ORIGINAL PAPER

HOMOTOPY PERTURBATION TRANSFORM METHOD FOR SOLVING SYSTEMS OF NONLINEAR PARTIAL FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. *In this paper, we apply an efficient method called the Homotopy perturbation transform method (HPTM) to solve systems of nonlinear fractional partial differential equations. The HPTM can easily be applied to many problems and is capable of reducing the size of computational work.*

Keywords: *Caputo's fractional derivative; systems of nonlinear fractional partial differential equations; homotopy perturbation transform method; Laplace transform; He's polynomials.*

1. INTRODUCTION

The study of non-linear evolution system of equations have attracted attention of many mathematicians and physicists. Many authors are interested to the research of the exact solutions [1-3], because the exact solutions to non-linear evolution equations are the key tool to understand the various physical phenomena that govern the real world today. Hence, searching for exact traveling wave solutions to nonlinear evolution equations plays an important role in the study of nonlinear physical phenomena in many fields such as fluid dynamics, water wave mechanics, meteorology, electromagnetic theory, plasma physics and nonlinear optics [2, 4].

In this paper, we will study an important nonlinear evolution system of equations [5, 6]. Many researchers have studied the of non-linear evolution system of equations in different contexts.

To investigate the traveling wave solutions, we propose in this work the (HPTM) method (Homotopy perturbation transform method), because it is a powerful technique to search for traveling waves coming out from one-dimensional nonlinear wave and evolution equations. In particular, in those problems where dispersive effects, reaction, diffusion and/or convection play an important role. To show the strength of the method, an overview is given to find out which kind of problems are solved with this technique and how in some nontrivial cases this method, adapted to the problem at hand, still can be applied. Single as well as coupled equations, arising from wave phenomena which appears in different scientific domains such as physics, chemical kinetics, geochemistry and mathematical biology [7-10].

Our goal is to obtained the approximate solutions of the systems of nonlinear partial fractional differential equations, and compare this solution (in particular case) with the

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traveling wave solution of the system of equations to show that the proposed algorithm (HPTM) is suitable for such problems and is very efficient.

Many numerical methods were used in the past to solve fractional systems of nonlinear partial differential equations, such as the fractional Sumudu transform [8, 11], the fractional Adomian decomposition method (FADM) [12, 13], the fractional Adomian decomposition method with Aboodh transform method [6], the new integral transform "AF transform" [14], the Mahgoub Homotopy perturbation Transform Sceme [15] and the fractional homotopy perturbation method (FHPM) [16].

2. BASIC DEFINITIONS

Before the beginning of this research, we are trying in a hurry to get to know the supporting materials to accomplish this work.

Definition 2.1 [17] *Let $f(t)$ be a function of t specified for $t > 0$. Then the Laplace transform of $f(t)$, denoted by $\mathcal{L}\{f(t)\}$, is defined by*

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt, \tag{1}$$

where we assume at present that the parameter s is strictly positive real number.

Theorem 2.2 [17] *If c_1 and c_2 are any constants while $f_1(t)$ and $f_2(t)$ are functions with Laplace* transforms $F_1(s)$ and $F_2(s)$ respectively, then

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = c_1\mathcal{L}\{f_1(t)\} + c_2\mathcal{L}\{f_2(t)\} = c_1F_1(s) + c_2F_2(s). \tag{2}$$

Theorem 2.3 [18] *If*

$$f(t) = \sum_{n=0}^{\infty} a_n t^n \tag{3}$$

converges for $t \geq 0$, with

$$|a_n| \leq \frac{K\alpha^n}{n!}, \tag{4}$$

for all n sufficiently large and $\alpha > 0$, $K > 0$, then

$$\mathcal{L}\{f(t)\} = \sum_{n=0}^{+\infty} a_n \mathcal{L}\{t^n\} = \sum_{n=0}^{+\infty} \frac{a_n n!}{s^{n+1}} (\mathcal{R}e(s) > \alpha). \tag{5}$$

Definition 2.4 [17] *A real function $f(t)$, $t \succ 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$ if there exists a real number $p \succ \mu$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C([0,\infty))$, and it is said to be in the space C_μ^m, $m \in \mathbb{N}$ if $f^{(m)} \in C_\mu$.*

Definition 2.5 [17] *The fractional derivative of $f \in C_{-1}^m$ in the Caputo's sense is defined as:*

$$D_*^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau)d\tau, \text{if} \quad m-1 < \alpha < m \\ \frac{d^m}{dt^m} f(t), \text{if} \alpha = m, \end{cases}$$

where $m \in \mathbb{N}^*$.

