ORIGINAL PAPER

ON INVERSE OF A REGULAR SURFACE WITH RESPECT TO THE UNIT SPHERE S^2 IN E^3

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Manuscript received: 16.11.2020; Accepted paper: 04.02.2021;
Published online: 30.03.2021.

Abstract. *The purpose of this paper, first, is to give a definition of the inverse surface of a given regular surface with respect to a unit sphere in E^3. Second, some characteristic properties of the inverse surface are to express depending on the algebraic invariants of the original surface. In the last part of the study, we gave examples supporting our claims and plotted their graphics with the help of Maple software program.*

Keywords: *inversion; surface; support function; principal curvatures; fundamental forms; Christoffel symbols of second kind.*

1. INTRODUCTION

Let's begin this section by introducing the concept of inversion with erespect to a geometric object (circle and sphere). It is well known that the circle inversion is one of the most important transformations in egeometry. Inversion has a history ranging from synthetic geometry to differential geometry. But the history of the inversion transformations is complex and not clear. In the sixteenth century, inversely related points were known by French mathematician François Vieta. In 1749, Robert Simson, in his restoration of the work Plane Loci of Apollonius, included (on the basis of commentary made by Pappus) one of the basic theorems of the theory of inversion, namely that the inverse of a straight line or a circle is a straight line or a circle [1-3].

The nineteenth-century has great importance in view of the history of geometry. In this century, it has occurred important developments in geometry. In the early years of this century, French engineer and mathematician Jean-Victor Poncelet defined the inverse points with respect to a circle and called the reciprocal points. Swiss mathematician Jacob Steiner considered the inverse points in connection with the power of a point for a circle. The invention of the transformation of inversion is sometimes credited to Ludwig Immanuel Magnus, who published his work on the subject in 1831 [4].

A systematic development of inversion in a circle was first given by German mathematician and physicist Julius Plücker in his 1834 paper entitled " Analytisch-geometrische Aphorismen". Julius Plücker approximated the problem analytically and showed that the inversion preserved the magnitude of the angles between lines and circles. From here, it can be said that the inversion is a conform transformation [5, 6]. On rhe other hand, conformal transformations of the plane appeared in Swiss mathematican Leonhard Euler's 1770 paper [2, 6].

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In this paper, Euler considered linear fractional transformations of the complex plane. In 1845, Sir William Thomson (Lord Kelvin) used inversion to give geometrical proofs of some difficult propositions in the mathematical theory of elasticity. In 1847, French mathematician Joseph Liouville called inversion the transformation by reciprocal radii [6, 7]. With the aim of getting further information about inversion geometry, the interested readers are referred to [8-19].

Definition 1: Let C be a circle with centered O and radius r, and let P be any point other than O. If R is the point on the line OP that lies on the same side of O as P and satisfies the equation

$$\overline{OP}\,\overline{OR}=\overline{OQ}^2=r^2 \tag{1}$$

then we call R the inverse of P with respect to the circle C. Similarly, we can say that P is the inverse of R with respect to the circle C, (Fig. 1). On the other hand, if R lies on the opposite side of O from P then we have

$$\overline{OP}\,\overline{OR}=-\overline{OQ}^2=-r^2. \tag{2}$$

The point O is called the center of inversion, and C is called the circle of inversion, [19].

From here we can write the following properties:

1. If P is inside in C then its inverse R is outside and vice versa.
2. Every point on C is its own inverse.
3. As P moves closer to the center of C, the inverse point R approaches infinity.

In this case, the Euclidean plane with the added point of infinity is called inversive plane. Then inversion can be defined as a bijective transformation.

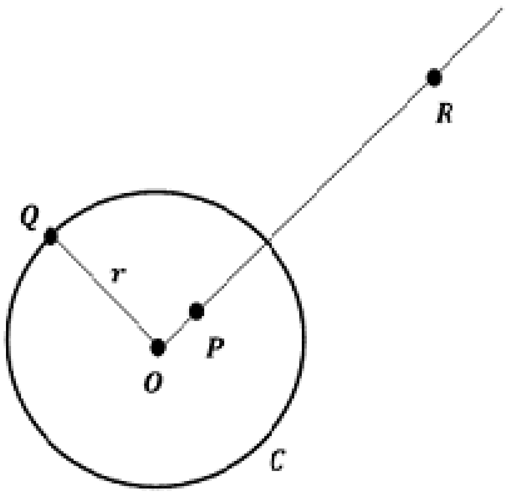

Figure 1. Circle inversion.

