ORIGINAL PAPER

POST QUANTUM-HERMITE-HADAMARD INEQUALITIES FOR DIFFERENTIABLE CONVEX FUNCTIONS WITH A CRITICAL POINT

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Manuscript received: 01.04.2020; Accepted paper: 02.05.2021.
Published online: 30.06.2021.

Abstract. *In this study, we obtained some new post quantum-Hermite-Hadamard inequalities for differentiable convex function with critical point by using generalized (p, q)-Hermite-Hadamard Inequality. The perseverance of this article is to establish different results on the left-hand side of (p, q)-Hermite-Hadamard inequality for differentiable convex function along with critical point. Special cases were obtained for different (p, q)-Hermite Hadamard inequalies with the critical point c for some special values of q.*

Keywords: (p, q)*-derivative;* (p, q)*-integerals; Hermit–Hadamard's inequality; critical point; convex functions.*

INTRODUCTION

1.1. PRESENT STATE OF CRITICAL POINT AND CONVEXITY

Critical point is an extensive term used in various branches of mathematics, but is always connected to the derivative of a function or mapping.

Critical point deals with functions in different ascepts are defined as:

For functions of a real variabale, a critical point is defined as a point in the domain of function where the function is either not differentiable or the derivative is equal to zero.

When dealing with complex variables, a critical point is a point in the domain of a function where it is either not holomorphic or the derivative is equal to zero.

Likewise, for a function of several real variables, a critical point is a value in its domain where the gradient is undefined or is equal to zero. The value of function at a critical point is a critical value.

Convex functions are essential and provide a basis for constructing literature on mathematical inequalities. A function $f: I \dot{\subseteq} \rightarrow \mathbb{R}$ is said to be convex function on I, if

$$f\big((1-t)x + ty\big) \le (1-t)f(x) + tf(y) \ \ \forall\, x, y \in I, t \in [0,1].$$

In recent years much attention has been given in studying numerous aspects of convex functions. This concept has been extended and generalized in different directions, see [1-8].

A large number of inequalities were obtained by means of convex functions see [9-12]. A classical inequality for convex functions is the Hermite–Hadamard inequality, and is given as follows:

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$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

where $f: I \dot{\subseteq} \mathbb{R} \rightarrow \mathbb{R}$ is a convex function and $a, b \in I$ with (see [13]).

1.2. INSPIRTATION OF QUANTUM ASSESSMENTS

Quantum calculus or q −calculus is a approach pertinent to the classic study of calculus but it is mainly arranged on the idea of derivation of q -analogous results without the use of limits. Post-quantum or $(p, q) -$ calculus is a generalization of q −calculus and it is the next stage ahead of the q-calculus. The idea of q calculus was first introduced by Euler who started his study in the earlier years of the 18th century. In q −calculus, the classical derivative is replaced by the q -difference operator in order to deal with non-differentiable functions; see [14-15] for more details. In recent years, the topic of q −calculus has attracted the attention of several scholars. Therefore q -calculus bridges a connection between mathematics and physics Applications of q −calculus can be found in various fields of mathematics and physics, and the interested readers are referred to [16–19].

In 2014, Tariboon and Ntouyas [20] investigated the q −analogue of Hermite-Hadamard's inequality, that is

$$f(\frac{qa+b}{1+q}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{qf(a)+f(b)}{1+q}.$$

In 2018, Alp et al [21] proved the $(p.q)$ −Hermite –Hadamard inequality.

$$f(\frac{qa+pb}{p+q}) \leq \frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x)dx \leq \frac{qf(a)+pf(b)}{p+q}.$$

Moreover, they studied the generalized $(p.q)$ −Hermite–Hadamard inequality for differentiable convex functions, that is,

$$max\{I_1, I_2, I_3\} \leq \frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x) \quad {}_a d_{p,q}x \leq \frac{qf(a)+pf(b)}{p+q}.$$

where

$$I_1 = f\left(\frac{qa+pb}{p+q}\right)$$

$$I_2 = f\left(\frac{pa+qb}{p+q}\right) + \frac{(p-q)(b-a)}{p+q} f'\left(\frac{pa+qb)}{p+q}\right)$$

$$I_3 = f\left(\frac{a+b}{2}\right) + \frac{(p-q)(b-a)}{2(p+q)} f'\left(\frac{a+b)}{2}\right)$$

2. PRELIMINARIES & DEFINATIONS OF (p, q) − CALCULUS:

Throughout this paper, we let f = [a, b] ⊂ ℝ be an interval and $0 < q < p \leq 1$ be a constant. The definations for (p, q)- derivative and (p, q)- integeral were given in [22-23].