Definition 2.6 [17] *The Laplace transform,* $\mathcal{L}\{D_*^{\alpha} f(t); s\}$ *of the Caputo's fractional derivative is* defined as:

$$\mathcal{L}\{D_*^{\alpha} f(t); s\} = s^{\alpha} F(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(0), \tag{6}$$

where $m-1 < \alpha \leq m, m \in \mathbb{N}^*$.

3. ANALYSIS OF HOMOTOPY PERTURBATION TRANSFORM METHOD (HPTM)

To illustrate the basic idea of (HPTM), we consider the general system of nonlinear fractional partial differential equations with the initial conditions of the form

$$\begin{cases} D_{*t}^{\alpha} u(x,t) + R_1 u(x,t) + N_1(u,v) = h_1(x,t), 0 < \alpha \leq 1 \\ D_{*t}^{\beta} v(x,t) + R_2 v(x,t) + N_2(u,v) = h_2(x,t), 0 < \beta \leq 1, \end{cases} \tag{7}$$

subject to the initial conditions

$$\begin{cases} u(x,0) = g_1(x), \\ v(x,0) = g_2(x), \end{cases} \tag{8}$$

where $D_{*t}^{\alpha} u(x,t)$, $D_{*t}^{\beta} v(x,t)$ are the caputo fractional derivatives of the functions $u(x,t)$, $v(x,t)$, respectively, R_1, R_2 are linear operators, N_1 and N_2 are nonlinear operators, and h_1 and h_2 are source terms. Taking the Laplace transform on both sides of (7) to get:

$$\begin{cases} \mathcal{L}\{D_{*t}^{\alpha} u(x,t)\} + \mathcal{L}\{R_1 u(x,t) + N_1(u,v)\} = \mathcal{L}\{h_1(x,t)\}, 0 < \alpha \leq 1 \\ \mathcal{L}\left\{D_{*t}^{\beta} v(x,t)\right\} + \mathcal{L}\{R_2 v(x,t) + N_2(u,v)\} = \mathcal{L}\{h_2(x,t)\}, 0 < \beta \leq 1, \end{cases} \tag{9}$$

Now, using the differentiation property of the Laplace transform, we have:

$$\begin{cases} \mathcal{L}\{u(x,t)\} = s^{-1} g_1(x) + s^{-\alpha} \mathcal{L}\{h_1(x,t)\} - s^{-\alpha} \mathcal{L}\{R_1 u(x,t) + N_1(u,v)\}, 0 < \alpha \leq 1 \\ \mathcal{L}\{v(x,t)\} = s^{-1} g_2(x) + s^{-\beta} \mathcal{L}\{h_2(x,t)\} - s^{-\alpha} \mathcal{L}\{R_2 v(x,t) + N_2(u,v)\}, 0 < \beta \leq 1, \end{cases} \tag{10}$$

Operating the inverse Laplace transform on both sides in (10), we get:

$$\begin{cases} u(x,t) = G_1(x,t) - \mathcal{L}^{-1}\left\{s^{-\alpha} \mathcal{L}\{R_1 u(x,t) + N_1(u,v)\}\right\}, 0 < \alpha \leq 1 \\ v(x,t) = G_2(x,t) - \mathcal{L}^{-1}\left\{s^{-\beta} \mathcal{L}\{R_2 v(x,t) + N_2(u,v)\}\right\}, 0 < \beta \leq 1, \end{cases} \tag{11}$$

Note that $G_1(x,t)$ and $G_2(x,t)$ are arising from the non homogeneous term and the prescribed initial conditions. Now, applying the classical perturbation technique, we can assume that the solution can be expressed as a power series in p, as given below:

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t), \qquad v(x,t) = \sum_{n=0}^{\infty} p^n v_n(x,t) \tag{12}$$

$$N_1(u,v) = \sum_{n=0}^{\infty} p^n A_n, \qquad N_2(u,v) = \sum_{n=0}^{\infty} p^n B_n, \tag{13}$$

where A_n and B_n are He's polynomials representing the nonlinear term $N_1(u,v)$, $N_2(u,v)$, respectively, which are given by:

$$A_n = \frac{1}{n!}\frac{\partial^n}{\partial \lambda^n}\left[N_1\left(\sum_{i=0}^{n} \lambda^i u_i, \sum_{i=0}^{n} \lambda^i v_i\right)\right]_{\lambda=0}, n = 0,1,2,3, \dots \tag{14}$$

$$B_n = \frac{1}{n!}\frac{\partial^n}{\partial \lambda^n}\left[N_2\left(\sum_{i=0}^{n} \lambda^i u_i, \sum_{i=0}^{n} \lambda^i v_i\right)\right]_{\lambda=0}, n = 0,1,2,3, \dots\ . \tag{15}$$

Some other approaches to obtain He's polynomials can be found in [19].