Definition 2: For all $P=(x_1,y_1)\in\mathbb{R}^2-\{O\}$, the function

$$\begin{aligned} f:\mathbb{R}^2-\{O\} &\to \mathbb{R}^2 \\ P \quad &\to f(P)=\left(\frac{x_1}{x_1^{\,2}+y_1^{\,2}},\frac{y_1}{x_1^{\,2}+y_1^{\,2}}\right) \end{aligned} \tag{3}$$

is called an inversion transformation where [11, 14],

$$C=\left\{Q=(x,y):x^2+y^2=r^2\right\}\subset\mathbb{R}^2.$$

Circle inversion can be used to study several well-known problems and theorems in geometry such as the problem of Appollonius, Steiner porism, Feuerbach's theorem, The Pappus' chain theorem, Peaucellier's cell, Construction problems, Ptolemy's theorem, among others. In addition, inversion has important applications in areas of physics, engineering, astronomy, medicine, geometric modeling, etc.

Circle inversion is generalizable in E^3. Inversion in a sphere was considered by Italian mathematician Giusto Bellavitis in his 1836 paper entitled "Teoria della figure inverse, e loro uso nella geometria elementar". In this paper, Bellavitis showed that stereographic projection is the inversion of a sphere into a plane [6, 10, 20-24]. The Riemann sphere was introduced by Riemann in his 1857 paper [10]. In 1852, German mathematician August Ferdinand Mobius introduced the cross ratio of four points in the plane and studied the Mobius transformations of plane by Mobius in 1855. Liouville considered the Mobius transformations of 3-space in 1847. Subsequently, Mobius transforms were used by Henri Poincare, Eugenio Beltrami etc. to create a geometric model for non-Euclidean geometries [10, 24-28]. Because of the inversion transformation has a conformal structure in plane (or space), many scientists studied in this subject. For example; Considering the central conics in plane Allen [7], studied circles. Childress Ottens studied geometry of Dupin cyclides under the inversion transformation [28]. Sarıoğlugil amd Kuruoğlu [29, 30] investigated inverse surfaces in E^3. The, inverse surface surface was studied by Röthe [31].

The inversion of a point P in E^3 with respect to a sphere centered at a point O with radius r is R such that

$$\overline{OP}\;\overline{OR}=\overline{OQ}^2=r^2$$

where the points P and R lie on the same side of OP ray. Then, we can give the definition of spherical inversion.

Definition 3: For all $P=(x_1,y_1,z_1)\in\mathbb{R}^3-\{O\}$, the function

$$\begin{aligned} f:\mathbb{R}^3-\{O\} &\to \mathbb{R}^3 \\ P \quad &\to f(P)=\left(\frac{r^2x_1}{x_1^{\,2}+y_1^{\,2}+z_1^{\,2}},\frac{r^2y_1}{x_1^{\,2}+y_1^{\,2}+z_1^{\,2}},\frac{r^2z_1}{x_1^{\,2}+y_1^{\,2}+z_1^{\,2}}\right) \end{aligned} \tag{4}$$

is called an inversion transformation where (Fig. 2) [5, 8],

$$S_r^2 = \{Q = (x, y, z) : x^2 + y^2 + z^2 = r^2\} \subset \mathbb{R}^3.$$

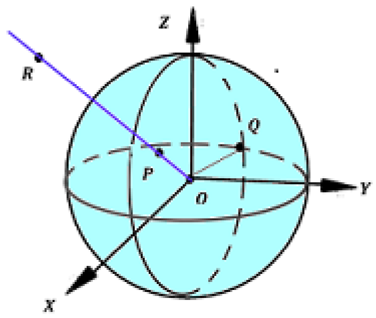

Figure 2. Spherical inversion.

This paper is organized as follows: Section 2 occurs of the basic concepts of the curves and surfaces theory to be used during throughout our study. In section 3, some characteristic properties of the inverse surface of a regular surface in E^3 are given related to the support function and the length of the position vector of M. Finally, invers surfaces of some surfaces are found and their graphics are drawn with the help of the maple software program.

2. PRELIMINARIES

In this section, we will give the basic concepts of the differential geometry for later use. Let us consider the three dimensional Euclidean space E^3 with the inner product $g = \langle , \rangle$. Here, g is a metric and is called the Euclidean metric. This metric is given by

$$g = ds^2 = dx^2 + dy^2 + dz^2 \tag{5}$$

where $\{x, y, z\}$ is the local coordinate system in E^3. On the other hand, the vectoral product of the vectors $X = \sum_{i=1}^{3} x_i e_i$ and $Y = \sum_{j=1}^{3} y_j e_j$ is defined by

$$X \wedge Y = \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \tag{6}$$

where $\left\{ e_i = \left.\frac{\partial}{\partial x_i}\right|_P : 1 \le \dot{I} \le 3 \right\}$ is a standard basis of E^3 [32].

