ORIGINAL PAPER

HOLDITCH-TYPE THEOREMS FOR THE POLAR MOMENT OF INERTIA UNDER THE 1-PARAMETER CLOSED PLANAR HOMOTHETIC MOTION

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Abstract. *In this study, during the 1-parameter closed homothetic motion, the Holditch-Type Theorems are presented for the polar moments of inertia of the closed orbit curves of three non-collinear points.*

Keywords: *Holditch Theorem; Homothetic Motion; Polar Moment of Inertia.*

1. INTRODUCTION

Harmonic evolute surface of quasi binormal surface associated with quasi frame was studied [1]. The N-Bishop frame for timelike curves in Minkowski space was investigated [2]. Authors defined Fermi-Walker derivative in Galilean space Fermi-Walker transport and non-rotating frame by using Fermi-Walker derivative were given [3].

Let E and E' be moving and fixed Euclidean planes $\left(E = E' = E^2\right)$ and $\{O;\mathbf{e_1},\mathbf{e_2}\}$ and $\{O';\mathbf{e}'_1,\mathbf{e}'_2\}$ be their rectangular coordinate systems, respectively. By taking $\mathbf{OO'} = \mathbf{u} = u_1\mathbf{e_1} + u_2\mathbf{e_2}$, the motion defined by the transformation

$$\mathbf{x}' = h\mathbf{x} - \mathbf{u}, \tag{1}$$

is called **1-parameter planar homothetic motion** and denoted by $H = E/E'$, where h is a homothetic scale of the motion $H = E/E'$, and $\mathbf{x},\mathbf{x}'$ are the position vectors with respect to the moving and fixed rectangular coordinate systems of a point $X = (x_1, x_2) \in E$, respectively. The homothetic scale h and the vectors $\mathbf{x},\mathbf{x}'$ and $\mathbf{u}$ are continuously differentiable functions of a real parameter t, (Fig. 1). Furthermore, at the initial time $t = t_0$ the coordinate systems $\{O;\mathbf{e_1},\mathbf{e_2}\}$ and $\{O';\mathbf{e}'_1,\mathbf{e}'_2\}$ are coincident [4].

Taking $\theta = \theta(t)$ as the rotation angle between $\mathbf{e_1}$ and $\mathbf{e}'_1$, the equation

$$\begin{aligned} \mathbf{e_1} &= \cos\theta\mathbf{e}'_1 + \sin\theta\mathbf{e}'_2 \\ \mathbf{e_2} &= -\sin\theta\mathbf{e}'_1 + \cos\theta\mathbf{e}'_2 \end{aligned} \tag{2}$$

can be written.

If the equation (1) is differentiated with respect to t, the **sliding velocity** of a fixed point $X = (x_1, x_2) \in E$ is gotten as

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$$\begin{aligned}\mathbf{V_f} &= \left\{-\dot{u}_1 + \left(u_2 - hx_2\right)\dot{\theta} + \dot{h}x_1\right\}\mathbf{e_1} \\ &+ \left\{-\dot{u}_2 + \left(-u_1 + hx_1\right)\dot{\theta} + \dot{h}x_2\right\}\mathbf{e_2}.\end{aligned} \tag{3}$$

To avoid the cases of the pure translation and the pure rotation, it must be assumed that $\dot{\theta} = \dot{\theta}(t) \neq 0,\; h = h(t) \neq \text{constant}$.

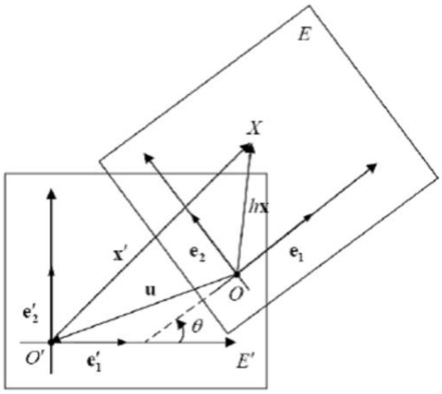


Figure 1. Parameter planar homothetic motion.