Definition 2.1. Let f : [a; b] → ℝ be a continuous function. The (p, q) −derivative of f at t ∈[a,b] is characterized by the expression

$$ {}_aD_{p,q} f\,(t)=\frac{f(pt+(1-p)a)-f(qt+(1-q)a)}{(p-q)(t-a)}\,,t\neq a. \tag{2.1}$$

Since f:[a,b]→ ℝ is a continuous function, thus we have

$$ {}_aD_{p,q} f\,(a)=\lim_{t\to a} {}_aD_{p,q} f(t)\,. $$

The function f is said to be (p,q)-differentiable on [a,b].
${}_aD_{p,q} f$ (t) exists for all t ∈[a,b]. If a=0 in (2.1), then ${}_0D_{p,q} f$ (t)= $D_{p,q} f$ (t) is familiar (p ,q) derivative of f at t ∈[a,b] defined by the expression

$$ D_{p,q} f\,(t)=\frac{f(pt)-f(qt)}{(p-q)t}\,, \qquad t\neq 0 \tag{2.2}$$

Note also that if p = 1 in (2.2), then Dq f (x) is familiar q-derivative of f at x ∈ [a,b] defined by the expression

$$ D_q f(t) = \frac{f(t)-f(qt)}{(1-q)t}\,, \qquad t\neq 0 \tag{2.3}$$

Definition 2.2 Let f : [a; b] → ℝ be a continuous function. The definite (p, q) integral on $[a, b]$ is defined as:

$$ \int_a^t f(x)_a\, d_{(p,q)}x = (p-q)(t-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}} f\left\{\frac{q^n}{p^{n+1}}t + \left(1-\frac{q^n}{p^{n+1}}\right)a\right\} \tag{2.4}$$

For t ∈ [*a, b*]. If *c* ∈ *(a, t)*, then the (p, q)- definite integral on [*c, t*] is expressed as

$$ \int_c^t f(x)_a\, d_{(p,q)}x = \int_a^t f(x)_a\, d_{(p,q)}x - \int_a^c f(x)_a\, d_{(p,q)}x \tag{2.5}$$

If p = 1 in (2.4), then one can get the classical q-definite integral on $[a, b]$ defined by

$$ \int_a^t f(x)_a\, d_{(1,q)}x = (1-q)(t-a)\sum_{n=0}^{\infty} q^n f\{q^n t + (1-q^n)a\} \tag{2.6}$$

If *a* = 0 in (2.4), then one can get the classical (p, q)-definite integral defined by

$$ \int_0^t f(x)_0\, d_{(p,q)}x = (p-q)t\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\, f\,\left(\frac{q^n}{p^{n+1}}t\right). $$

The proofs of the following theorems were given in [24].

Theorem 2.1. Suppose that f:[a,b]→ ℝ is a differentiable convex function on (a,b) such that $f'(c) = 0$ for $c \in (a,b)$ and let q be a constant with $0 < q < 1$.Then we have

$$f\left(\frac{q(a+c)+(1-q)b}{p+q}\right)+f'\left(\frac{q(a+c)+(1-q)b}{1+q}\right)\left(\frac{q(b-c)}{1+q}\right)$$
$$\leq \frac{1}{(b-a)}\int_a^b f(x) \quad {}_a d_q x \leq \frac{qf(a)+f(b)}{1+q}.$$

Theorem 2.2. let f:[a,b]→ ℝ is a differentiable convex function on (a,b) such that $f'(c) = 0$ for $c \in (a,b)$ and let q be a constant with $0 < q < 1$.Then we have

$$f\left(\frac{(1-q)a+q(c+b)}{p+q}\right)+f'\left(\frac{(1-q)a+q(c+b)}{p+q}\right)\left(\frac{q(2a-b-c)+(b-a)}{1+q}\right)$$
$$\leq \frac{1}{(b-a)}\int_a^b f(x) \quad {}_a d_q x \leq \frac{qf(a)+f(b)}{1+q}$$