Substituting (12) and (13) in (11) and using HPM by He see ([20, 21]), we get:

$$\begin{cases} \sum_{n=0}^{\infty} p^n u_n(x,t) = G_1(x,t) - \mathcal{L}^{-1}\left\{s^{-\alpha}\mathcal{L}\{R_1 \sum_{n=0}^{\infty} p^n u_n(x,t) + \sum_{n=0}^{\infty} p^n A_n\}\right\}, 0 < \alpha \le 1 \\ \sum_{n=0}^{\infty} p^n v_n(x,t) = G_2(x,t) - \mathcal{L}^{-1}\left\{s^{-\beta}\mathcal{L}\{R_2 \sum_{n=0}^{\infty} p^n v_n(x,t) + \sum_{n=0}^{\infty} p^n B_n\}\right\}, 0 < \beta \le 1, \end{cases} \tag{16}$$

Now, equating the coefficient of corresponding power of p on both sides, the following approximations are obtained as:

p^0:

$$\begin{cases} u_0(x,t) = G_1(x,t) \\ v_0(x,t) = G_2(x,t), \end{cases} \tag{17}$$

p^n:

$$\begin{cases} u_n(x,t) = -\mathcal{L}^{-1}\left\{s^{-\alpha}\mathcal{L}\{R_1 u_{n-1}(x,t) + A_{n-1}\}\right\}, \\ v_n(x,t) = -\mathcal{L}^{-1}\left\{s^{-\beta}\mathcal{L}\{R_2 v_{n-1}(x,t) + B_{n-1}\}\right\}, \end{cases} \tag{18}$$

where $n \in \mathbb{N}^*$.

Finally, the approximate solutions are given by

$$\begin{cases} u(x,t) = \lim_{N\to+\infty} \sum_{n=0}^{N} u_n(x,t), \\ v(x,t) = \lim_{N\to+\infty} \sum_{n=0}^{N} v_n(x,t), \end{cases} \tag{19}$$

The above series solutions generally converge very rapidly. A classical approach of convergence of this type of series is already presented by Abbaoui and Charruault [22].

4. NUMERICAL EXAMPLES

Example 4.1. Consider the time-fractional nonlinear coupled Burgers' system of equations:

$$\begin{cases} D_{*t}^{\alpha} u(x,t) = u_{xx} + 2uu_x - (uv)_x, 0 < \alpha \le 1 \\ D_{*t}^{\beta} v(x,t) = v_{xx} + 2vv_x - (uv)_x, 0 < \beta \le 1, \end{cases} \tag{20}$$

subject to the initial conditions

$$\begin{cases} u(x,0) = cosx, \\ v(x,0) = cosx. \end{cases} \tag{21}$$

For $\alpha = \beta = 1$, the exact solution of (20) is given by

$$\begin{cases} u(x,t) = e^{-t}cosx, \\ v(x,t) = e^{-t}cosx. \end{cases} \tag{22}$$

Taking the Laplace transform on both sides of (20), we get:

$$\begin{cases} \mathcal{L}\{D_{*t}^{\alpha}u(x,t)\} = \mathcal{L}\{u_{xx} + 2uu_x - (uv)_x\}, \\ \mathcal{L}\left\{D_{*t}^{\beta}v(x,t)\right\} = \mathcal{L}\{v_{xx} + 2vv_x - (uv)_x\}. \end{cases} \tag{23}$$

An application of Eq. (6), yields:

$$\begin{cases} \mathcal{L}\{u(x,t)\} = s^{-1}cosx + s^{-\alpha}\mathcal{L}\{u_{xx} + 2uu_x - (uv)_x\}, \\ \mathcal{L}\{v(x,t)\} = s^{-1}cosx + s^{-\beta}\mathcal{L}\{v_{xx} + 2vv_x - (uv)_x\}. \end{cases} \tag{24}$$

Applying the inverse Laplace transform on both sides of (24), we get:

$$\begin{cases} u(x,t) = cosx + \mathcal{L}^{-1}\{s^{-\alpha}\mathcal{L}\{u_{xx} + 2uu_x - (uv)_x\}\}, \\ v(x,t) = cosx + \mathcal{L}^{-1}\left\{s^{-\beta}\mathcal{L}\{v_{xx} + 2vv_x - (uv)_x\}\right\}. \end{cases} \tag{25}$$

Now, applying the classical perturbation technique, we can assume that the solution can be expressed as a power series in p, as given below:

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t), \qquad v(x,t) = \sum_{n=0}^{\infty} p^n v_n(x,t), \tag{26}$$

$$uu_x = \sum_{n=0}^{\infty} p^n, \quad vv_x = \sum_{n=0}^{\infty} p^n B_n, \quad (uv)_x = \sum_{n=0}^{\infty} p^n C_n, \tag{27}$$

where A_n, B_n, C_n are the Adomian polynomials that represent the nonlinear terms. By applying the aforesaid homotopy perturbation method, we have:

$$\begin{cases} \sum_{n=0}^{\infty} p^n u_n(x,t) = cosx + \mathcal{L}^{-1}\{s^{-\alpha}\mathcal{L}\{\sum_{n=0}^{\infty} p^n (u_n)_{xx} + 2\sum_{n=0}^{\infty} p^n A_n - \sum_{n=0}^{\infty} p^n C_n\}\}, \\ \sum_{n=0}^{\infty} p^n v_n(x,t) = cosx + \mathcal{L}^{-1}\left\{s^{-\beta}\mathcal{L}\{\sum_{n=0}^{\infty} p^n (v_n)_{xx} + 2\sum_{n=0}^{\infty} p^n B_n - \sum_{n=0}^{\infty} p^n C_n\}\right\}. \end{cases} \tag{28}$$

Equating the coefficient of the like power of p on both sides in (28), we get:
p^0:

$$\begin{cases} u_0(x,t) = cosx \\ v_0(x,t) = cosx, \end{cases} \tag{29}$$

p^1:

$$\begin{cases} u_1(x,t) = \mathcal{L}^{-1}\{s^{-\alpha}\mathcal{L}\{u_{0xx} + 2A_0 - C_0\}\}, \\ v_1(x,t) = \mathcal{L}^{-1}\left\{s^{-\beta}\mathcal{L}\{v_{0xx} + 2B_0 - C_0\}\right\}, \end{cases} \tag{30}$$

$$\begin{cases} u_1(x,t) = \mathcal{L}^{-1}\{s^{-\alpha}\mathcal{L}\{-cosx + 2u_{0x}u_0 - u_0v_{0x} - u_{0x}v_0\}\} = -cosx\frac{t^\alpha}{\Gamma(\alpha+1)}, \\ v_1(x,t) = \mathcal{L}^{-1}\left\{s^{-\beta}\mathcal{L}\{-cosx + 2v_{0x}v_0 - u_0v_{0x} - u_{0x}v_0\}\right\} = -cosx\frac{t^\beta}{\Gamma(\beta+1)}, \end{cases} \quad (31)$$

we continue to get

p^2:

$$\begin{cases} u_2(x,t) = \mathcal{L}^{-1}\{s^{-\alpha}\mathcal{L}\{u_{1xx} + 2A_1 - C_1\}\}, \\ v_2(x,t) = \mathcal{L}^{-1}\left\{s^{-\beta}\mathcal{L}\{v_{1xx} + 2B_1 - C_1\}\right\}, \end{cases} \quad (32)$$

$$\begin{cases} u_2(x,t) = \mathcal{L}^{-1}\left\{s^{-\alpha}\mathcal{L}\left\{\frac{t^\alpha}{\Gamma(\alpha+1)}cosx + 2u_{0x}u_1 + 2u_0u_{1x} - (u_1v_0 + u_0v_1)_x\right\}\right\}, \\ v_2(x,t) = \mathcal{L}^{-1}\left\{s^{-\alpha}\mathcal{L}\left\{\frac{t^\beta}{\Gamma(\beta+1)}cosx + 2v_{0x}v_1 + 2v_0v_{1x} - (u_1v_0 + u_0v_1)_x\right\}\right\}, \end{cases} \quad (33)$$

$$\begin{cases} u_2(x,t) = cosx\left(\frac{(1+2sinx)}{\Gamma(2\alpha+1)}t^{2\alpha} - \frac{2sinx}{\Gamma(\alpha+\beta+1)}t^{\alpha+\beta}\right), \\ v_2(x,t) = cosx\left(\frac{(1+2sinx)}{\Gamma(2\beta+1)}t^{2\beta} - \frac{2sinx}{\Gamma(\alpha+\beta+1)}t^{\alpha+\beta}\right). \end{cases} \quad (34)$$

Finally, the series solution of the unknown functions $u(x,t)$ and $v(x,t)$ of (20) are given by

$$\begin{cases} u(x,t) = \sum_{n=0}^{+\infty} u_n(x,t) = cosx - cosx\frac{t^\alpha}{\Gamma(\alpha+1)} + cosx\left(\frac{(1+2sinx)}{\Gamma(2\alpha+1)}t^{2\alpha} - \frac{2sinx}{\Gamma(\alpha+\beta+1)}t^{\alpha+\beta}\right) + \cdots, \\ v(x,t) = \sum_{n=0}^{+\infty} v_n(x,t) = cosx - cosx\frac{t^\beta}{\Gamma(\beta+1)} + cosx\left(\frac{(1+2sinx)}{\Gamma(2\beta+1)}t^{2\beta} - \frac{2sinx}{\Gamma(\alpha+\beta+1)}t^{\alpha+\beta}\right) + \cdots. \end{cases} \quad (35)$$

