1}\mathrm{B}$ respectively. A relation between $\lambda,\omega,\alpha$and μ is given by equation (11) then,

$$(1-\lambda)[\lambda(1-\omega+\omega\alpha)-(1-\omega\alpha)(1-\omega)]=4\lambda\alpha\omega^2\mu$$

$$2\lambda+\omega-2\lambda\omega+\alpha\omega+\lambda^2\omega-\alpha\omega^2-\lambda^2+\lambda\alpha\omega^2-\lambda^2\alpha\omega-4\lambda\alpha\mu\omega^2-1=0$$

$$(\alpha\omega+1-\omega)\lambda^2+(-2+2\omega-\alpha\omega^2+4\alpha\mu\omega^2)\lambda+(-\omega-\alpha\omega+\alpha\omega^2+1)=0$$

$$\lambda^2 - \frac{(2-2\omega+\alpha\omega^2-4\alpha\mu\omega^2)}{(\alpha\omega+1-\omega)}\lambda + \frac{(-\omega-\alpha\omega+\ \alpha\omega^2+1)}{(\alpha\omega+1-\omega)} = 0$$

$$\lambda = .5\left[\frac{(2-2\omega+\alpha\omega^2-4\alpha\mu\omega^2)}{(\alpha\omega+1-\omega)} \pm \sqrt{\left(\left(\frac{(2-2\omega+\alpha\omega^2-4\alpha\mu\omega^2)}{(\alpha\omega+1-\omega)}\right)^2 - 4\frac{(-\omega-\alpha\omega+\ \alpha\omega^2+1)}{(\alpha\omega+1-\omega)}\right)}\right] \quad (15)$$

$$b = \frac{(2-2\omega+\alpha\omega^2-4\alpha\mu\omega^2)}{(\alpha\omega+1-\omega)}$$

$$c = \frac{(-\omega-\alpha\omega+\ \alpha\omega^2+1)}{(\alpha\omega+1-\omega)}$$

According to above lemma we have, from (14)

$$\left|\frac{(-\omega-\alpha\omega+\ \alpha\omega^2+1)}{(\alpha\omega+1-\omega)}\right| < 1 \quad (16)$$

$$\left|\frac{(2-2\omega+\alpha\omega^2-4\alpha\mu\omega^2)}{(\alpha\omega+1-\omega)}\right| < 1 + \frac{(-\omega-\alpha\omega+\ \alpha\omega^2+1)}{(\alpha\omega+1-\omega)} \quad (17)$$

From inequality (16) we have,

$$-1 < \frac{(-\omega-\alpha\omega+\ \alpha\omega^2+1)}{(\alpha\omega+1-\omega)} < 1$$

$$-(\alpha\omega+1-\omega) < (-\omega-\alpha\omega+\ \alpha\omega^2+1) < (\alpha\omega+1-\omega)$$

$$\alpha\omega^2 - 2\omega + 2 > 0 \text{and} \alpha\omega^2 - 2\,\alpha\omega < 0$$

So, from above inequality we have,

$$0 < \omega < 2$$

From inequality (17) we have

$$-1 - \frac{(-\omega-\alpha\omega+\ \alpha\omega^2+1)}{(\alpha\omega+1-\omega)} < \frac{(2-2\omega+\alpha\omega^2-4\alpha\mu\omega^2)}{(\alpha\omega+1-\omega)} < 1 + \frac{(-\omega-\alpha\omega+\ \alpha\omega^2+1)}{(\alpha\omega+1-\omega)}$$

So, from above inequality we have,

$$(-4+4\omega-2\alpha\omega^2+4\alpha\mu\omega^2) < 0 \ \text{ and } \ -4\alpha\mu\omega^2 < 0$$

$$\text{i.e.,} 0 < \alpha < \frac{2(1-\omega)}{\omega^2(2\mu-1)}.$$

Which is the required condition for $\omega \& \alpha$.

Calculation for optimal parameters [11]

$$h(\alpha,\omega,\mu) = \frac{(2-2\omega+\alpha\omega^2-4\alpha\mu\omega^2)}{(\alpha\omega+1-\omega)}$$

$$h_\mu(\alpha,\omega,\mu) = \frac{(-4\alpha\omega^2)}{(\alpha\omega+1-\omega)}$$

$$h_\alpha(\alpha,\omega,\mu) = \frac{\omega(4\mu\omega^2-\omega^2+3\omega-4\mu\omega-2)}{(\alpha\omega+1-\omega)^2}$$

$$c(\alpha,\omega,\mu) = \frac{(-\omega-\ \alpha\,\omega+\ \alpha\omega^2+1)}{(\alpha\omega+1-\omega)}$$

$$c_\alpha(\alpha,\omega,\mu) = \frac{-\omega(\omega-2)(\omega-1)}{(\alpha\omega+1-\omega)^2}$$

$$g(\alpha,\omega,\mu) = \left(\frac{(2-2\omega+\alpha\omega^2-4\alpha\mu\omega^2)}{(\alpha\omega+1-\omega)}\right)^2 - 4\frac{(-\omega-\ \alpha\omega+\ \alpha\omega^2+1)}{(\alpha\omega+1-\omega)}$$

$$= \frac{16\alpha^2\mu^2\omega^4-8\alpha^2\mu\omega^4+\alpha^2\omega^4-4\alpha^2\omega^3+4\alpha^2\omega^2+16\alpha\mu\omega^3-16\alpha\mu\omega^2}{(\alpha\omega+1-\omega)^2}$$

From theorem (3.2) we have,

$$0<\omega<2\ \&0<\alpha<\frac{2(1-\omega)}{\omega^2(2\mu-1)}$$

From equation (15) some assumptions are taken as,

$$g(\alpha,\omega,\mu)\le 0 \quad \text{when, } 0<\alpha\le\frac{16\mu(1-\omega)}{8\mu\omega^2(2\mu-1)+(\omega-2)^2}$$

$$g(\alpha,\omega,\mu)>0 \text{ and } h(\alpha,\omega,\mu)\ge 0 \text{ when , } \frac{16\mu(1-\omega)}{8\mu\omega^2(2\mu-1)+(\omega-2)^2}<\alpha\le\frac{2(1-\omega)}{\omega^2(4\mu-1)} \text{ and } \omega<\frac{2}{4\mu+1}$$

$$g(\alpha,\omega,\mu)>0 \text{ and } h(\alpha,\omega,\mu)<0 \text{ when , } \alpha>\frac{2(1-\omega)}{\omega^2(4\mu-1)} \text{ and } \omega<\frac{2}{4\mu+1} \text{ or,}$$

$$\alpha>\frac{16\mu(1-\omega)}{8\mu\omega^2(2\mu-1)+(\omega-2)^2} \text{ and } \omega\ge\frac{2}{4\mu+1}$$

$$|\lambda| = \sqrt{\frac{(1-\omega)(1-\ \omega\alpha)}{(\omega\alpha+1-\omega)}}\text{when,}0<\alpha\le\frac{16\mu(1-\omega)}{8\mu\omega^2(2\mu-1)+(\omega-2)^2}$$

$$|\lambda| = \frac{1}{2}\left[h(\alpha,\omega,\mu)+\sqrt{g(\alpha,\omega,\mu)}\right]\text{when, }\frac{16\mu(1-\omega)}{8\mu\omega^2(2\mu-1)+(\omega-2)^2}<\alpha\le\frac{2(1-\omega)}{\omega^2(4\mu-1)} \text{ and } \omega<\frac{2}{4\mu+1}$$

$$|\lambda| = \frac{1}{2}\left[-h(\alpha,\omega,\mu)+\sqrt{g(\alpha,\omega,\mu)}\right]\text{when, }\alpha>\frac{2(1-\omega)}{\omega^2(4\mu-1)} \text{ and } \omega<\frac{2}{4\mu+1} \text{ or,}$$

$$\alpha>\frac{16\mu(1-\omega)}{8\mu\omega^2(2\mu-1)+(\omega-2)^2} \text{ and } \omega\ge\frac{2}{4\mu+1}$$