Theorem 2.3. [Generalized q-Hermite-Hadamard Inequality for convex differentiable functions]
Let $f: [a,b] \to \mathbb{R}$ be a differentiable convex function on (a,b) such that $f'(c) = 0$ for $c \in (a,b)$ and $0 < q < 1$, Then we have

$$max\{I_1, I_2\} \leq \frac{1}{(b-a)}\int_a^b f(x) \quad {}_a d_q x \leq \frac{qf(a)+f(b)}{1+q}.$$

where

$$I_1 = f\left(\frac{q(a+c)+(1-q)b}{1+q}\right)+f'\left(\frac{q(a+c)+(1-q)b}{1+q}\right)\left(\frac{q(b-c)}{1+q}\right),$$

$$I_2 = f\left(\frac{(1-q)a+q(c+b)}{1+q}\right)+f'\left(\frac{(1-q)a+q(c+b)}{1+q}\right)\left(\frac{q(2a-b-c)+(b-a)}{1+q}\right)$$

3. MAIN OUTCOMES

In this section, we present some new post quantum Hermite –Hadamard integeral inequalities for differentiable convex functions along with critical point and also present some examples which satisfied main outcomes.

Theorem 3.1. Suppose that f:[a,b]→ ℝ is a differentiable convex function on (a,b) such that $f'(c) = 0$ for $c \in (a,b)$ and let q be a constant with $0 < q < p \leq 1$.Then we have

$$f\left(\frac{q(a+c)+(p-q)b}{p+q}\right)+f'\left(\frac{q(a+c)+(p-q)b}{p+q}\right)\left(\frac{q(b-c)}{p+q}\right)$$
$$\leq \frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x) \quad {}_a d_{p,q} x \leq \frac{qf(a)+pf(b)}{p+q} \tag{3.1}$$

Proof:

Since the function f is differentiable on (a,b), there exist a tangent line at the point $\frac{q(a+c)+(p-q)b}{p+q} \in (a,b)$, given by

$$h(x) = f\left(\frac{q(a+c)+(p-q)b}{p+q}\right) + f'\left(\frac{q(a+c)+(p-q)b}{p+q}\right)\left(x - \frac{q(a+c)+(p-q)b}{p+q}\right). \tag{3.2}$$

Since f is convex function on [a, b], it follows that $h(x) \le f(x)$ for all $x \in [a,b]$. Applying (p,q)- integerating of eq (3.2) on $[a,b]$, we have

$$\int_{\boldsymbol{a}}^{\boldsymbol{pb+(1-p)a}} h(x))_a d_{p,q}x$$
$$= \int_{\boldsymbol{a}}^{\boldsymbol{pb+(1-p)a}} \left[f\left(\frac{q(a+c)+(p-q)b}{p+q}\right) + f'\left(\frac{q(a+c)+(p-q)b}{p+q}\right)\left(x - \frac{q(a+c)+(p-q)b}{p+q}\right)\right] {}_a d_{p,q}x$$
$$= p(b-a) f\left(\frac{q(a+c)+(p-q)b}{p+q}\right) + f'\left(\frac{q(a+c)+(p-q)b}{p+q}\right)\left(x - \frac{q(a+c)+(p-q)b}{p+q}\right)\left(\int_{\boldsymbol{a}}^{\boldsymbol{pb+(1-p)a}} \boldsymbol{x} \;{}_a d_{p,q}x - p(b-a)\frac{q(a+c)+(p-q)b}{p+q}\right)$$
$$= p(b-a) f\left(\frac{q(a+c)+(p-q)b}{p+q}\right) + f'\left(\frac{q(a+c)+(p-q)b}{p+q}\right)\left(p(b-a)\frac{(qa+pb)}{p+q} - p(b-a)\frac{q(a+c)+(p-q)b}{p+q}\right)$$
$$= p(b-a)\left[f\left(\frac{q(a+c)+(p-q)b}{p+q}\right) + f'\left(\frac{q(a+c)+(p-q)b}{p+q}\right)\left(\frac{q(b-c)}{p+q}\right)\right]$$
$$\le \int_{\boldsymbol{a}}^{\boldsymbol{pb+(1-p)a}} f(x))_a d_{p,q}x$$

On the other hand, meanwhile f is convex function, we obtain

$$\frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x))_a d_{p,q}$$
$$= \frac{1}{p(b-a)}\left[(p-q)p(b-a)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}}(pb+(1-p)a) + \left(1-\frac{q^n}{p^{n+1}}\right)a\right)\right]$$
$$\le (p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left[\frac{q^n}{p^{n+1}} pf(b) + (1-p)f(a) + \left(1-\frac{q^n}{p^{n+1}}\right)f(a)\right]$$
$$= (p-q)\left[\frac{pf(b)}{p^2-q^2} + \frac{qf(a)}{p^2-q^2} - \frac{q^2 f(a)}{p(p^2-q^2)}\right]$$
$$= \frac{qf(a)+pf(b)}{p+q}.$$

The proof is complete.