Suppose that $\mathbf{V_f} = \mathbf{0}$ in the motion $H = E/E'$, then at the pole point $P = (p_1, p_2)$ of the motion

$$\begin{aligned}p_1 &= \frac{\dot{h}\left(\dot{u}_1 - u_2\dot{\theta}\right) + h\dot{\theta}\left(\dot{u}_2 + u_1\dot{\theta}\right)}{\dot{h}^2 + h^2\dot{\theta}^2} \\ p_2 &= \frac{\dot{h}\left(\dot{u}_2 + u_1\dot{\theta}\right) - h\dot{\theta}\left(\dot{u}_1 - u_2\dot{\theta}\right)}{\dot{h}^2 + h^2\dot{\theta}^2}\end{aligned} \tag{4}$$

are found. During the motion $H = E/E'$, the locus of the pole points $P = (p_1, p_2) \in E$ (which are fixed in both planes at all "t") are called **moving** and **fixed pole curves** and will be denoted by (P) and (P'), on moving and fixed planes, respectively.

If $\dot{u}_1$ and $\dot{u}_2$ are solved from the equation (4), then

$$\begin{aligned}\dot{u}_1 &= p_1\dot{h} - p_2 h\dot{\theta} + u_2\dot{\theta} \\ \dot{u}_2 &= p_2\dot{h} + p_1 h\dot{\theta} - u_1\dot{\theta}.\end{aligned} \tag{5}$$

is obtained. If these expressions are replaced in the equation (4), for the sliding velocity

$$\begin{aligned}\mathbf{V_f} &= \left\{\left(x_1 - p_1\right)\dot{h} - \left(x_2 - p_2\right)h\dot{\theta}\right\}\mathbf{e_1} \\ &+ \left\{\left(x_1 - p_1\right)h\dot{\theta} + \left(x_2 - p_2\right)\dot{h}\right\}\mathbf{e_2}\end{aligned} \tag{6}$$

is gotten [4, 5]. During the 1-parameter planar homothetic motion, if there exists a number $T > 0$ such that

$$\begin{aligned}&u_j\left(t+T\right) = u_j\left(t\right), \quad j = 1,2 \\ &\theta\left(t+T\right) = \theta\left(t\right) + 2\pi\nu, \quad \nu \in \mathbb{Z} \\ &h\left(t+T\right) = h\left(t\right),\; h\left(0\right) = h\left(T\right) = 1,\; \forall t \in \mathbb{R}\end{aligned} \tag{7}$$

for all t (the smallest such number T is called the **period of motion**), then the motion $H = E/E'$ is called a **1-parameter closed planar homothetic motion**, where the integer ν is the **number of rotations** of the motion $H = E/E'$, [4].

Suppose that $\nu > 0$ throughout this study.

The Steiner point S, which is the center of gravity of the moving pole curve (P) for the distribution of mass with density $h^2 d\theta$, is given by

$$s_j = \frac{\oint h^2 p_j d\theta}{\oint h^2 d\theta}, \qquad j = 1, 2 \tag{8}$$

where the integrations are taken along the closed pole curve (P).

Furthermore, using the mean-value theorem for integration of a continuous function and the equation (7)

$$\int_0^T h^2(t)\, d\theta(t) = 2h^2(t_0)\pi\nu, \tag{9}$$

is had where $h := h(t_0),\ t_0 \in [0, T]$ [4].

2. THE POLAR MOMENT OF INERTIA OF THE ORBIT CURVE

Let $X = (x_1, x_2)$ be a fixed point in E and (X) be the orbit curve of X. Then, the **polar moment of inertia (PMI)** T_X of (X) is given by

$$T_X = \oint \|\mathbf{x}'\|^2 d\theta \tag{10}$$

where $\mathbf{x}'$ is given by the equation (1) and the integration is taken along the closed orbit curve (X) in E' [6].

Using the equation (1)

$$\begin{aligned} T_X = {} & \left(x_1^2 + x_2^2\right)\oint h^2 d\theta - 2x_1 \oint h u_1 d\theta \\ & - 2x_2 \oint h u_2 d\theta + \oint \left(u_1^2 + u_2^2\right) d\theta \end{aligned} \tag{11}$$

is obtained. If $X = O$ $(x_1 = x_2 = 0)$ then, for the PMI of the origin point O

$$T_O = \oint \left(u_1^2 + u_2^2\right) d\theta. \tag{12}$$

is had. However, from the equation (5)

$$\begin{aligned} u_1 d\theta &= p_1 h d\theta + p_2 dh - du_2 \\ u_2 d\theta &= p_2 h d\theta - p_1 dh - du_1 \end{aligned} \tag{13}$$

is gotten. Substituting the equations (8), (9), (12) and (13) into the equation (11) yields

$$T_X = T_O + 2h^2(t_0)\pi\nu\left(x_1^2 + x_2^2 - 2x_1s_1 - 2x_2s_2\right) \\ + 2x_1\eta_1 + 2x_2\eta_2 \tag{14}$$

where

$$\eta_1 = \oint\left(-p_2hdh + hdu_2\right), \\ \eta_2 = \oint\left(p_1hdh - hdu_1\right). \tag{15}$$

Then, it may be given the following theorem, [5].