When $\alpha = 1$ and $\beta = 1$, the serie solutions of (20) are

$$\begin{cases} u(x,t) = \sum_{n=0}^{+\infty} u_n(x,t) = cosx(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots) = e^{-t}cosx, \\ v(x,t) = \sum_{n=0}^{+\infty} v_n(x,t) = cosx(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots) = e^{-t}cosx, \end{cases} \quad (36)$$

which is an exact solution of nonlinear system given in (22).

We plotted these surfaces by using Maple software:

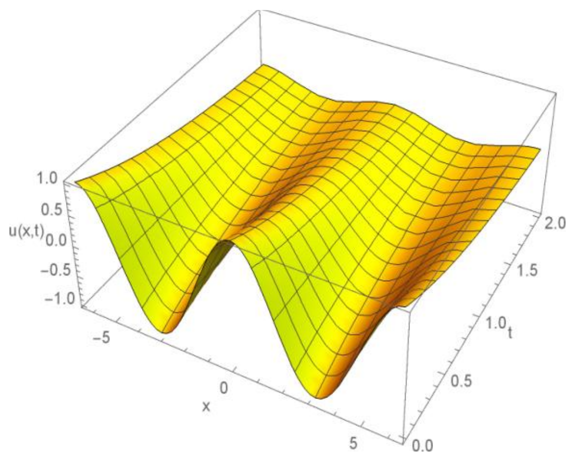

Figure 1. Exact solution of nonlinear system given in (20), when α = β = 1.

Example 4.2. *Consider the system of nonlinear partial differential equations with time-fractional derivatives:*

$$\begin{cases} D_{*t}^{\alpha}u(x,y,t) = -u - v_x w_y + v_y w_x, 0 < \alpha \leq 1 \\ D_{*t}^{\beta}v(x,y,t) = v, 0 < \beta \leq 1 \\ D_{*t}^{\gamma}w(x,y,t) = w - u_x v_y - u_y v_x, 0 < \gamma \leq 1, \end{cases} \tag{37}$$

subject to the initial conditions

$$\begin{cases} u(x,y,0) = e^{x+y}, \\ v(x,y,0) = e^{x-y}, \\ w(x,y,0) = e^{y-x}. \end{cases} \tag{38}$$

For $\alpha = \beta = \gamma = 1$, the exact solution of (37) is given by

$$\begin{cases} u(x,t) = e^{x+y-t}, \\ v(x,t) = e^{x-y+t}, \\ w(x,t) = e^{y-x+t}. \end{cases} \tag{39}$$

Taking the Laplace transform on both sides of Eq. (37), we get:

$$\begin{cases} \mathcal{L}\{D_{*t}^{\alpha}u(x,y,t)\} = \mathcal{L}\{-u - v_x w_y + v_y w_x\}, \\ \mathcal{L}\{D_{*t}^{\beta}v(x,y,t)\} = \mathcal{L}\{v\}, \\ \mathcal{L}\{D_{*t}^{\gamma}w(x,y,t)\} = \mathcal{L}\{w - u_x v_y - u_y v_x\}. \end{cases} \tag{40}$$

An application of Eq. (6), yields:

$$\begin{cases} \mathcal{L}\{u(x,y,t)\} = s^{-1}e^{x+y} + s^{-\alpha}\mathcal{L}\{-u - v_x w_y + v_y w_x\}, \\ \mathcal{L}\{v(x,y,t)\} = s^{-1}e^{x-y} + s^{-\beta}\mathcal{L}\{v\}, \\ \mathcal{L}\{w(x,y,t)\} = s^{-1}e^{y-x} + s^{-\gamma}\mathcal{L}\{w - u_x v_y - u_y v_x\}. \end{cases} \tag{41}$$

Applying the inverse Laplace transform of both sides in Eq. (41), we get:

$$\begin{cases} u(x,y,t) = e^{x+y} + \mathcal{L}^{-1}\left\{s^{-\alpha}\mathcal{L}\{-u - v_x w_y + v_y w_x\}\right\}, \\ v(x,y,t) = e^{x-y} + \mathcal{L}^{-1}\left\{s^{-\beta}\mathcal{L}\{v\}\right\}, \\ w(x,y,t) = e^{y-x} + \mathcal{L}^{-1}\left\{s^{-\gamma}\mathcal{L}\{w - u_x v_y - u_y v_x\}\right\}. \end{cases} \tag{42}$$