For simplification we take following assumptions.

$$\begin{cases} \alpha_1(\omega,\mu) = \frac{16\mu(1-\omega)}{8\mu\omega^2(2\mu-1)+(2-\omega)^2} \\ \alpha_2(\omega,\mu) = \frac{2(1-\omega)}{\omega^2(4\mu-1)} \end{cases}$$

And,

$$\begin{cases} f_1(\alpha,\omega,\mu) = \sqrt{\frac{(1-\omega)(1-\omega\alpha)}{(\omega\alpha+1-\omega)}} \\ f_2(\alpha,\omega,\mu) = \frac{1}{2}\left[h(\alpha,\omega,\mu) + \sqrt{g(\alpha,\omega,\mu)}\right] \\ f_3(\alpha,\omega,\mu) = \frac{1}{2}\left[-h(\alpha,\omega,\mu) + \sqrt{g(\alpha,\omega,\mu)}\right] \end{cases}$$

We can also write,

$$\begin{cases} |\lambda| = f_1(\alpha,\omega,\mu) \text{when } 0 < \alpha \leq \frac{16\mu(1-\omega)}{8\mu\omega^2(2\mu-1)+(\omega-2)^2} \\ |\lambda| = f_2(\alpha,\omega,\mu) \text{when } \alpha_1(\omega,\mu) < \alpha < \alpha_2(\omega,\mu) \text{ and } \omega < \frac{2}{4\mu+1} \\ |\lambda| = f_3(\alpha,\omega,\mu) \text{when} \alpha > \alpha_2(\omega,\mu) \text{and } \omega < \frac{2}{4\mu+1} \\ \text{or}, \alpha > \frac{16\mu(1-\omega)}{8\mu\omega^2(2\mu-1)+(\omega-2)^2} \text{ and } \omega \geq \frac{2}{4\mu+1} \end{cases} \quad (18)$$

$$\begin{cases} \frac{\partial f_1(\alpha,\omega,\mu)}{\partial\mu} = 0 \\ \frac{\partial f_2(\alpha,\omega,\mu)}{\partial\mu} = \frac{1}{2}\left[h_\mu(\alpha,\omega,\mu) + \frac{h_\mu(\alpha,\omega,\mu)h(\alpha,\omega,\mu)}{\sqrt{g(\alpha,\omega,\mu)}}\right] \\ \frac{\partial f_3(\alpha,\omega,\mu)}{\partial\mu} = \frac{1}{2}\left[-h_\mu(\alpha,\omega,\mu) + \frac{h_\mu(\alpha,\omega,\mu)h(\alpha,\omega,\mu)}{\sqrt{g(\alpha,\omega,\mu)}}\right] \end{cases}$$

$$\begin{cases} \frac{\partial f_2(\alpha,\omega,\mu)}{\partial\mu} < 0 \text{when } \alpha_1(\omega,\mu) < \alpha < \alpha_2(\omega,\mu) \text{and } \omega < \frac{2}{4\mu+1} \\ \frac{\partial f_3(\alpha,\omega,\mu)}{\partial\mu} > 0 \quad \text{when} \alpha > \alpha_2(\omega,\mu) \text{ and } \omega < \frac{2}{4\mu+1} \\ \text{or } \alpha > \frac{16\mu(1-\omega)}{8\mu\omega^2(2\mu-1)+(\omega-2)^2} \text{ and } \omega \geq \frac{2}{4\mu+1} \end{cases} \quad (19)$$

Here we can observe that $f_2(\alpha,\omega,\mu)$ is decreasing with respect to μ when $\alpha_1(\omega,\mu) < \alpha < \alpha_2(\omega,\mu)$ and $\omega < \frac{2}{4\mu+1}$and $f_3(\alpha,\omega,\mu)$ is increasing with respect to μ when $\alpha >$ $\alpha 2\omega,\mu$ and $\omega<2 4\mu+1$.

$$\begin{cases} \frac{\partial f_1(\alpha,\omega,\mu)}{\partial\alpha} = -\frac{1}{2}\sqrt{\frac{(\omega\alpha+1-\omega)}{(1-\omega)(1-\omega\alpha)}}\left[\frac{\omega(1-\omega)(\omega-2)}{(\omega\alpha+1-\omega)^2}\right] \\ \frac{\partial f_2(\alpha,\omega,\mu)}{\partial\alpha} = \frac{1}{2}\left[h_\alpha(\alpha,\omega,\mu) + \frac{h_\alpha(\alpha,\omega,\mu)h(\alpha,\omega,\mu)-2c(\alpha,\omega,\mu)}{\sqrt{g(\alpha,\omega,\mu)}}\right] \\ \frac{\partial f_3(\alpha,\omega,\mu)}{\partial\alpha} = \frac{1}{2}\left[-h_\alpha(\alpha,\omega,\mu) + \frac{h_\alpha(\alpha,\omega,\mu)h(\alpha,\omega,\mu)-2c(\alpha,\omega,\mu)}{\sqrt{g(\alpha,\omega,\mu)}}\right] \end{cases}$$

$$\begin{cases} \frac{\partial f_1(\alpha,\omega,\mu)}{\partial \alpha} < 0 & \text{when } 0 < \alpha \leq \alpha_1(\omega,\mu) \\ \frac{\partial f_2(\alpha,\omega,\mu)}{\partial \alpha} > 0 & \text{when } \alpha_1(\omega,\mu) < \alpha < \alpha_2(\omega,\mu) \\ \frac{\partial f_3(\alpha,\omega,\mu)}{\partial \alpha} > 0 & \text{when } \alpha > \alpha_2(\omega,\mu) \text{ and } \omega < \frac{2}{4\mu+1} \\ & \alpha > \alpha_1(\omega,\mu) \text{and } \omega \geq \frac{2}{4\mu+1} \end{cases} \tag{20}$$

Further we can see that , $f_1(\alpha,\omega,\mu)$ is a decreasing function with respect to α when $\alpha_1(\omega,\mu) < \alpha < \alpha_2(\omega,\mu)$ and $f_2(\alpha,\omega,\mu)$ is increasing with respect to α when $\alpha >$ $\alpha 2\omega,\mu$ and $\omega<2 4\mu+1$.