Let M be a regular surface in E^3 and let

$$\begin{aligned} X : U \subset E^2 &\to X(D) \subset E^3 \\ (u,v) &\to X(u,v) = \left(x(u,v), y(u,v), z(u,v)\right) \end{aligned} \tag{7}$$

be the parametric equation. Here

$$X_u \wedge X_v \neq \vec{0} \tag{8}$$

where $X_u = \dfrac{\partial}{\partial u}(X)$ and $X_v = \dfrac{\partial}{\partial v}(X)$ are the partial derivates of *X*. From here, the unit normal vector of *M* is

$$N = \frac{1}{\left\| X_u \wedge X_v \right\|}\left(X_u \wedge X_v\right). \tag{9}$$

Then we can say that

$$T_M\left(P\right) = Span\left\{X_u, X_v\right\}. \tag{10}$$

On the other hand, the regular curve on M is defined by

$$\begin{aligned} \beta = X \circ \alpha : I \subset \mathbb{R} &\to M \subset E^3 \\ s &\to \beta\left(s\right) = X\left(\alpha\left(s\right)\right) \end{aligned} \tag{11}$$

where

$$\begin{aligned} \alpha : I \subset \mathbb{R} &\to E^2 \\ s &\to \alpha\left(s\right) = \left(u\left(s\right), v\left(s\right)\right). \end{aligned} \tag{12}$$

If we keep the first parameter u constant, $v \to X\left(u,v\right)$ is a curve on *M*. Similarly, if v is constant, $u \to X\left(u,v\right)$ is a curve on *M*. These curves called the parameter (or grid) curves of *M*. Here, the vectors X_u and X_v are the velocity vectors of the parameter curves (Fig. 3).

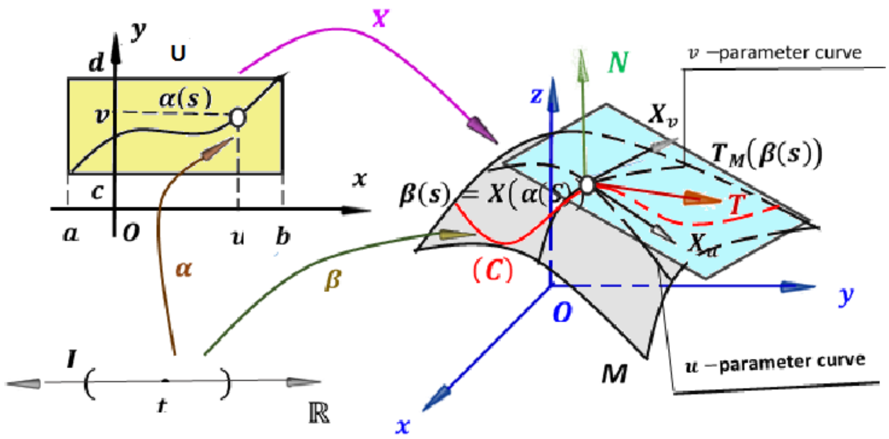

Figure 3. The curves on surfaces.

Definition 4: Let M be a regular surface in E^3. The support function of M is defined by

$$\begin{aligned} h : M &\to \mathbb{R} \\ P &\to h(P) = \langle X, N \rangle \end{aligned} \tag{13}$$

where X is the position vector of M [32]. From here, it can be said that h is the length of the projection of X in direction of N (Fig. 4).

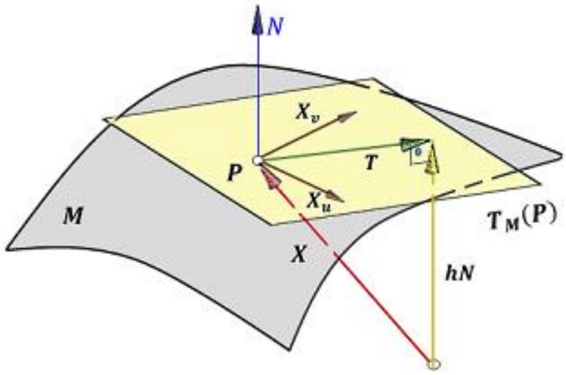

Figure 4. The support function of *M*.

Definition 5: Let M be a regular surface in E^3 and N be a unit normal vector of M. Then, the shape operator of M is defined by

$$\begin{aligned} S : T_M(P) &\to T_M(P) \\ X &\to S(X) = -D_X N \end{aligned} \tag{14}$$

where D is the affine connection on M. Furthermore, the shape operator is linear and self-adjoint. Then the matrix of the shape oprator S is

$$S = \begin{bmatrix} -\dfrac{\det(X_{uu}, X_u, X_v)}{\|X_u\|^3 \|X_v\|} & -\dfrac{\det(X_{uv}, X_u, X_v)}{\|X_u\|^2 \|X_v\|^2} \\ -\dfrac{\det(X_{uv}, X_u, X_v)}{\|X_u\|^2 \|X_v\|^2} & -\dfrac{\det(X_{vv}, X_u, X_v)}{\|X_u\| \|X_v\|^3} \end{bmatrix} \tag{15}$$

where $T_M(P) = Sp\{X_u, X_v\}$ [32].