Remark 3.2. Consider Theorem 3.1, if $q \in \left(0, \frac{c-b}{a-b}\right), then\ \frac{q(a+c)+(p-q)b}{p+q} \in [c,b)$. We can reduce the left hand side of Theorem 3.1 as:

$$f\left(\frac{q(a+c)+(p-q)b}{p+q}\right) \leq \frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x) \quad {}_a d_{p,q}x \leq \frac{qf(a)+pf(b)}{p+q}$$

Since $f'\left(\frac{q(a+c)+(p-q)b}{p+q}\right)\left(\frac{q(b-c)}{p+q}\right) \geq 0.$

Remark 3.3. In Remark 3.2, if $c \to a^{+}$, then $\frac{c-b}{a-b} \to 1^{-}$. Since $q \in (0,1)$, we have

$$\frac{q(a+c)+(p-q)b}{p+q} \in (a,b).$$

We can reduce the left hand side of Theorem 3.1 as:

$$f\left(\frac{qa+(p-q)b}{p+q}\right) \leq \frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x) \quad {}_a d_{p,q}x \leq \frac{qf(a)+pf(b)}{p+q}$$

Since $f'\left(\frac{q(a+c)+(p-q)b}{p+q}\right)\left(\frac{q(b-c)}{p+q}\right) \geq 0.$

Corollary 3.4. Assume that $f:[a,b] \to \mathbb{R}$ is a differentiable convex function on (a,b) such that $f'\left(\frac{a+b}{2}\right) = 0$ for $0 < q < p \leq 1$.Then we have

$$f\left(\frac{q\left(a+\frac{\boldsymbol{a}+\boldsymbol{b}}{\mathbf{2}}\right)+(p-q)b}{p+q}\right) + f'\left(\frac{q\left(a+\frac{\boldsymbol{a}+\boldsymbol{b}}{\mathbf{2}}\right)+(p-q)b}{p+q}\right)\left(\frac{q(b-a)}{2(p+q)}\right) \leq \frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x) \quad {}_a d_{p,q}x \leq \frac{qf(a)+pf(b)}{p+q}.$$

Corollary 3.5. Assume that $f:[a,b] \to \mathbb{R}$ is a differentiable convex function on (a,b) such that $f'(0) = 0$ for $0 \in (a,b)$ *and* $0 < q < p \leq 1$.Then we have

$$f\left(\frac{qa+(p-q)b}{p+q}\right) + f'\left(\frac{q(a)+(p-q)b}{p+q}\right)\left(\frac{qb}{(p+q)}\right) \leq \frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x) \quad {}_a d_{p,q}x \leq \frac{qf(a)+pf(b)}{p+q}.$$

Theorem 3.6. let f:[a,b]→ $\mathbb{R}$ is a differentiable convex function on (a,b) such that $f'(c) = 0$ for $c \in (a,b)$ and let q be a constant with $0 < q < p \leq 1$.Then we have

$$f\left(\frac{(p-q)a+q(c+b)}{p+q}\right) + f'\left(\frac{(p-q)a+q(c+b)}{p+q}\right)\left(\frac{q(2a-b-c)+p(b-a)}{p+q}\right) \leq \frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x) \quad {}_a d_{p,q}x \leq \frac{qf(a)+pf(b)}{p+q} \tag{3.3}$$

Proof: Meanwhile the function f is differentiable on (a,b), there exist a tangent line at the point

$$\frac{(p-q)a+q(c+b)}{p+q} \in (a,b),$$

which is given by

$$k(x) = f\left(\frac{(p-q)a+q(c+b)}{p+q}\right) + f'\left(\frac{(p-q)a+q(c+b)}{p+q}\right)\left(x - \frac{(p-q)a+q(c+b)}{p+q}\right). \tag{3.4}$$