For fix value of ω and μ we have

$$\begin{cases} f_1(\alpha,\tilde{\omega},\tilde{\mu}) = f_2(\alpha,\tilde{\omega},\tilde{\mu}) \text{for} \alpha = \alpha_1(\omega,\mu) \\ f_1(\alpha,\tilde{\omega},\tilde{\mu}) = f_3(\alpha,\tilde{\omega},\tilde{\mu}) \text{ for} \alpha = \alpha_1(\omega,\mu) \\ f_2(\alpha,\tilde{\omega},\tilde{\mu}) = f_3(\alpha,\tilde{\omega},\tilde{\mu}) \text{ for} \alpha = \alpha_2(\omega,\mu) \end{cases} \tag{21}$$

For fix value of w we have

$$f_2(\alpha,\tilde{\omega},\mu_u) = f_3(\alpha,\tilde{\omega},\mu_v) \quad \text{for} \quad \alpha = \alpha_2\left(\omega, \frac{\mu_u+\mu_v}{2}\right) \tag{22}$$

$$\alpha_-(\omega) = \alpha_1(\omega,\mu_{min}), \quad \alpha_+(\omega) = \alpha_1(\omega,\mu_{max}), \quad \alpha_0(\omega) = \alpha_2\left(\omega, \frac{\mu_{min}+\mu_{max}}{2}\right)$$

Case 1. $\omega < \frac{2}{4\mu+1}$

$$|\lambda| = \begin{cases} f_1(\alpha,\omega,\mu) \text{when } 0 < \alpha \leq \alpha_1(\omega,\mu) \\ f_2(\alpha,\omega,\mu) \quad \text{when } \alpha_1(\omega,\mu) < \alpha < \alpha_2(\omega,\mu) \\ f_3(\alpha,\omega,\mu) \quad \text{when } \alpha > \alpha_2(\omega,\mu) \end{cases} \tag{23}$$

Case 2. $\omega \geq \frac{2}{4\mu+1}$

$$|\lambda| = \begin{cases} f_1(\alpha,\omega,\mu) \text{when } 0 < \alpha \leq \alpha_1(\omega,\mu) \\ f_3(\alpha,\omega,\mu) \quad \text{when } \alpha > \alpha_1(\omega,\mu) \end{cases} \tag{24}$$

Let ρ(M(α, ω)) be the spectral radius of M(α, ω). From (23)and (24), in order to compute ρ(M(α, ω)) we need toconsider the following three cases about the parameter ω.

(a) $\omega \leq \frac{2}{4\mu_{max}+1}$ (b) $\omega \geq \frac{2}{4\mu_{min}+1}$ (c) $\frac{2}{4\mu_{max}+1} < \omega < \frac{2}{4\mu_{min}+1}$.

Theorem 3.3. Let A and Q be symmetric positive definite, and B be full column rank. Denote the smallest and largest eigenvalues of the matrix $Q^{-1}B^T A^{-1} B$ by $\mu_{min}$and μ_{max}, respectively.

Case 1. $0 < \omega \leq \frac{2}{[4\sqrt{\mu_{min}\mu_{max}}+1]}$

$$\rho(\mathrm{M}(\alpha,\omega)) = \begin{cases} \sqrt{\dfrac{(1-\omega)(1-\omega\alpha)}{(\omega\alpha+1-\omega)}}, & \text{when } 0<\alpha<\alpha_{-}(\omega) \\ .5\left[\dfrac{(2-2\omega+\alpha\omega^2-4\alpha\mu_{min}\omega^2)}{(\alpha\omega+1-\omega)}+\sqrt{\left(\left(\dfrac{(2-2\omega+\alpha\omega^2-4\alpha\mu_{min}\omega^2)}{(\alpha\omega+1-\omega)}\right)^2-4\dfrac{(-\omega-\alpha\omega+\alpha\omega^2+1)}{(\alpha\omega+1-\omega)}\right)}\right] & \text{when } \alpha_{-}(\omega)\le\alpha\le\alpha_0(\omega) \\ .5\left[\dfrac{(2\omega-2-\alpha\omega^2+4\alpha\mu_{max}\omega^2)}{(\alpha\omega+1-\omega)}+\sqrt{\left(\left(\dfrac{(2-2\omega+\alpha\omega^2-4\alpha\mu_{max}\omega^2)}{(\alpha\omega+1-\omega)}\right)^2-4\dfrac{(-\omega-\alpha\omega+\alpha\omega^2+1)}{(\alpha\omega+1-\omega)}\right)}\right] & \text{when } \alpha_0(\omega)<\alpha<\dfrac{2(1-\omega)}{\omega^2(2\mu_{max}-1)} \end{cases}$$

Case 2. $\frac{2}{[4\sqrt{\mu_{min}\mu_{max}}+1]} < \omega < 2$

$$\rho\big(\mathrm{M}(\alpha,\omega)\big) = \begin{cases} \sqrt{\dfrac{(1-\omega)(1-\omega\alpha)}{(\omega\alpha+1-\omega)}} & \text{when } 0<\alpha<\alpha_{+}(\omega) \\ .5\left[\dfrac{(2\omega-2-\alpha\omega^2+4\alpha\mu_{max}\omega^2)}{(\alpha\omega+1-\omega)}+\sqrt{\left(\left(\dfrac{(2-2\omega+\alpha\omega^2-4\alpha\mu_{max}\omega^2)}{(\alpha\omega+1-\omega)}\right)^2-4\dfrac{(-\omega-\alpha\omega+\alpha\omega^2+1)}{(\alpha\omega+1-\omega)}\right)}\right] & \text{when } \alpha_{+}(\omega)\le\alpha<\dfrac{2(1-\omega)}{\omega^2(2\mu_{max}-1)} \end{cases}$$

Moreover, the optimal parameters are given by,

$$\omega_{opt} = \frac{2}{[4\sqrt{\mu_{min}\mu_{max}}+1]}, \alpha_{opt} = \frac{[16\mu_{min}\mu_{max}-1]}{(4\mu_{min}+4\mu_{max}-2)}$$

And the corresponding optimal convergence factor is,

$$\rho(\mathrm{M}(\alpha_{opt},\omega_{opt}) = \frac{\sqrt{\mu_{min}}-\sqrt{\mu_{max}}}{\sqrt{\mu_{min}}+\sqrt{\mu_{max}}}$$

Proof: Let's consider the following three cases of ω,

Case 1. $\omega \le \frac{2}{4\mu_{max}+1}$

For this case $\omega \le \frac{2}{4\mu+1}$ holds for all the eigen values of the matrix $Q^{-1}B^T\mathrm{A}^{-1}\mathrm{B}$. Moreover,

$$\alpha_{-}(\omega) < \alpha_{+}(\omega) < \alpha_0(\omega)$$

Obviously, the absolute values of the eigenvalues of $\mathrm{M}(\alpha,\omega)$ are given by (23). For any fixed α,ω>0, according to the monotone property of the functions $f_j(\alpha,\omega,\mu)$ (j = 1, 2, 3) with respect to μshown in (19), we get

$$\begin{cases} \max_{\mu}\{f_1(\alpha,\omega,\mu) \mid & \text{when } 0 < \alpha \le \alpha_1(\omega,\mu)\} = f_1(\alpha,\omega,\mu) \\ \max_{\mu}\{f_2(\alpha,\omega,\mu) \mid & \text{when } \alpha_1(\omega,\mu) < \alpha \le \alpha_2(\omega,\mu)\} = f_2(\alpha,\omega,\mu_{min}) \\ \max_{\mu}\{f_3(\alpha,\omega,\mu) \mid & \text{when } \alpha > \alpha_2(\omega,\mu)\} = f_3(\alpha,\omega,\mu_{max}) \end{cases}$$