Theorem 1. Let us consider the 1-parameter closed planar homothetic motions. All the fixed points of the moving plane E whose orbit curves have equal the PMI lie on the same circle with the center

$$C = (c_1, c_2) = \left(s_1 - \frac{\eta_1}{2h^2(t_0)\pi\nu}, s_2 - \frac{\eta_2}{2h^2(t_0)\pi\nu}\right) \tag{16}$$

in the moving plane, [5].

Theorem 2. (Holditch Type Theorem): Let us consider a line segment **XY** with constant length. If the endpoints X and Y trace the same closed convex curve in the fixed plane during the 1-parameter planar homothetic motion $H = E/E'$, then, the point Z on this line segment traces another closed curve. The difference between the polar moments of inertia (PMIs) of these curves depends on the distances of Z from the endpoints and the homothetic scale of the motion, [5].

Theorem 3. Let three collinear points X, Y and Z in the moving plane E such that $\overline{XZ} = \lambda a, \overline{ZY} = \lambda b$. During the 1-parameter closed planar homothetic motion $H = E/E'$, for the PMIs of the points,

$$T_Z = \frac{bT_X + aT_Y}{a+b} - 2h^2(t_0)\pi\nu\lambda^2 ab \tag{17}$$

is obtained [5].

Special Case 1. In the case of $h(t) \equiv 1$, $\eta_1 = \eta_2 = 0$ is obtained. Thus,

$$T_X = T_O + 2\pi\nu\left(x_1^2 + x_2^2 - 2x_1s_1 - 2x_2s_2\right) \tag{18}$$

is gotten which was given by [4]. Also, the center C and the Steiner point S coincide, [6].

3. THE HOLDITCH TYPE THEOREMS FOR POLAR MOMENTS OF INERTIA

PART I

Let the endpoints X and Y of a line segment with constant length d trace the closed curves k_X and k_Y, respectively, during the closed planar homothetic motion. Now, let's construct a frame $\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}\}$ by means of the normal vector $\mathbf{a_3}$ of the moving plane E as follows (Fig. 2):

$$\mathbf{a_1} := \frac{\mathbf{y}-\mathbf{x}}{d}, \quad \mathbf{a_2} = \mathbf{a_3} \times \mathbf{a_1}.$$

Then,

$$\oint \langle d\mathbf{a_1}, \mathbf{a_2} \rangle = -\oint \langle d\mathbf{a_2}, \mathbf{a_1} \rangle = 2\pi\nu$$

is had.

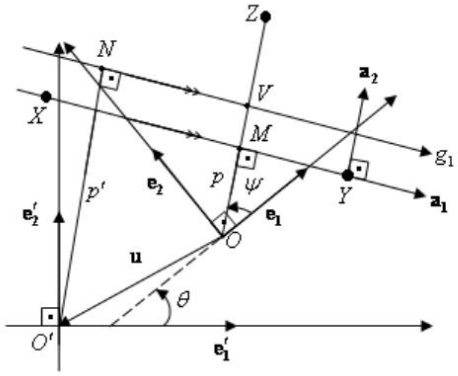


Figure 2. The Holditch type theorems for polar moments of inertia.

For Hessian form of line segment $\mathbf{XY}$ in the moving plane E, it can be written $\mathbf{XY}...x\cos\psi + y\sin\psi = p, \quad p = \overline{OM}$.