Definition 6: Let M be a regular surface in E^3 and S be shape operator of M. Then, the q^{th} fundamental form of M is defined by

$$\begin{aligned} I^q : T_M(P) \times T_M(P) &\to C^\infty(M, \mathbb{R}),\ 1 \le q \le 3, \\ (X, Y) &\to I^q(X, Y) = \langle S^{q-1}(X), Y \rangle \end{aligned} \tag{16}$$

where N is the unit normal vector of M [33].

For $q=1$, the map

$$\begin{aligned} I: T_M(P)\times T_M(P) &\to C^\infty(M,\mathbb{R}) \\ (X,Y) \quad &\to \quad I(X,Y)=\langle X,Y\rangle \end{aligned} \tag{17}$$

is called the first fundamental form of *M*.

For $q=2$, the map

$$\begin{aligned} II: T_M(P)\times T_M(P) &\to C^\infty(M,\mathbb{R}) \\ (X,Y) \quad &\to \quad II(X,Y)=\langle S(X),Y\rangle \end{aligned} \tag{18}$$

is called the second fundamental form of *M*.

For $q=3$, the map

$$\begin{aligned} III: T_M(P)\times T_M(P) &\to C^\infty(M,\mathbb{R}) \\ (X,Y) \quad &\to \quad III(X,Y)=\langle S^2(X),Y\rangle \end{aligned} \tag{19}$$

is called the third fundamental form of *M*. Then, for the coefficients of the first, second and third fundamental forms we have [4],

$$g_{11}=\langle X_u,X_u\rangle,\ g_{12}=g_{21}=\langle X_u,X_v\rangle,\ g_{22}=\langle X_v,X_v\rangle \tag{20}$$

$$b_{11}=-\langle X_u,N_u\rangle,\ b_{12}=b_{21}=-\langle X_u,N_v\rangle,\ b_{22}=-\langle X_v,N_v\rangle \tag{21}$$

$$n_{11}=\langle N_u,N_u\rangle,\ n_{12}=n_{21}=\langle N_u,N_v\rangle,\ n_{22}=\langle N_v,N_v\rangle. \tag{22}$$

From here, the first, second and third fundamental forms are [34]:

$$I=g_{11}du^2+2\ g_{12}dudv+g_{22}dv^2 \tag{23}$$

$$II=b_{11}du^2+2\ b_{12}dudv+b_{22}dv^2 \tag{24}$$

$$III=n_{11}du^2+2\ n_{12}dudv+n_{22}dv^2. \tag{25}$$

Theorem 1: Let *M* be a regular surface in E^3. Then the Christoffel symbols of the second kind are

$$\begin{cases} \Gamma_{11}^{1} = \dfrac{g_{22}\left(g_{11}\right)_u - 2g_{12}\left(g_{12}\right)_u + g_{12}\left(g_{11}\right)_v}{2W^2} \\ \Gamma_{12}^{1} = \Gamma_{21}^{1} = \dfrac{g_{22}\left(g_{11}\right)_v - g_{12}\left(g_{22}\right)_u}{2W^2} \\ \Gamma_{22}^{1} = -\dfrac{g_{22}\left(g_{22}\right)_u - 2g_{22}\left(g_{12}\right)_v + g_{12}\left(g_{22}\right)_v}{2W^2} \\ \Gamma_{11}^{2} = -\dfrac{g_{11}\left(g_{11}\right)_v - 2g_{11}\left(g_{12}\right)_u + g_{12}\left(g_{11}\right)_u}{2W^2} \\ \Gamma_{12}^{2} = \Gamma_{21}^{2} = \dfrac{g_{11}\left(g_{22}\right)_u - g_{12}\left(g_{11}\right)_v}{2W^2} \\ \Gamma_{22}^{2} = \dfrac{g_{11}\left(g_{22}\right)_v - 2g_{12}\left(g_{12}\right)_v + g_{12}\left(g_{22}\right)_u}{2W^2} \end{cases} \tag{26}$$

where [32],

$$W^2 = g_{11}g_{22} - \left(g_{12}\right)^2 \neq 0. \tag{27}$$

If g_{12} is identically zero, then we have

$$\begin{cases} \Gamma_{11}^{1} = \dfrac{\left(g_{11}\right)_u}{2g_{11}} \quad , \quad \Gamma_{11}^{2} = -\dfrac{\left(g_{11}\right)_v}{2g_{11}} \\ \Gamma_{12}^{1} = \dfrac{\left(g_{11}\right)_v}{2g_{11}} = \Gamma_{21}^{1} \quad , \quad \Gamma_{12}^{2} = \dfrac{\left(g_{22}\right)_u}{2g_{22}} = \Gamma_{21}^{2} \\ \Gamma_{22}^{1} = -\dfrac{\left(g_{22}\right)_u}{2g_{11}} \quad , \quad \Gamma_{22}^{2} = \dfrac{\left(g_{22}\right)_v}{2g_{22}} \end{cases} \tag{28}$$

Definition 6: Let M be a regular surface in E^3 and S be shape operator of M. The characteristic values of S are called the principal curvatures of M. Non-zero vectors corresponding to these characteristic values are called the principal directions of M. Then we have

$$S(X) = kX \tag{29}$$

where X is a principal direction corresponding to the principal curvature k [32].