In addition, from (21) and (22)we know that the intersection point of the curves $f_1(\alpha,\omega,\mu)$and $f_2(\alpha,\omega,\mu_{min})$is $\alpha\_(\omega)$,and the intersection point of the curves $f_2(\alpha,\omega,\mu_{min})$and $f_3(\alpha,\omega,\mu_{max})$is $\alpha_0(\omega)$. By considering the monotone propertyof the functions $f_j(\alpha,\omega,\mu)(j = 1, 2, 3)$ with respect to αshown in (20)we obtain,

$$\rho(\mathrm{M}(\alpha,\omega)) = \begin{cases} f_1(\alpha,\omega,\mu) & \text{when } 0 < \alpha < \alpha\_(\omega) \\ f_2(\alpha,\omega,\mu_{min}) & \text{when } \alpha\_(\omega) \le \alpha \le \alpha_0(\omega) \\ f_3(\alpha,\omega,\mu_{max}) & \text{when } \alpha_0(\omega) < \alpha < \frac{2(1-\omega)}{\omega^2(2\mu_{max}-1)} \end{cases}$$

It follows that,

$$\alpha\_(\omega) = arg\min_{\alpha}\rho(\mathrm{M}(\alpha,\omega))$$

For any fixed ω. Because,

$$\rho\big(\mathrm{M}(\alpha\_(\omega),\omega)\big) = \sqrt{\frac{(1-\omega)(1-\omega\alpha\_(\omega))}{(\omega\alpha\_(\omega)+1-\omega)}}$$

$$= \sqrt{\frac{(1-\omega)[8\mu_{min}\omega^2(2\mu_{min}-1)+(\omega-2)^2-16\mu_{min}\omega(1-\omega)]}{16\mu_{min}\omega(1-\omega)+(1-\omega)(8\mu_{min}\omega^2(2\mu_{min}-1)+(\omega-2)^2)}}$$

$$= \sqrt{\frac{(8\mu_{min}\omega^2(2\mu_{min}-1)+(\omega-2)^2)-16\mu_{min}\omega(1-\omega)}{(8\mu_{min}\omega^2(2\mu_{min}-1)+(\omega-2)^2)+16\mu_{min}\omega}}$$

$$= \sqrt{\frac{16\mu_{min}{}^2\omega^2+8\mu_{min}\omega^2+\omega^2+4-4\omega-16\mu_{min}\omega}{16\mu_{min}{}^2\omega^2-8\mu_{min}\omega^2+\omega^2+4-4\omega+16\mu_{min}\omega}}$$

$$= \frac{4\mu_{min}\omega+\omega-2}{4\mu_{min}\omega-\omega+2}$$

One can see that the parameter ωsuch that $\rho\big(\mathrm{M}(\alpha_(\omega),\omega)\big)$attains the minimum is $\omega = \frac{2}{4\mu_{max}+1}$. Therefore, in this case,the optimal parameters are,

$$\omega_{opt}^{(1)} = \frac{2}{4\mu_{max}+1},\ \alpha_{opt}^{(1)} = \frac{16\mu_{min}(1-\frac{2}{4\mu_{max}+1})}{8\mu_{min}(\frac{2}{4\mu_{max}+1})^2(2\mu_{min}-1)+(\frac{2}{4\mu_{max}+1}-2)^2}$$

$$= \frac{4\mu_{min}(4\mu_{max}-1)(4\mu_{max}+1)}{8\mu_{min}(2\mu_{min}-1)+(-4\mu_{max})^2}$$

$$= \frac{4\mu_{min}(16\mu_{max}{}^2-1)}{(16\mu_{min}{}^2+16\mu_{max}{}^2)-8\mu_{min}}$$

$$= \frac{\mu_{min}(16\mu_{max}{}^2-1)}{(4\mu_{min}{}^2+4\mu_{max}{}^2)-2\mu_{min}}$$

and the corresponding optimal convergence factor is,

$$\rho(\mathrm{M}(\alpha_{opt}^{(1)}, \omega_{opt}^{(1)})) = \sqrt{\frac{(1-\omega_{opt}^{(1)})(1-\omega_{opt}^{(1)}\alpha_{opt}^{(1)})}{(\omega_{opt}^{(1)}\alpha_{opt}^{(1)}+1-\omega_{opt}^{(1)})}}$$

$$= \sqrt{\frac{\left(1-\frac{2}{4\mu_{max}+1}\right)\left(1-\frac{2}{(4\mu_{max}+1)}\frac{4\mu_{min}(16\mu_{max}{}^2-1)}{(16\mu_{min}{}^2+16\mu_{max}{}^2-8\mu_{min})}\right)}{\frac{4\mu_{min}(16\mu_{max}{}^2-1)}{(16\mu_{min}{}^2+16\mu_{max}{}^2-8\mu_{min})}\frac{2}{(4\mu_{max}+1)}+1-\frac{2}{4\mu_{max}+1}}}$$

$$= \sqrt{\frac{64\mu_{max}^3-128\mu_{max}^2\mu_{min}+16\mu_{max}^2+64\mu_{max}\mu_{min}^2-32\mu_{max}\mu_{min}+16\mu_{min}^2}{256\mu_{max}^4+512\mu_{max}^3\mu_{min}+256\mu_{max}^2\mu_{min}^2-16\mu_{max}^2-32\mu_{max}\mu_{min}-16\mu_{min}^2}}$$

$$= \sqrt{\frac{4\mu_{max}^3-8\mu_{max}^2\mu_{min}+\mu_{max}^2+4\mu_{max}\mu_{min}^2-2\mu_{max}\mu_{min}+\mu_{min}^2}{16\mu_{max}^4+32\mu_{max}^3\mu_{min}+16\mu_{max}^2\mu_{min}^2-\mu_{max}^2-2\mu_{max}\mu_{min}-\mu_{min}^2}}$$

$$= \frac{\mu_{max}-\mu_{min}}{\mu_{max}+\mu_{min}}\frac{1}{\sqrt{4\mu_{max}-1}}$$

Case 2. $\omega \geq \frac{2}{4\mu_{min}+1}$

For this case $\omega > \frac{2}{4\mu+1}$ holds for all the eigen values of the matrix $Q^{-1}B^T\mathrm{A}^{-1}\mathrm{B}$. Obviously, the absolute values of the eigen values $\mathrm{M}(\alpha, \omega)$ are given by (24). For any fixed $\alpha, \omega > 0$, according to the monotone property of the function $f_j(\alpha, \omega, \mu)(j = 1,3)$ with respect to μ shown in (19) we get,

$$\begin{cases} \max\limits_{\mu}\{f_1(\alpha,\omega,\mu) \mid \text{ when } 0 < \alpha \leq \alpha_1(\omega,\mu)\} = f_1(\alpha,\omega,\mu) \\ \max\limits_{\mu}\{f_3(\alpha,\omega,\mu) \mid \text{ when } \alpha > \alpha_1(\omega,\mu)\} = f_3(\alpha,\omega,\mu_{max}) \end{cases}$$