Let us choose a fixed point $Z\left(\overline{ZM} = c_2, \overline{XM} = \mu d, \overline{MY} = \lambda d, \overline{OV} = hp, \overline{O'N} = hp'\right)$. Then

$$\begin{aligned} &\mathbf{z} = \lambda\mathbf{x} + \mu\mathbf{y} + c_2\mathbf{a_2}, \\ &\lambda + \mu = 1, \quad \lambda, \mu, \mathrm{c}_2 = \text{constant}. \end{aligned} \tag{19}$$

can be written. If the equation (19) is substituted in the equation (1),

$$\mathbf{z}' = \lambda\mathbf{x}' + \mu\mathbf{y}' + hc_2\mathbf{a_2} \tag{20}$$

is obtained. Then, for the PMI of the closed orbit curve k_Z,

$$T_Z = \oint \|\mathbf{z}'\|^2 d\theta$$

is gotten or

$$\begin{aligned} T_Z = {} & \lambda^2 T_X + 2\lambda\mu T_{XY} + \mu^2 T_Y + 2h^2(t_0)\pi\nu c_2^2 \\ & + 2c_2 \oint \langle \lambda\mathbf{x}' + \mu\mathbf{y}', h\mathbf{a}_2 \rangle d\theta \end{aligned} \tag{21}$$

where $T_{XY} = T_{YX} = \oint \langle \mathbf{x}', \mathbf{y}' \rangle \, d\theta$ is the mixture the PMIs of the curves k_X and k_Y.

Let (g_1) be the closed envelope curve of the line g_1 which is parallel to the line segment $\mathbf{XY}$. For Hessian form of the line g_1

$$x\cos\psi + y\sin\psi = hp$$

can be written. Then, the length of (g_1) is given by

$$L_{g_1} = \oint hp' d\theta = \oint \langle \lambda \mathbf{x}' + \mu \mathbf{y}', h\mathbf{a_2} \rangle d\theta \tag{22}$$

(see [7]).

Then, it may be given the following theorem using the equations (21) and (22).

Theorem 4. During the closed planar homothetic motion $H = E/E'$ with homothetic scale h, if the endpoints of a line segment $\mathbf{XY}$ with constant length d move along the closed curves k_X and k_Y, respectively, then a point Z which is fixed according to the line segment $\mathbf{XY}$ traces another closed curve k_Z. The PMI-T_Z of k_Z depends not only T_X and T_Y but also the lengths of the line segment $\mathbf{XY}$ and the envelope curve of the line g_1 which is parallel to the line segment $\mathbf{XY}$. That is,

$$\begin{aligned} T_Z &= \lambda T_X + \mu T_Y - 2h^2(t_0)\lambda\mu\pi\nu d^2 \\ &\quad + 2c_2\left(\pi\nu c_2 + L_{g_1}\right), \end{aligned} \tag{23}$$

where c_2 is the distance between point Z and point M and L_{g_1} is the length of closed envelope curve of the line g_1.

Special Case 2. In the case of $h(t) \equiv 1$, the result given by [8] is gotten. Also, $L_{g_1} = L_{XY}$ is obtained [8].

Special Case 3. In the case of $c_2 = 0$, i.e. the points X, Y and Z are collinear, the result given by [5] is obtained.

PART II

Under the closed planar homothetic motion $H = E/E'$, if there non-collinear points $X, Y, Z \in E$ move along the same closed trajectory curve k (with orientation), then $T_X = T_Y = T_Z = T$ can be written and the circumcenter of the triangle ΔXYZ is the point C.
If the point C is chosen instead of the origin of the moving orthonormal frame on the moving plane E, then from equation (14), for the PMIs of the points $X = (r, 0), Q = (x, y) \in E$

$$T_Q = T_C + 2h^2(t_0)\pi\nu\left(x^2 + y^2\right)$$

is obtained and

$$T_X = T_C + 2h^2(t_0)\pi\nu r^2.$$

Theorem 5. During the closed planar homothetic motion $H = E/E'$, let three non-collinear points $X, Y, Z \in E$ trace the same closed curve with the PMI-T. Then for the PMI of any point $Q = (x, y) \in E$

$$T - T_Q = 2h^2(t_0)\pi\nu\left(r^2 - R^2\right)$$

is gotten where r is the circumradius of triangle with the vertices X, Y, Z and R is the distance between the points Q and circumcenter.

Special Case 4. In the case of $h(t) \equiv 1$, the result given by [8] is gotten.

Now, it can be given the following theorem which is the general form of Holditch Theorem for closed planar homothetic motion:

Theorem 6. During the closed planar homothetic motion $H = E/E'$, let T_X, T_Y and T_Z be the PMIs of the $X = (0,0)$, $Y = (b,0)$, $Z = (c,d) \in E$, respectively.