Definition 7: Let C be a regular curve on M and T be a unit tangent vector of C. If $S(T) = kT$, then C is called the line of curvature on M [15].

The equation of the lines of curvature is

$$\begin{vmatrix} dv^2 & -dudv & du^2 \\ g_{11} & g_{12} & g_{22} \\ b_{11} & b_{12} & b_{22} \end{vmatrix} = 0 \tag{30}$$

where g_{ij} and $b_{ij}, 1\leq i,j\leq 2,$ arethe coefficients of the first and the second fundamental forms of M, respectively [32].

Defintion 8: Let C be a regular curve on M and T be a unit tangent vector of C. The tangent vector T is called the asymptotic direction satifsfied by

$$\langle S(T),T\rangle = II\langle T,T\rangle = 0 \tag{31}$$

The differential equation of the asymptotic lines is

$$b_{11}du^2 + 2b_{12}dudv + b_{22}dv^2 = 0 \tag{32}$$

where $b_{ij}, 1\leq i,j\leq 2,$ is the coefficients of the second fundamental forms of *M*, respectively [32].

Theorem 2 (Gauss Equations): Let *M* be a regular surface in E^3. Then we have

$$\begin{cases} X_{uu} = \Gamma_{11}^1 X_u + \Gamma_{11}^2 X_v + b_{11}N \\ X_{uv} = X_{vu} = \Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v + b_{12}N \\ X_{vv} = \Gamma_{22}^1 X_u + \Gamma_{22}^2 X_v + b_{22}N \end{cases} \tag{33}$$

where b_{ij} and $\Gamma_{ij}^k, 1\leq i,j,k\leq 2,$ are the coefficients of the second fundamental form and, the Christoffel symbols of the second kind of M, respectively [32].

Theorem 3 (Weingarten Equations): Let *M* be a regular surface in E^3. Then the shape operator S of *M* is given in terms of the basis $\{X_u, X_v\}$ by

$$\begin{cases} -S(X_u) = N_u = \dfrac{b_{12}g_{12} - b_{11}g_{22}}{W^2} X_u + \dfrac{b_{11}g_{12} - b_{12}g_{11}}{W^2} X_v \\ -S(X_v) = N_v = \dfrac{b_{22}g_{12} - b_{12}g_{22}}{W^2} X_u + \dfrac{b_{12}g_{12} - b_{22}g_{11}}{W^2} X_v \end{cases} \tag{34}$$

where g_{ij} and $b_{ij}, 1\leq i,j\leq 2,$ are the coefficients of the first and the second fundamental forms of *M*, respectively [32].

On the other hand, if the parameter curves of *M* are the curvature lines then we can write $g_{12} = b_{12} = 0$. Thus, by (31) we have

$$\begin{cases} N_u = -\dfrac{b_{11}}{g_{11}} X_u \\ N_v = -\dfrac{b_{22}}{g_{22}} X_v \end{cases} \tag{35}$$

and is called the Olinde Rodrigues curvature formula. Furthermore, from (26) we get

$$\begin{cases} k_1 = \dfrac{b_{11}}{g_{11}} \\ k_2 = \dfrac{b_{22}}{g_{22}} \end{cases}. \tag{36}$$

Here, k_1 and k_2 are called the principal curvatures of *M*. The curvatures k_1 and k_2 are the principal values of the shape operator S . Thus, the matrix of the shape operrator S

$$S = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}. \tag{37}$$

Then, the Gauss curvature and the mean curvature of *M* are defined by

$$K = \det(S) = k_1 k_2 \tag{38}$$

and

$$H = \frac{1}{2} iz(S) = \frac{1}{2}\left(k_1 + k_2\right) \tag{39}$$

On the other hand, from the definition K and H, the principal curvaturesare the roots of the equation

$$k^2 - 2Hk + K = 0 \tag{40}$$

from which

$$\begin{cases} k_1 = H + \sqrt{H^2 - K} \\ k_2 = H - \sqrt{H^2 - K} \end{cases}. \tag{41}$$

Theorem 4: Let *M* be a regular surface in E^3. Then the Gauss curvature and the mean curvature of *M*, respectively, are

$$K = \frac{b_{11}b_{22} - b_{12}{}^2}{g_{11}g_{22} - g_{12}{}^2} \tag{42}$$

and

$$H = \frac{b_{11}g_{22} - 2b_{12}g_{12} + b_{22}g_{11}}{2(g_{11}g_{22} - g_{12}{}^2)} \tag{43}$$

where g_{ij} and $b_{ij}, 1 \leq i, j \leq 2,$ are the coefficients of the first and second fundametal forms of *M*, respectively [32].