Meanwhile f is convex on$[a,b]$, it follows that $k(x) \leq f(x)$ for all $x \in [a,b]$. Applying (p,q)-integrating, we get

$$\int_a^{pb+(1-p)a} k(x) \quad {}_a d_{p,q}x$$

$$=\int_a^{pb+(1-p)a} \left[f\left(\frac{(p-q)a+q(c+b)}{p+q}\right) + f'\left(\frac{(p-q)a+q(c+b)}{p+q}\right)\left(x - \frac{(p-q)a+q(c+b)}{p+q}\right) \quad {}_a d_{p,q}x\right]$$

$$=p(b-a)f\left(\frac{(p-q)a+q(c+b)}{p+q}\right) + f'\left(\frac{(p-q)a+q(c+b)}{p+q}\right)$$

$$\left(\int_a^{pb+(1-p)a} x \quad {}_a d_{p,q}x - (b-a)\frac{(p-q)a+q(c+b)}{p+q}\right)$$

$$=p(b-a)\left(\frac{(p-q)a+q(c+b)}{p+q}\right) + f'\left(\frac{(p-q)a+q(c+b)}{p+q}\right)\left[p(b-a)\left(\left(\frac{qa+pb}{p+q}\right) - \frac{(p-q)a+q(c+b)}{p+q}\right)\right]$$

$$= p(b-a)\left[\begin{array}{l} f\left(\frac{(p-q)a+q(c+b)}{p+q}\right) + f'\left(\frac{(p-q)a+q(c+b)}{p+q}\right) \\ \qquad\qquad \left(\frac{q(2a-b-c)+p(b-a)}{p+q}\right) \end{array}\right]$$

$$\leq \int_a^{pb+(1-p)a} f(x) \quad {}_a d_{p,q}x$$

The proof is complete.

Remark 3.7. Consider Theorem 3.6, if

$$q \in \left(\frac{1}{2}, \frac{c-a}{b-a}\right), then\ f'\left(\frac{(p-q)a+q(c+b)}{p+q}\right) \leq 0 \quad and \quad \left(\frac{q(2a-b-c)+p(b-a)}{p+q}\right) \leq 0$$

We can reduce the left hand side of Theorem 3.6 as:

$$f\left(\frac{(p-q)a+q(c+b)}{p+q}\right) \leq \frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x) \quad {}_a d_{p,q}x \leq \frac{qf(a)+pf(b)}{p+q},$$

Meanwhile

$$f'\left(\frac{(p-q)a+q(c+b)}{p+q}\right)\left(\frac{q(2a-b-c)+p(b-a)}{p+q}\right) \geq 0.$$

Remark 3.8. In Remark 3.7 , if $c \to b^-$, then $q \to 1^-$,Since $q \in \left(\frac{1}{2}, 1\right)$, we have $\frac{(p-q)a+q(c+b)}{p+q} \in \left(\frac{a+2b}{3}, b\right)$. We can reduce the left hand side of Theorem 3.6 as:

$$f\left((p-q)(\frac{2a+b}{3}+qb)\right) \leq \frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x) \quad {}_a d_{p,q}x \leq \frac{qf(a)+pf(b)}{p+q},$$

Since $f'\left(\frac{(p-q)a+q(c+b)}{p+q}\right)\left(\frac{q(2a-b-c)+p(b-a)}{p+q}\right) \geq 0.$

Theorem 3.9. [Generalized (*p, q*)-Hermite-Hadamard Inequality for convex differentiable functions]
Let $f:[a,b]\to\mathbb{R}$ be a differentiable convex function on (a,b) such that $f'(c)=0$ for $c\in(a,b)$ and $0<q<p\leq 1$. Then we have:

$$max\{I_1, I_2\} \leq \frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x) \quad {}_a d_{p,q}x \leq \frac{qf(a)+pf(b)}{p+q}. \tag{3.5}$$

where

$$I_1 = f\left(\frac{q(a+c)+(p-q)b}{p+q}\right) + f'\left(\frac{q(a+c)+(p-q)b}{p+q}\right)\left(\frac{q(b-c)}{p+q}\right),$$

$$I_2 = f\left(\frac{(p-q)a+q(c+b)}{p+q}\right) + f'\left(\frac{(p-q)a+q(c+b)}{p+q}\right)\left(\frac{q(2a-b-c)+p(b-a)}{p+q}\right)$$

Proof: Combining of (3.1) and (3.3) yields (3.5).This completes the proof.