In addition, from (21) and (22)we know that the intersection point of the curves $f_1(\alpha, \omega, \mu)$and $f_2(\alpha, \omega, \mu_{max})$is $\alpha_+(\omega)$. By considering the monotone propertyof the functions $f_j(\alpha, \omega, \mu)$ (j = 1, 3) with respect to αshown in (20), we obtain,

$$\rho(\mathrm{M}(\alpha,\omega))=\begin{cases} f_1(\alpha,\omega,\mu) & \text{when } 0 < \alpha < \alpha_+(\omega) \\ f_3(\alpha,\omega,\mu_{max}) & \text{when } \alpha_+(\omega) < \alpha < \frac{2(1-\omega)}{\omega^2(2\mu_{max}-1)} \end{cases}$$

It follows that,

$$\alpha_+(\omega) = arg\min_{\alpha} \rho(\mathrm{M}(\alpha, \omega))$$

For any fixed ω. Because,

$$\rho\big(\mathrm{M}(\alpha_+(\omega), \omega)\big) = \sqrt{\frac{(1-\omega)(1-\omega\alpha_+(\omega))}{(\omega\alpha_+(\omega)+1-\omega)}}$$

$$= \sqrt{\frac{(1-\omega)(1-\frac{16\mu_{max}\omega(1-\omega)}{8\mu_{max}\omega^2(2\mu_{max}-1)+(\omega-2)^2})}{(\frac{16\mu_{max}\omega(1-\omega)}{8\mu_{max}\omega^2(2\mu_{max}-1)+(\omega-2)^2}+1-\omega)}}$$

$$= \sqrt{\frac{[8\mu_{max}\omega^2(2\mu_{max}-1)+(\omega-2)^2-16\mu_{max}\omega(1-\omega)]}{16\mu_{max}\omega+[8\mu_{max}\omega^2(2\mu_{max}-1)+(\omega-2)^2]}}$$

$$= \sqrt{\frac{16\mu_{max}^2\omega^2+8\mu_{max}\omega^2+\omega^2+4-4\omega-16\mu_{max}\omega}{16\mu_{max}^2\omega^2-8\mu_{max}\omega^2+\omega^2+4-4\omega+16\mu_{max}\omega}}$$

$$= \frac{4\mu_{max}\omega-\omega-2}{4\mu_{max}\omega-\omega-2}$$

One can see that the parameter ω such that $\rho\big(\mathrm{M}(\alpha_+(\omega), \omega)\big)$ attains the minimum is $\frac{2}{4\mu_{min}+1}$. Therefore, in this case,the optimal parameters are,

$$\omega_{opt}^{(2)} = \frac{2}{4\mu_{min}+1}, \alpha_{opt}^{(2)} = \frac{16\mu_{max}\left(1-\frac{2}{4\mu_{min}+1}\right)}{8\mu_{max}\left(\frac{2}{4\mu_{min}+1}\right)^2(2\mu_{max}-1)+\left(\frac{2}{4\mu_{min}+1}-2\right)^2}$$

$$= \frac{4\mu_{max}(4\mu_{min}-1)(4\mu_{min}+1)}{8\mu_{max}(2\mu_{max}-1)+16\mu_{min}{}^2}$$

$$= \frac{\mu_{max}(16\mu_{min}{}^2-1)}{2\mu_{max}(2\mu_{max}-1)+4\mu_{min}{}^2}$$

$$= \frac{\mu_{max}(16\mu_{min}{}^2-1)}{4\mu_{max}{}^2+4\mu_{min}{}^2-2\mu_{max}}.$$

And the corresponding convergence factor is,

$$\rho(\mathrm{M}(\alpha_{opt}^{(2)}, \omega_{opt}^{(2)})) = \sqrt{\frac{\left(1-\omega_{opt}^{(2)}\right)(1-\omega_{opt}^{(2)}\alpha_{opt}^{(2)})}{(\omega_{opt}^{(2)}\alpha_{opt}^{(2)}+1-\omega_{opt}^{(2)})}}$$

$$= \sqrt{\frac{\left(1-\frac{2}{4\mu_{min}+1}\right)(1-\frac{2\mu_{max}(16\mu_{min}{}^2-1)}{[(4\mu_{min}+1)(4\mu_{max}{}^2+4\mu_{min}{}^2-2\mu_{max})]})}{(\frac{2\mu_{max}(16\mu_{min}{}^2-1)}{(4\mu_{min}+1)(4\mu_{max}{}^2+4\mu_{min}{}^2-2\mu_{max})}+1-\frac{2}{(4\mu_{min}+1)})}}$$

$$= \sqrt{\frac{16\mu_{max}^2\mu_{min}^2-\mu_{max}^2-32\mu_{max}\mu_{min}^3+2\mu_{max}\mu_{min}+16\mu_{min}^4-\mu_{min}^2}{4\mu_{max}^2\mu_{min}-\mu_{max}^2+8\mu_{max}\mu_{min}^2-2\mu_{max}\mu_{min}+4\mu_{min}^3-\mu_{min}^2}}$$

$$= \frac{\mu_{max}-\mu_{min}}{\mu_{max}+\mu_{min}}\sqrt{(4\mu_{min}} + 1)$$

Case 3. $\frac{2}{4\mu_{max}+1} < \omega < \frac{2}{4\mu_{min}+1}$

We first assume that, $\widehat{\omega} = \frac{2}{[4\sqrt{\mu_{min}\mu_{max}}+1]}$

$$\alpha_-(\widehat{\omega}) = \alpha_+(\widehat{\omega}) = \alpha_0(\widehat{\omega}) = \frac{16\mu_{min}\mu_{max} - 1}{4(\mu_{min}+\mu_{max}) - 2}$$

when $\omega \in \left(\frac{2}{4\mu_{max}+1}, \widehat{\omega}\right]$ it holds that,

$$\alpha_-(\omega) \le \alpha_+(\omega) \le \alpha_0(\omega)$$

Let μ_u =max$\left\{\mu \mid \mu < \frac{2-\omega}{4\omega} \, and \, \frac{2}{4\mu_{max}+1} < \omega \le \widehat{\omega}\right\}$

For any fixed α, ω According to the monotone property of the functions $f_j(\alpha,\omega,\mu)(j = 1,2,3$with respect to μ shown in (19) we get,

$$\begin{cases} \max\limits_{\mu_{min}\le\mu\le\mu_u} \{ f_1(\alpha,\omega,\mu) | & 0 < \alpha \le \alpha_1(\omega,\mu)\} = f_1(\alpha,\omega,\mu) \\ \max\limits_{\mu_{min}\le\mu\le\mu_u} \{ f_2(\alpha,\omega,\mu) | & \alpha_1(\omega,\mu) < \alpha \le \alpha_2(\omega,\mu)\} = f_2(\alpha,\omega,\mu_{min}) \\ \max\limits_{\mu_{min}\le\mu\le\mu_u} \{ f_3(\alpha,\omega,\mu) | & \alpha > \alpha_2(\omega,\mu)\} = f_3(\alpha,\omega,\mu_u) \end{cases}$$