Now, applying the classical perturbation technique, we can assume that the solution can be expressed as a power series in p, as given below:

$$u(x,y,t) = \sum_{n=0}^{\infty} p^n u_n(x,y,t),\ v(x,y,t) = \sum_{n=0}^{\infty} p^n v_n(x,y,t),\ w(x,y,t) = \sum_{n=0}^{\infty} p^n w_n(x,y,t) \tag{43}$$

$$v_x w_y = \sum_{n=0}^{\infty} p^n A_n,\ v_y w_x = \sum_{n=0}^{\infty} p^n B_n, u_x v_y = \sum_{n=0}^{\infty} p^n C_n,\ u_y v_x = \sum_{n=0}^{\infty} p^n D_n, \tag{44}$$

where A_n, B_n, C_n,D_n are the Adomian polynomials that represent the nonlinear terms.

By applying the aforesaid homotopy perturbation method, we have:

$$\begin{cases} \sum_{n=0}^{\infty} p^n u_n(x,y,t) = e^{x+y} + \mathcal{L}^{-1}\{s^{-\alpha}\mathcal{L}\{-\sum_{n=0}^{\infty} p^n u_n(x,y,t) - \sum_{n=0}^{\infty} p^n A_n + \sum_{n=0}^{\infty} p^n B_n\}\}, \\ \sum_{n=0}^{\infty} p^n v_n(x,y,t) = e^{x-y} + \mathcal{L}^{-1}\left\{s^{-\beta}\mathcal{L}\{\sum_{n=0}^{\infty} p^n v_n(x,y,t)\}\right\}, \\ \sum_{n=0}^{\infty} p^n w_n(x,y,t) = e^{y-x} + \mathcal{L}^{-1}\{s^{-\gamma}\mathcal{L}\{\sum_{n=0}^{\infty} p^n w_n(x,y,t) - \sum_{n=0}^{\infty} p^n C_n - \sum_{n=0}^{\infty} p^n D_n\}\}. \end{cases} \tag{45}$$

Equating the coefficient of the like power of p on both sides in Eq. (45), we get:

p^0:

$$\begin{cases} u_0(x,y,t) = e^{x+y} \\ v_0(x,y,t) = e^{x-y} \\ v_0(x,y,t) = e^{y-x} \end{cases} \tag{46}$$

p^1:

$$\begin{cases} u_1(x,y,t) = \mathcal{L}^{-1}\{s^{-\alpha}\mathcal{L}\{-u_0 - A_0 + B_0\}\}, \\ v_1(x,y,t) = \mathcal{L}^{-1}\left\{s^{-\beta}\mathcal{L}\{v_0\}\right\}, \\ w_1(x,y,t) = \mathcal{L}^{-1}\{s^{-\gamma}\mathcal{L}\{w_0 - C_0 - D_0\}\} \end{cases} \tag{47}$$

$$\begin{cases} u_1(x,y,t) = \mathcal{L}^{-1}\left\{s^{-\alpha}\mathcal{L}\{-e^{x+y} - v_{0x}w_{0y} + v_{0y}w_{0x}\}\right\} = -e^{x+y}\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \\ v_1(x,y,t) = \mathcal{L}^{-1}\left\{s^{-\beta}\mathcal{L}\{e^{x-y}\}\right\} = e^{x-y}\frac{t^{\beta}}{\Gamma(\beta+1)}, \\ w_1(x,y,t) = \mathcal{L}^{-1}\left\{s^{-\gamma}\mathcal{L}\{e^{y-x} - u_{0x}v_{0y} - u_{0y}v_{0x}\}\right\} = e^{y-x}\frac{t^{\gamma}}{\Gamma(\gamma+1)}, \end{cases} \tag{48}$$

we continue to get

p^2:

$$\begin{cases} u_2(x,y,t) = \mathcal{L}^{-1}\{s^{-\alpha}\mathcal{L}\{-u_1 - A_1 + B_1\}\}, \\ v_2(x,y,t) = \mathcal{L}^{-1}\left\{s^{-\beta}\mathcal{L}\{v_1\}\right\}, \\ w_2(x,y,t) = \mathcal{L}^{-1}\{s^{-\gamma}\mathcal{L}\{w_1 - C_1 - D_1\}\} \end{cases} \tag{49}$$