Theorem 5: Let *M* be a regular surface in E^3.Then we have

$$\textbf{1.} \quad N \Lambda X_u = \frac{1}{W}\left[g_{11}X_v - g_{12}X_u\right] \tag{44}$$

2. $N\Lambda X_v = \frac{1}{W}\left[g_{12}X_v - g_{22}X_u \right]$ (45)

3. $N\Lambda\left[N\Lambda X_u \right] = -X_u$ (46)

4. $N\Lambda\left[N\Lambda X_v \right] = -X_v$ (47)

6. $X_v\Lambda\left[N\Lambda X_u \right] = g_{12}N$ (48)

7. $X_u\Lambda\left[N\Lambda X_v \right] = g_{12}N$ (49)

8. $X_v\Lambda\left[N\Lambda X_v \right] = g_{22}N$ (50)

where N is the unit normal vector of *M* [34].

Theorem 6: Let *M* be a regular surface in E^3.Then we have

$$X = hN - \frac{\rho}{W^2}\left[\left(\rho_v g_{12} - \rho_u g_{22} \right) X_u + \left(\rho_u g_{12} - \rho_v g_{11} \right) X_v \right] \tag{51}$$

where $T_M\left(P\right) = Span\left\{ X_u, X_v \right\}$.

Proposition 1: Let *M* be a regular surface in E^3.Then we have

$$\rho^2 = \frac{h^2}{1 - \overset{I}{\nabla}\left(\rho, \rho \right)} \tag{52}$$

where

$$\overset{I}{\nabla}\left(\rho, \rho \right) = \frac{\left(\rho_v \right)^2 g_{11} - 2\rho_u\rho_v g_{12} + \left(\rho_u \right)^2 g_{22}}{g_{11}g_{22} - (g_{22})^2} \tag{53}$$

Here $\overset{I}{\nabla}\left(\rho, \rho \right)$ is the Beltrami operator with respect to the coefficients of the first fundamental form of *M*.

3. ON INVERSE OF A REGULAR SURFACE WITH RESPECT TO A UNIT SPHERE S^2 IN E^3

In this section we will investigated the geometry of the inverse surface of a regular surface in 3-dimensional Euclidean space E^3 with respect to the unit sphere S^2. Then, let us starting with the definition of the inverse surface of a given surface with respect to the unit sphere S^2.

Definition 9: Let M be a regular surface in E^3 and let S^2 be a unit sphere centered O. The surface $\overline{M}$ is called the inverse of M with respect to the unt sphere S^2 defined by

$$\overline{X} = \frac{1}{\rho^2} X \tag{53}$$

where ρ denotes the length of the position vector X of M [26], (Fig. 5). From here, we can write

$$\rho^2 = \langle X, X \rangle . \tag{54}$$

Differentiating (54) with respect to the parameters u and v we get

$$\begin{cases} \rho_u = \langle X, X_u \rangle \\ \rho_v = \langle X, X_v \rangle \end{cases} . \tag{55}$$

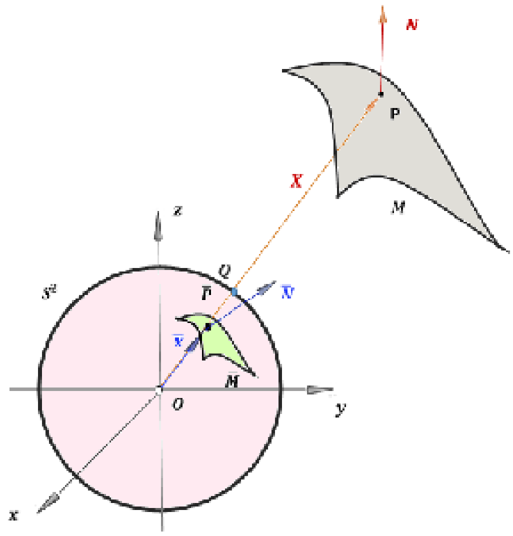

Figure 5. The inverse surface of the surface M with respect to the unit sphere S^2 in E^3.

Theorem 7: Let $\overline{M}$ be inverse of the regular surface M in E^3. Then the coefficients of the first fundamental form of $\overline{M}$ are

$$\begin{cases} \overline{g}_{11} = \frac{1}{\rho^4} g_{11} \\ \overline{g}_{12} = \frac{1}{\rho^4} g_{12} \\ \overline{g}_{22} = \frac{1}{\rho^4} g_{22} \end{cases} . \tag{56}$$

Proof: From (20) we may write

$$\begin{cases} \overline{g}_{11} = \left\langle \overline{X}_u, \overline{X}_u \right\rangle \\ \overline{g}_{12} = \left\langle \overline{X}_u, \overline{X}_v \right\rangle . \\ \overline{g}_{22} = \left\langle \overline{X}_v, \overline{X}_v \right\rangle \end{cases} \tag{57}$$

Differentiating (53) with respect to the parameters u and v we get

$$\begin{cases} \overline{X}_u = -2\dfrac{\rho_u}{\rho^3}X + \dfrac{1}{\rho^2}X_u \\ \overline{X}_v = -2\dfrac{\rho_v}{\rho^3}X + \dfrac{1}{\rho^2}X_v \end{cases} . \tag{58}$$

Substituting (58) into (57) the desired result is found.
Then, the following corollaries can be given.