In addition, from (21) and (22)we know that the intersection point of the curves $f_1(\alpha,\omega,\mu)$and $f_2(\alpha,\omega,\mu_{min})$ is $\alpha_-(\omega)$, and the intersection point of the curves $f_2(\alpha,\omega,\mu_{min})$ and $f_3(\alpha,\omega,\mu_u)$ is

$$\hat{a}_0(\omega) = \frac{16(\mu_{min}+\mu_u)(1-\omega)}{8(\mu_{min}+\mu_u)\omega^2(2(\mu_{min}+\mu_u) - 1) + (2-\omega)^2}$$

According to the monotone property of the functions $f_j(\alpha,\omega,\mu)(j = 1,2,3)$ with respect to αshown in (20), we obtain

$$\max\{|\lambda| | \mu_{min} \le \mu \le \mu_u\} = \begin{cases} f_1(\alpha,\omega,\mu) \text{when } 0 < \alpha \le \alpha_-(\omega) \\ f_2(\alpha,\omega,\mu_{min}) \quad \text{when } \alpha_-(\omega) \le \alpha \le \hat{a}_0(\omega) \\ f_3(\alpha,\omega,\mu_u) \quad \text{when } \hat{a}_0(\omega) < \alpha < \frac{16\mu_{max}(1-\omega)}{8\mu_{max}\omega^2(2\mu_{max} - 1) + (2-\omega)^2} \end{cases}$$

Similarly, Let μ_v =max$\left\{\mu \mid \mu < \frac{2-\omega}{4\omega} \, and \, \frac{2}{4\mu_{max}+1} < \omega \le \widehat{\omega}\right\}$

Then, for any fixed α, ω, according to the monotone property of the functions $f_j(\alpha, \omega, \mu)$ (j= 1, 3) with respect to μshownin (19), we get

$$\begin{cases} \max\limits_{\mu_v \le \mu \le \mu_{max}} \{f_1(\alpha, \omega, \mu) \mid & \text{when } 0 < \alpha \le \alpha_1(\omega, \mu)\} = f_1(\alpha, \omega, \mu) \\ \max\limits_{\mu_v \le \mu \le \mu_{max}} \{f_3(\alpha, \omega, \mu) \mid & \text{when } \alpha > \alpha_1(\omega, \mu)\} = f_3(\alpha, \omega, \mu_{max}) \end{cases}$$

In addition, from (21) and (22)we know that the intersection point of the curves $f_1(\alpha, \omega, \mu_{max})$and $f_3(\alpha, \omega, \mu)$is $\alpha_+(\omega)$.By considering the monotone property of the functions $f_j(\alpha, \omega, \mu)$ (j= 1, 3) with respect to αshown in (20), we obtain

$$\max\{|\lambda| | \mu_v \le \mu \le \mu_{max}\}$$
$$= \begin{cases} f_1(\alpha, \omega, \mu) & \text{when } 0 < \alpha \le \alpha_+(\omega) \\ f_3(\alpha, \omega, \mu_{max}) & \text{when } \alpha_+(\omega) < \alpha < \dfrac{16\mu_{max}(1-\omega)}{8\mu_{max}\omega^2(2\mu_{max}-1) + (2-\omega)^2} \end{cases}$$

Since,

$$\alpha_-(\omega) \le \alpha_+(\omega) \le \alpha_0(\omega) \le \hat{\alpha}_0(\omega)$$

According to the monotone increasing property of the function $f_3(\alpha, \omega, \mu)$with respect to α, we see that

$$f_3(\alpha, \omega, \mu_{max}) \ge f_3(\alpha, \omega, \mu_v)$$

Moreover, as $f_2(\alpha, \omega, \mu_{min})$and $f_3(\alpha, \omega, \mu_{max})$intersects at $\alpha_0(\omega)$, we can obtain

$$\rho(\mathrm{M}(\alpha, \omega)) = \begin{cases} f_1(\alpha, \omega, \mu) & \text{when } 0 < \alpha < \alpha_-(\omega) \\ f_2(\alpha, \omega, \mu_{min}) & \text{when } \alpha\_(\omega) \le \alpha \le \alpha_0(\omega) \\ f_3(\alpha, \omega, \mu_{max}) & \text{when } \alpha_0(\omega) < \alpha < \dfrac{2(1-\omega)}{\omega^2(2\mu_{max}-1)} \end{cases}$$

It follows that,

$$\alpha_-(\omega) = arg\min_{\alpha} \rho(\mathrm{M}(\alpha, \omega))$$

For any fixed, $\omega \in \left(\frac{2}{4\mu_{max}+1}, \hat{\omega}\right]$ because,

$$\rho(\mathrm{M}(\alpha_-(\omega), \omega)) = \sqrt{\frac{(8\mu_{min}\omega^2(2\mu-1) + (\omega-2)^2) - 16\mu_{min}\omega}{(8\mu_{min}\omega^2(2\mu-1) + (\omega-2)^2) + 16\mu_{min}\omega}}$$

One can see that the parameter ω such that $\rho(\mathrm{M}(\alpha_-(\omega), \omega))$ attains the minimum is $\hat{\omega}$. Therefore, in this case, the optimalparameters are

$$\omega_{opt}^{(3)} = \frac{2}{[4\sqrt{\mu_{min}\mu_{max}} + 1]}$$

$$\alpha_{opt}^{(3)} = \frac{16\mu_{min}(1-\frac{2}{[4\sqrt{\mu_{min}\mu_{max}}+1]})}{8\mu_{min}\left[\frac{2}{[4\sqrt{\mu_{min}\mu_{max}}+1]}\right]^2(2\mu_{min}-1)+\left(2-\frac{2}{[4\sqrt{\mu_{min}\mu_{max}}+1]}\right)^2}$$

$$= \frac{16\mu_{min}[4\sqrt{\mu_{min}\mu_{max}}-1][4\sqrt{\mu_{min}\mu_{max}}+1]}{32\mu_{min}(2\mu_{min}-1)+4[4\sqrt{\mu_{min}\mu_{max}}]^2}$$

$$= \frac{\mu_{min}[16\mu_{min}\mu_{max}-1]}{2\mu_{min}(2\mu_{min}-1)+4\mu_{min}\mu_{max}} = \frac{16\mu_{min}\mu_{max}-1}{4(\mu_{min}+\mu_{max})-2}$$

And the corresponding optimal convergence factor is

$$\rho(\mathrm{M}(\alpha_{opt}^{(3)}, \omega_{opt}^{(3)})) = \sqrt{\frac{\left(1-\omega_{opt}^{(3)}\right)\left(1-\omega_{opt}^{(3)}\alpha_{opt}^{(3)}\right)}{\left(\omega_{opt}^{(3)}\alpha_{opt}^{(3)}+1-\omega_{opt}^{(3)}\right)}}$$

$$= \sqrt{\frac{16\left(\mu_{min}\sqrt{\mu_{max}}-\mu_{max}\sqrt{\mu_{min}}\right)^2-(\sqrt{\mu_{min}}-\sqrt{\mu_{max}})^2}{16\left(\mu_{min}\sqrt{\mu_{max}}+\mu_{max}\sqrt{\mu_{min}}\right)^2-(\sqrt{\mu_{min}}+\sqrt{\mu_{max}})^2}}$$

$$= \sqrt{\frac{16\mu_{min}\mu_{max}(\sqrt{\mu_{min}}-\sqrt{\mu_{max}})^2-(\sqrt{\mu_{min}}-\sqrt{\mu_{max}})^2}{16\mu_{min}\mu_{max}(\sqrt{\mu_{min}}+\sqrt{\mu_{max}})^2-(\sqrt{\mu_{min}}+\sqrt{\mu_{max}})^2}}$$

$$= \frac{\sqrt{\mu_{min}}-\sqrt{\mu_{max}}}{\sqrt{\mu_{min}}+\sqrt{\mu_{max}}}$$