Then for the PMI of any point $Q = (x, y) \in E$

$$\begin{aligned} T_Q = &\left(1 - \frac{x}{b} + \frac{c-b}{bd}y\right)T_X + \left(\frac{x}{b} - \frac{cy}{bd}\right)T_Y \\ &+ \frac{y}{d}T_Z + 2h^2(t_0)\pi\nu \\ &\left(x^2 + y^2 - bx - \frac{c^2+d^2}{d}y + \frac{bc}{d}y\right) \end{aligned} \tag{24}$$

is obtained.

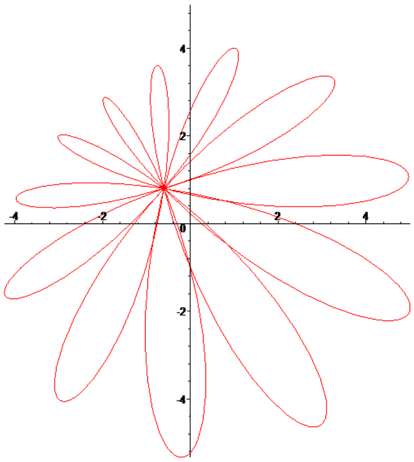
Proof: One can also prove the theorem 6, using the equation (14) for T_X, T_Y, T_Z and T_Q.

Special Case 5. In the case of $h(t) \equiv 1$, the result given by [8] is had.

4. MAPLE EXAMPLES

Let $X = (2,4)$ be a fixed point in E and (X) be the orbit curve of X during 1-parameter closed planar homothetic motion such that $u_1(t) = cost, u_2(t) = sint$, $h(t) = cos(6t)$, $\theta(t) = t$, $\nu = 1$. Here are (X) and PMI of T_X using by Maple Software:

```
> restart:with(plottools):with(plots):
> Motion:=proc(x1,x2,u1,u2,h,b,c,d) local k1;
>k1:=animate(plot,[[h*(x1*cos(b)-x2*sin(b))-
u1*cos(b)+u2*sin(b),h*(x1*sin(b)+x2*cos(b))-u1*sin(b)-
u2*cos(b),t=c..A],color=red],A=c..0+d,color=red,scaling=CONSTRAINED);
> end proc:
> Motion(2,4,cos(t),sin(t),cos(6*t),t,0,
2*Pi);
```




```
> restart:
> PMI:=proc(x1,x2,u1,u2,h,b,n,c,d) local T,H;
>
H:=evalf(int((u1**2+u2**2)*diff(b,t),t=c..d)+2*(int(h**2*diff(b,t),t=c..d)/
(2*Pi*n))*(Pi*n)*(x1**2+x2**2-2*x1*(int((((diff(h,t))*(diff(u1,t)-
u2*diff(b,t))+h*diff(b,t)*(diff(u2,t)+u1*diff(b,t)))/(diff(h,t)**2+(h*diff(
b,t))**2))*h**2*diff(b,t),t=c..d)/(int(h**2*diff(b,t),t=c..d))-
2*x2*int((((diff(h,t))*(diff(u2,t)+u1*diff(b,t))-h*diff(b,t)*(diff(u1,t)-
u2*diff(b,t)))/(diff(h,t)**2+(h*diff(b,t))**2))*h**2*diff(b,t),t=c..d)/int(
h**2*diff(b,t),t=c..d)))+2*x1*(int(-
(((diff(h,t))*(diff(u2,t)+u1*diff(b,t))-h*diff(b,t)*(diff(u1,t)-
u2*diff(b,t)))/(diff(h,t)**2+(h*diff(b,t))**2))*h*diff(h,t),t=c..d)+int(h*d
iff(u2,t),t=c..d))+2*x2*(int((((diff(h,t))*(diff(u1,t)-
u2*diff(b,t))+h*diff(b,t)*(diff(u2,t)+u1*diff(b,t)))/(diff(h,t)**2+(h*diff(
b,t))**2))*h*diff(h,t),t=c..d)+int(-h*diff(u1,t),t=c..d))):
> print(T[X]=H);
> end:
> PMI(2,4,cos(t),sin(t),cos(6*t),t,1,0,2*Pi);
```

$$T_X = 69.11503839$$

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