Corollary 1: Let $\overline{M}$ be inverse of the regular surface M in E^3. Then we have

$$\overline{W}^2 = \frac{1}{\rho^4}W^2. \tag{59}$$

Corollary 2: Let $\overline{M}$ be inverse of the regular surface M in E^3. Then we have

$$\overline{I} = \frac{1}{\rho^4}I \tag{60}$$

where I is the first fundamental forms of M.

Corollary 3: Let $\overline{M}$ be inverse of the regular surface M in E^3.If the parameter curves of M are orthogonal, then we have

$$\begin{cases} \overline{g}_{11} = \dfrac{1}{\rho^4}g_{11} \\ \overline{g}_{12} = 0 \\ \overline{g}_{22} = \dfrac{1}{\rho^4}g_{22} \end{cases} . \tag{61}$$

Theorem 8: Let $\overline{M}$ be inverse of the regular surface M in E^3. Then we have

$$\overline{N} = 2\frac{h}{\rho^2}X - N \tag{62}$$

where N is the unit normal vector of M .

Proof: By (9) we may write

$$\overline{N} = \frac{1}{\left\|\overline{X}_u \wedge \overline{X}_v\right\|}\left(\overline{X}_u \wedge \overline{X}_v\right). \tag{63}$$

Computing vectoral product of the vectors $\overline{X}_u$ and $\overline{X}_v$ we obtain

$$\overline{X}_u \wedge \overline{X}_v = -2\frac{\rho_u}{\rho^5} X \wedge X_v - 2\frac{\rho_v}{\rho^5} X_u \wedge X + \frac{W}{\rho^4} N. \tag{64}$$

On the other hand, using (50) we get

$$X_u \wedge X = \frac{1}{W}\left[h\left(g_{12}X_u - g_{11}X_v\right) + \rho\left(\rho_u g_{11} - \rho_u g_{12}\right)N\right] \tag{65}$$

and

$$X \wedge X_v = \frac{1}{W}\left[h\left(g_{12}X_v - g_{22}X_u\right) + \rho\left(\rho_u g_{22} - \rho_v g_{12}\right)N\right] \tag{66}$$

where h is the support function of M.
Substituting (65) and (66) into (64) we obtain

$$\overline{X}_u \wedge \overline{X}_v = -\frac{1}{W\rho^4}\begin{bmatrix} 2\frac{h}{\rho^2}\left[\left(\rho_v g_{12} - \rho_u g_{22}\right)X_u + \left(\rho_u g_{12} - \rho_v g_{11}\right)X_v\right] + \\ \left[2\left[\left(\rho_v\right)^2 g_{11} - 2\rho_u\rho_v g_{12} + \left(\rho_u\right)^2 g_{22}\right] - W^2\right]N \end{bmatrix}. \tag{67}$$

Substituting (50),(51) and (52) into (67) and rearranging we get

$$\overline{X}_u \wedge \overline{X}_v = \frac{W}{\rho^4}\left(2\frac{h}{\rho^2}X - N\right). \tag{68}$$

From here we obtain

$$\left\|\overline{X}_u \wedge \overline{X}_v\right\| = \frac{W}{\rho^4}. \tag{69}$$

Combining (68) and (69) with (63) we have

$$\overline{N} = 2\frac{h}{\rho^2}X - N.$$

This is completed the proof.

Theorem 9: Let $\overline{M}$ be inverse of the regular surface M in E^3. Then we have

$$\overline{h} = \frac{h}{\rho^2}. \tag{70}$$

where h is the support function of M .

Proof: By (13) we may write

$$\overline{h}=\left\langle \overline{X},\overline{N}\right\rangle .. \tag{71}$$

Substituting (53) and (62) into (71) we get

$$\overline{h}=\frac{h}{\rho^{2}}.$$

This is completed the proof.

Theorem 10: Let $\overline{M}$ be inverse of the regular surface M in E^3. Then the coefficients of the second fundamental form of $\overline{M}$ are

$$\begin{cases}\overline{b}_{11}=-\dfrac{1}{\rho^{4}}\left(2hg_{11}+\rho^{2}b_{11}\right)\\ \overline{b}_{12}=-\dfrac{1}{\rho^{4}}\left(2hg_{12}+\rho^{2}b_{12}\right)\\ \overline{b}_{22}=-\dfrac{1}{\rho^{4}}\left(2hg_{22}+\rho^{2}b_{22}\right)\end{cases}. \tag{72}$$

where g_{ij} and $b_{ij}, 1\le i,j\le 2,$ are the coefficients of the first and the second fundamental forms of M.