When $\omega \in \left[\widehat{\omega}, \frac{2}{4\mu_{min}+1}\right)$ it holds that

$$\alpha_0(\omega) \le \alpha_+(\omega) \le \alpha_-(\omega)$$

Let μ_u =max$\left\{\mu \mid \mu < \frac{2-\omega}{4\omega} and \widehat{\omega} \le \omega < \frac{2}{4\mu_{min}+1}\right\}$

For any fixed α, ω, according to the monotone property of the functions $f_j(\alpha, \omega, \mu)$ (j = 1, 2, 3) with respect to μshown in(19), we get

$$\begin{cases} \max\limits_{\mu_{min}\le\mu\le\mu_u} f_1(\alpha,\omega,\mu)| & 0 < \alpha \le \alpha_1(\omega,\mu) = f_1(\alpha,\omega,\mu) \\ \max\limits_{\mu_{min}\le\mu\le\mu_u} f_2(\alpha,\omega,\mu)| & \alpha_1(\omega,\mu) < \alpha \le \alpha_2(\omega,\mu) = f_2(\alpha,\omega,\mu_{min}) \\ \max\limits_{\mu_{min}\le\mu\le\mu_u} f_3(\alpha,\omega,\mu)| & \alpha > \alpha_1(\omega,\mu) = f_3(\alpha,\omega,\mu_u) \end{cases}$$

In addition, from (21) and (22) we know that the intersection point of the curves $f_1(\alpha, \omega, \mu)$and $f_2(\alpha, \omega, \mu_{min})$is $\alpha_-(\omega)$, and the intersection point of the curves $f_2(\alpha, \omega, \mu_{min})$and $f_3(\alpha, \omega, \mu_u)$is

$$\hat{a}_0(\omega) = \frac{16(\mu_{min}+\mu_u)(1-\omega)}{8(\mu_{min}+\mu_u)\omega^2(2(\mu_{min}+\mu_u)-1)+(2-\omega)^2}$$

By considering the monotone property of the functions $f_j(\alpha,\omega,\mu)$ (j= 1, 2, 3) with respect to αshown in (20), we obtain

$$\max\{|\lambda|| \mu_{min} \le \mu \le \mu_u\} = \begin{cases} f_1(\alpha,\omega,\mu) \text{when } 0<\alpha<\alpha_-(\omega) \\ f_2(\alpha,\omega,\mu_{min}) \quad \text{when } \alpha_-(\omega) \le \alpha \le \hat{a}_0(\omega) \\ f_3(\alpha,\omega,\mu_u) \quad \text{when } \hat{a}_0(\omega) < \alpha < \dfrac{16\mu_{max}(1-\omega)}{8\mu_{max}\omega^2(2\mu_{max}-1)+(2-\omega)^2} \end{cases}$$

Let μ_v =max$\left\{\mu \mid \mu < \frac{2-\omega}{4\omega} and \hat{\omega} \le \omega < \frac{2}{4\mu_{min}+1}\right\}$

Then, for any fixed α, ω, according to the monotone property of the functions $f_j(\alpha,\omega,\mu)$ (j= 1, 3) with respect to μshownin (19), we get

$$\begin{cases} \max\limits_{\mu_v \le \mu \le \mu_{max}} \{f_1(\alpha,\omega,\mu) \mid \text{ when } 0<\alpha \le \alpha_1(\omega,\mu)\} = f_1(\alpha,\omega,\mu) \\ \max\limits_{\mu_v \le \mu \le \mu_{max}} \{f_3(\alpha,\omega,\mu) \mid \text{when } \alpha > \alpha_1(\omega,\mu)\} = f_3(\alpha,\omega,\mu_{max}) \end{cases}$$

In addition, from (21) and (22) we know that the intersection point of the curves $f_1(\alpha,\omega,\mu_{max})$and $f_3(\alpha,\omega,\mu)$is $\alpha_+(\omega)$.By considering the monotone property of the functions $f_j(\alpha,\omega,\mu)$ (j= 1, 3) with respect to αshown in (20), we obtain

$$\max\{|\lambda|| \mu_v \le \mu \le \mu_{max}\}$$

$$= \begin{cases} f_1(\alpha,\omega,\mu) \quad \text{when } 0<\alpha<\alpha_+(\omega) \\ f_3(\alpha,\omega,\mu_{max}) \quad \text{when } \alpha_+(\omega) < \alpha < \dfrac{16\mu_{max}(1-\omega)}{8\mu_{max}\omega^2(2\mu_{max}-1)+(2-\omega)^2} \end{cases}$$

$$\alpha_0(\omega) \le \alpha_+(\omega) \le \alpha_-(\omega) < \widehat{\alpha_0}(\omega)$$

According to the monotone increasing property of the function $f_3(\alpha,\omega,\mu)$with respect to α, we see that

$$f_3(\alpha,\omega,\mu_{max}) \ge f_3(\alpha,\omega,\mu_u)$$

Moreover, as $f_2(\alpha,\omega,\mu_{min})$and $f_3(\alpha,\omega,\mu_{max})$intersects at $\alpha_0(\omega)$, we can obtain

$$\rho(\mathrm{M}(\alpha,\omega)) \begin{cases} f_1(\alpha,\omega,\mu) \quad \text{when } 0<\alpha \le \alpha_+(\omega) \\ f_3(\alpha,\omega,\mu_{max}) \quad \text{when } \alpha_+(\omega) < \alpha < \dfrac{16\mu_{max}(1-\omega)}{8\mu_{max}\omega^2(2\mu_{max}-1)+(2-\omega)^2} \end{cases}$$

$$\alpha_+(\omega) = arg\min_{\alpha} \rho(\mathrm{M}(\alpha,\omega))$$

For any fixed $\omega \in [\hat{\omega}, \frac{2}{4\mu_{max}+1})$. Because

$$\rho(\mathrm{M}(\alpha_{+}(\omega),\omega)) = \sqrt{\frac{[8\mu\omega^2(2\mu-1)+(\omega-2)^2-16\mu\omega(1-\omega)]}{16\mu\omega+[8\mu\omega^2(2\mu-1)+(\omega-2)^2]}}$$