$$\begin{cases} u_2(x,y,t) = \mathcal{L}^{-1}\left\{s^{-\alpha}\mathcal{L}\left\{e^{x+y}\frac{t^{\alpha}}{\Gamma(\alpha+1)} - v_{0x}w_{1y} - v_{1x}w_{0y} + v_{0y}w_{1x} + v_{1y}w_{0x}\right\}\right\} = e^{x+y}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ v_2(x,y,t) = \mathcal{L}^{-1}\left\{s^{-\beta}\mathcal{L}\left\{e^{x-y}\frac{t^{\beta}}{\Gamma(\beta+1)}\right\}\right\} = e^{x-y}\frac{t^{2\beta}}{\Gamma(2\beta+1)}, \\ w_2(x,y,t) = \mathcal{L}^{-1}\left\{s^{-\gamma}\mathcal{L}\left\{e^{y-x}\frac{t^{\gamma}}{\Gamma(\gamma+1)} - u_{1x}v_{0y} - u_{0x}v_{1y} - v_{0x}u_{1y} - v_{1x}u_{0y}\right\}\right\} = e^{y-x}\frac{t^{2\gamma}}{\Gamma(2\gamma+1)}, \end{cases} \tag{50}$$

we continue to get

p^n:

$$\begin{cases} u_n(x,y,t) = (-1)^n e^{x+y}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \\ v_n(x,y,t) = e^{x-y}\frac{t^{n\beta}}{\Gamma(n\beta+1)}, \\ w_n(x,y,t) = e^{y-x}\frac{t^{n\gamma}}{\Gamma(n\gamma+1)}, \end{cases} \tag{51}$$

where $n \in \mathbb{N}^*$.

Finally, the series solution of the unknown functions $u(x,t)$ and $v(x,t)$ of (37) are given by

$$\begin{cases} u(x,y,t) = \sum_{n=0}^{+\infty} u_n(x,y,t) = e^{x+y} - e^{x+y}\frac{t^\alpha}{\Gamma(\alpha+1)} + e^{x+y}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots + (-1)^n e^{x+y}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)} + \cdots, \\ v(x,y,t) = \sum_{n=0}^{+\infty} v_n(x,y,t) = e^{x-y} + e^{x-y}\frac{t^\beta}{\Gamma(\beta+1)} + e^{x-y}\frac{t^{2\beta}}{\Gamma(2\beta+1)} + \cdots + e^{x-y}\frac{t^{n\beta}}{\Gamma(n\beta+1)} + \cdots, \\ w(x,y,t) = \sum_{n=0}^{+\infty} w_n(x,y,t) = e^{y-x} + e^{y-x}\frac{t^\gamma}{\Gamma(\gamma+1)} + e^{y-x}\frac{t^{2\gamma}}{\Gamma(2\gamma+1)} + \cdots + e^{y-x}\frac{t^{n\gamma}}{\Gamma(n\gamma+1)} + \cdots, \end{cases} \tag{52}$$

$$\begin{cases} u(x,y,t) = \sum_{n=0}^{+\infty} (-1)^n e^{x+y}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \\ v(x,y,t) = \sum_{n=0}^{+\infty} e^{x-y}\frac{t^{n\beta}}{\Gamma(n\beta+1)} \\ w(x,y,t) = \sum_{n=0}^{+\infty} e^{y-x}\frac{t^{n\gamma}}{\Gamma(n\gamma+1)}. \end{cases} \tag{53}$$

When $\alpha = \beta = \gamma = 1$, the serie solutions of (37) are

$$\begin{cases} u(x,y,t) = \sum_{n=0}^{+\infty} u_n(x,y,t) = e^{x+y}\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right) = e^{x+y-t}, \\ v(x,y,t) = \sum_{n=0}^{+\infty} v_n(x,y,t) = e^{x-y}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) = e^{x-y+t}, \\ w(x,y,t) = \sum_{n=0}^{+\infty} w_n(x,y,t) = e^{y-x}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) = e^{y-x+t}, \end{cases} \tag{54}$$

which is an exact solution of nonlinear system given in (39).

We plotted these surfaces by using Maple software:

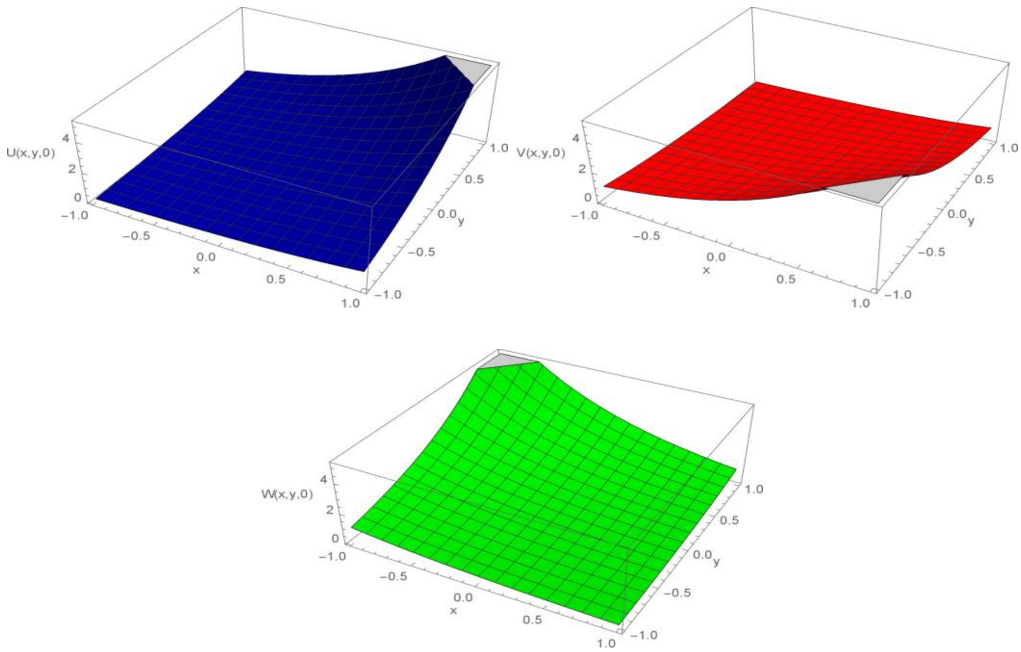

Figure 2. Exact solution of nonlinear system given in (37), when $\alpha = \beta = \gamma = 1$.

4. CONCLUSION

In this paper, it can be seen that the coupling of homotopy perturbation method (HPM) and the Laplace transform, proved very effective to solve certain type the system of partial and fractional partial differential equations. The proposed algorithm (HPTM) is suitable for such problems and is very user friendly. The advantage of this method is its ability to obtain exact solutions the system of partial and fractional partial differential equations. The result obtained in the examples presented shows that this modified method is very powerful and efficient technique in finding exact solutions for wide classes of problems.

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REFERENCES

[1] Fan, E.G,. *Chaos, Solitons and Fractals*, **16**, 819, 2003.
[2] Khan, K., Ali Akbar, M., Arnous, A. H., *Springer Plus*, **5**, 663, 2016.
[3] Qingling, G., *Applied Mathematical Sciences*, **5**(25), 1209, 2011.
[4] Fan, E.G., *Computers and Mathematics Applications*, **43**, 671, 2002.
[5] Spiegel, M.R., *Laplace Tranforms*, McGRAW-HILL, New York, 1965.
[6] Cherif, M.H., Ziane, D. M., *International Journal of Analysis and Applications*, **15**(2), 188, 2017.
[7] Helal, M.A., Mehanna, M.S., *Applied Mathematics and Computation*, **190**, 599, 2007.
[8] Katatbeh, Q.D., Belgacem, F.B., *Nonlinear Studies*, **18**(1), 99, 2011.
[9] Malfliet, W., *Mathematical Methods in the Applied Sciences*, **28,** 2031, 2005.
[10] Wazwaz, A.M., *Applied Mathematics and Computation*, **110**, 251, 2000.
[11] Kumar, D., Singh, J., Kiliçman, A., *Abstract and Applied Analysis*, **2013**, 608943, 2013.
[12] Garg, M.R., Sharma, A.J., *Journal of Inequalities and Special Functions*, **2**(1), 1, 2011.
[13] Ray, S.S., Bera, R.K., *Applied Mathematics and Computation*, **167**, 561, 2005.
[14] Pourreza Ahmadi, S.A., Pourreza Ahmadi, S.F., *Tamap Journal of Mathematics and Statistics*, **2020**, SI-TAMAP-SAP002, 2020.
[15] Khandelwal, Y., Chanchal, P., Khandelwal, R., *Advances in Mathematics: Scientific Journal*, **9**(4), 1983, 2020.
[16] Wang, Q., *Chaos Solitons Fractals*, **35**, 843, 2008.
[17] Kehaili, A., Hakem, A., Benali, A., *Global Journal of Pure Applied Sciences*, **26**, 35, 2020.
[18] Schiff, J.L., *The Laplace Tranform, Theory and Applications*, Springer-Verlag, New York, 1999.
[19] Ghorbani, A., *Choas Solitons Fractals*, **39,** 1486, 2009.
[20] He, J.H., *Computer Methods in Applied Mechanics and Engineering*, **178**, 257, 1999.
[21] He, J.H., *Journal of Sound and Vibration*, **229**, 1257, 2000.
[22] Abbaoui, K., Cherruault, Y., *Computers & Mathematics with Applications*, **29**, 103, 1995.