Example 3.10. Define the function $f(x)=x^2$ on $[-1,3]$, and let $q\in(0,1)$.Applying Theorem 3.1 with $a=-1, b=3$ and $c=0$,the left hand side becomes:

$$f\left(\frac{q(a+c)+(p-q)b}{p+q}\right) + f'\left(\frac{q(a+c)+(p-q)b}{p+q}\right)\left(\frac{q(b-c)}{p+q}\right) - \frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x) \quad {}_a d_{p,q}x$$

$$= f\left(\frac{3p-4q}{p+q}\right) + f'\left(\frac{3p-4q}{p+q}\right)\left(\frac{3q}{p+q}\right) - \frac{1}{4p}\left[4p(p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}} f\left(\frac{3q^n}{p^{n+1}} - \left(1-\frac{q^n}{p^{n+1}}\right)\right)\right]$$

$$= \frac{-9q^4-9pq^3+7p^2q^2-16pq^2+16p^3q-32p^2q}{(p^2+pq+q^2)(p+q)^2} \leq 0$$

For the right hand side, we have:

$$\frac{1}{3-(-1)}\int_{-1}^{3} x^2 \quad {}_a d_{p,q}x - \frac{qf(-1)+f(3)}{P+q} = \frac{16}{(p^2+pq+q^2)} - \frac{8}{P+q} + 1 - \frac{q+9p}{p+q} \leq 0 .$$

Example 3.11. Define the function $f(x)=x^2$ on$[-1,1]$, and let $q\in(0,1)$. Applying Corollary 3.5 with $a=-1, b=1$and $c=0$, the left side becomes:

$$f\left(\frac{q(a)+(p-q)b}{p+q}\right) + f'\left(\frac{q(a)+(p-q)b}{p+q}\right)\left(\frac{qb}{(p+q)}\right) - \frac{1}{p(b-a)}\int_a^{pb+(1-p)a} f(x) \quad {}_a d_{p,q}x$$

$$= f\left(\frac{p-2q}{p+q}\right)+f'\left(\frac{p-2q}{p+q}\right)\left(\frac{q}{p+q}\right)-(p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{2q^n}{p^{n+1}}-1\right)$$

$$=\frac{p^2+4q^2+4pq}{(p+q)^2}+\frac{2q(p-2q)}{(p+q)^2}$$
$$-\frac{4p+4q+p^3+p^2q+pq^2+p^2q+pq^2+q^3-4p^2-4pq-4q^2}{(p+q)(p^2+pq+q^2)}\leq 0$$

For the right hand side,we have:

$$\frac{1}{p(1-(-1))}\int_{-1}^{3}x^2 \quad {}_a d_{p,q}x-\frac{qf(-1)+f(1)}{P+q}=\frac{4}{(p^2+pq+q^2)}-\frac{4}{P+q}+1-\frac{q+p}{p+q}\leq 0.$$

Example 3.12. Define the function $f(x)=x^2$ on $[-3,1]$, and let $q\in(0,1)$. Applying Theorem 3.6 with $a=-3, b=1$ and $c=0$,the left side becomes:

$$f\left(\frac{(p-q)a+q(c+b)}{p+q}\right)+f'\left(\frac{(p-q)a+q(c+b)}{p+q}\right)\left(\frac{q(2a-b-c)+p(b-a)}{p+q}\right)$$
$$-\frac{1}{p(b-a)}\int_a^{pb+(1-p)a}f(x) \quad {}_a d_{p,q}x$$
$$=f\left(\frac{4q-3p}{p+q}\right)+f'\left(\frac{4q-3p}{p+q}\right)\left(\frac{4p-7q}{p+q}\right)$$
$$-\frac{1}{4p}\left[4(p-q)p\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}f\left(\frac{3q^n}{p^{n+1}}-\left(1-\frac{q^n}{p^{n+1}}\right)\right)\right]$$
$$=\frac{16q^2-24pq+9p^2}{(p+q)^2}+\frac{-56q^2+74pq-24p^2}{(p+q)^2}-\frac{16}{p^2+pq+q^2}+\frac{24}{p+q}-9\leq 0$$

For the right hand side, we have

$$\frac{1}{1-(-3)}\int_{-3}^{4p-3}x^2 \quad {}_a d_{p,q}x-\frac{qf(-3)+pf(1)}{P+q}$$
$$=\frac{16}{(p^2+pq+q^2)}-\frac{24}{P+q}+9-\frac{9q+p}{p+q}\leq 0$$

4. CONCLUSION

In this study, we have obtained some new results for the (p,q)- calculus of Hermite–Hadamard inequalities for differentiable convex function with the critical point along withsome examples. We expect that the ideas and techniques used in this paper may inspire interested readers to explore some new applications of these newly introduced functions in various fields of pure and applied sciences.

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