One can see that the parameter ωsuch that $\rho(\mathrm{M}(\alpha_{+}(\omega),\omega))$attains the minimum is $\widehat{\omega}$. Therefore, in this case, the optimal parameters are

$$\omega_{opt}^{(4)} = \frac{2}{[4\sqrt{\mu_{min}\mu_{max}}+1]}$$

$$\alpha_{opt}^{(4)} = \alpha_{+}(\widehat{\omega}) = \frac{16\mu_{max}(1-\frac{2}{[4\sqrt{\mu_{min}\mu_{max}}+1]})}{8\mu_{max}\left[\frac{2}{[4\sqrt{\mu_{min}\mu_{max}}+1]}\right]^2(2\mu_{max}-1)+\left(2-\frac{2}{[4\sqrt{\mu_{min}\mu_{max}}+1]}\right)^2}$$

$$= \frac{\mu_{max}[4\sqrt{\mu_{min}\mu_{max}}-1][4\sqrt{\mu_{min}\mu_{max}}+1]}{2\mu_{max}(2\mu_{max}-1)+4([\sqrt{\mu_{min}\mu_{max}}])^2}$$

$$= \frac{[4\sqrt{\mu_{min}\mu_{max}}-1][4\sqrt{\mu_{min}\mu_{max}}+1]}{2(2\mu_{max}-1)+4\mu_{min}}$$

$$= \frac{[16\mu_{min}\mu_{max}-1]}{(4\mu_{min}+4\mu_{max}-2)}$$

And the corresponding optimal convergence factor is

$$\rho(\mathrm{M}\,(\alpha_{opt}^{(4)},\omega_{opt}^{(4)})) = \sqrt{\frac{\left(1-\frac{2}{[4\sqrt{\mu_{min}\mu_{max}}+1]}\right)\left(1-\frac{2}{[4\sqrt{\mu_{min}\mu_{max}}+1]}\frac{[16\mu_{min}\mu_{max}-1]}{(4\mu_{min}+4\mu_{max}-2)}\right)}{\left(\frac{2}{[4\sqrt{\mu_{min}\mu_{max}}+1]}\frac{[16\mu_{min}\mu_{max}-1]}{(4\mu_{min}+4\mu_{max}-2)}+1-\frac{2}{[4\sqrt{\mu_{min}\mu_{max}}+1]}\right)}}$$

$$= \sqrt{\frac{16(\mu_{min}\sqrt{\mu_{max}}-\mu_{max}\sqrt{\mu_{min}})^2-(\sqrt{\mu_{min}}-\sqrt{\mu_{max}})^2}{16(\mu_{min}\sqrt{\mu_{max}}+\mu_{max}\sqrt{\mu_{min}})^2-(\sqrt{\mu_{min}}+\sqrt{\mu_{max}})^2}}$$

$$= \sqrt{\frac{(16\mu_{min}\mu_{max}-1)(\sqrt{\mu_{min}}-\sqrt{\mu_{max}})^2}{(16\mu_{min}\mu_{max}-1)(\sqrt{\mu_{min}}+\sqrt{\mu_{max}})^2}}$$

$$= \frac{\sqrt{\mu_{min}}-\sqrt{\mu_{max}}}{\sqrt{\mu_{min}}+\sqrt{\mu_{max}}}$$

Now, summarizing caseswe obtain that the optimal parameters are

$$\omega_{opt} = \frac{2}{[4\sqrt{\mu_{min}\mu_{max}}+1]}, \qquad \alpha_{opt} = \frac{[16\mu_{min}\mu_{max}-1]}{(4\mu_{min}+4\mu_{max}-2)}$$

And the corresponding optimal convergence factor is

$$\rho_{opt} = \frac{\sqrt{\mu_{min}} - \sqrt{\mu_{max}}}{\sqrt{\mu_{min}} + \sqrt{\mu_{max}}}.$$

The conclusion is obtained.

4. NUMERICALEXAMPLE

Consider the augmented system (1) where

$$A = \begin{pmatrix} I \otimes T + T \otimes I & 0 \\ 0 & I \otimes T + T \otimes I \end{pmatrix} \in \mathbb{R}^{2p^2 \times 2p^2}, \qquad B = \begin{pmatrix} I \otimes F \\ F \otimes I \end{pmatrix} \in \mathbb{R}^{2p^2 \times p^2}$$

and

$$T = \frac{1}{h^2} tridiag(-1,2,-1) \in \mathbb{R}^{p \times p}, \qquad F = \frac{1}{h} tridiag(-1,1,0) \in \mathbb{R}^{p \times p}$$

with ⊗ being the Kronecker product symbol and h = $\frac{1}{1+p}$ the discretization mesh size and S =tridiag(a, b, c) is a tridiagonal matrix with s_{ii} = b, $s_{i-1,i}$ = a and $s_{i,i+1}$ = c for the appropriate i. For this example, we set m = $2p^2$and n = p^2. Hence, the total number of variables is m + n = $3p^2$. Here we consider the following three cases when p = 8, p = 16, and p = 24. In our experiments, all runs with respect to the SOR-like method and the proposed MSSOR-like method are started from an initial vector $(x^{(0)t}, y^{(0)t})^t$= 0, and terminated when the current iteration satisfies RES <10^{-9}, where RES = norm$(x^{(k)t} - x^{(0)t})^t, (y^{(k)t} - y^{(0)t})^t$ with $(x^{(k)t}, y^{(k)t})^t$ the final approximate solution. Additionaly, we choose the right hand-side vector $(b^t, q^t)^t \in \mathbb{R}^{m \times n}$such that the exact solution of the augmented system (1) is $((x^*)^t, (y^*)^t)^t = (1,1, \ldots ,1)^t \in \mathbb{R}^{m \times n}$. The number of iterations (denoted by IT) and the RES defined as above are reported in the following tables in order to show the efficiency of the MSSOR-like method.

Table 1. Choice of matrix Q.

Case no.	Matrix Q	Description
I	$B^T\hat{A}^{-1}B$	$\hat{A}$ = tridiag (A)
II	$B^T\hat{A}^{-1}B$	$\hat{A}$ = diag (A)

Table 2. Optimal parameters, IT (no. of iterations) and spectral radius for the given example.

m			**128**	**512**	**1152**
n			**64**	**256**	**576**
m+n			**192**	**768**	**1728**
Case 1	SOR-Like	ω_{opt}	0.5958	0.3657	0.2620
		ρ_{opt}	0.6358	0.7964	0.8591
		IT	62	130	200
	MSSOR	ω_{opt}	0.2375	0.4237	0.5738
		α_{opt}	1.8631	0.5068	0.2136
		ρ_{opt}	0.59402	0.7562	0.6720
		IT	58	110	180

m			128	512	1152
n			64	256	576
m+n			192	768	1728
Case 1	SOR-Like	ω_{opt}	0.4644	0.2720	0.1915
		ρ_{opt}	0.7305	0.8533	0.8992
		IT	92	191	293
	MSSOR	ω_{opt}	0.1715	0.0982	0.0687
		α_{opt}	2.0442	2.0120	2.0054
		ρ_{opt}	0.6756	0.8112	0.8667
		IT	80	149	192

4. CONCLUSIONS

In this paper, we have considered the splitting of the coefficient matrix of the linear augmented system as shown above and solve the linear augmented system by SSOR like method. In the first section, we have introduced the linear augmented system and successive over-relaxation (SOR) method. After the introduction, we have developed the scheme of an iterative method for solving linear augmented systems by the SSOR method. We have also described the convergence criteria of the method. We have shown the convergence properties of this method along with optimal parameters.

For the two different types of matrix Q as shown in Table1, we have drawn Table 2 having a comparison between the SOR and SSOR method with the help of an example. We have calculated number of iterations, the radius of convergence, and optimal parameters of this method as shown in Table 2. Finally, we have concluded that the SSOR-Like method gives better results in comparison to the other existing methods.

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