Proof: From (21) we may write

$$\begin{cases}\overline{b}_{11}=-\left\langle \overline{X}_{u},\overline{N}_{u}\right\rangle\\ \overline{b}_{12}=-\left\langle \overline{X}_{u},\overline{N}_{v}\right\rangle\\ \overline{b}_{22}=-\left\langle \overline{X}_{v},\overline{N}_{v}\right\rangle\end{cases}. \tag{73}$$

Differentiating (62) with respect to the parameters *u* and *v* we get

$$\begin{cases}\overline{N}_{u}=\dfrac{2}{\rho^{3}}\left(\rho h_{u}-2h\rho_{u}\right)X+2\dfrac{h}{\rho^{2}}X_{u}-N_{u}\\ \overline{N}_{v}=\dfrac{2}{\rho^{3}}\left(\rho h_{v}-2h\rho_{v}\right)X+2\dfrac{h}{\rho^{2}}X_{v}-N_{v}\end{cases}. \tag{74}$$

Substituting (58) and (74) into (73) the desired result is found.
Then, the following corollaries can be given.

Corollary 4: Let $\overline{M}$ be inverse of the regular surface M in E^3. Then we have

$$\overline{II}=-\frac{1}{\rho^{4}}\left(2hI+\rho^{2}II\right) \tag{75}$$

where I and II are the first and second fundamental forms of *M.*

Corollary 5: Let $\overline{M}$ be inverse of the regular surface M in E^3. If the parameter curves of M are the lines of curvature, then the parameter curves of $\overline{M}$ are the lines of curvature, The opposite of this proposition is also true.

Theorem 11: Let $\overline{M}$ be inverse of the regular surface M in E^3. Then the coefficients of the third fundamental form of $\overline{M}$ are

$$\begin{cases} \overline{n}_{11} = 4\dfrac{h}{\rho^4}\left[hg_{11} + \rho^2 b_{11}\right] + n_{11} \\ \overline{n}_{12} = 4\dfrac{h}{\rho^4}\left[hg_{12} + \rho^2 b_{12}\right] + n_{12} \\ \overline{n}_{22} = 4\dfrac{h}{\rho^4}\left[hg_{22} + \rho^2 b_{22}\right] + n_{22} \end{cases}. \tag{76}$$

Proof: From (22) we may write

$$\begin{cases} \overline{n}_{11} = \left\langle \overline{N}_u, \overline{N}_u \right\rangle \\ \overline{n}_{12} = \left\langle \overline{N}_u, \overline{N}_v \right\rangle \\ \overline{n}_{22} = \left\langle \overline{N}_v, \overline{N}_v \right\rangle \end{cases}. \tag{77}$$

Substituting (74) into (77) the desired result is found.
Then, the following corollaries can be given.

Corollary 6: Let $\overline{M}$ be inverse of the regular surface M in E^3. Then we have

$$\overline{II} = 4\frac{h}{\rho^4}\left(hI + \rho^2 II\right) + III \tag{78}$$

where I II and III are the first, second and third fundamental forms of

Corollary 7: Let $\overline{M}$ be inverse of the regular surface M in E^3 .If the parameter curves of M are the lines of curvature, then we have

$$\overline{g}_{12} = \overline{b}_{12} = \overline{n}_{12} = 0. \tag{79}$$

Then, we can give the following theorem

Theorem 11: Let $\overline{M}$ be inverse of the regular surface M in E^3. Then the curvature lines of M sends the curvature lines of M.

Proof: From (30) the differential equatıon of the curvature lines of $\overline{M}$ is

$$\begin{vmatrix} dv^2 & -dudv & du^2 \\ \overline{g}_{11} & \overline{g}_{12} & \overline{g}_{22} \\ \overline{b}_{11} & \overline{b}_{12} & \overline{b}_{22} \end{vmatrix} = 0 \tag{80}$$

or

$$\left[\overline{b}_{12}\overline{g}_{11} - \overline{b}_{11}\overline{g}_{12}\right]du^2 - \left[\overline{b}_{22}\overline{g}_{11} - \overline{b}_{11}\overline{g}_{11}\right]dudv + \left[\overline{b}_{22}\overline{g}_{12} - \overline{b}_{12}\overline{g}_{22}\right]dv^2 = 0 \tag{81}$$

Substituting (61) and (72) into (81) and rearranging we find

$$\left[b_{12}g_{11} - b_{11}g_{12}\right]du^2 + \left[b_{22}g_{11} - b_{11}g_{22}\right]dudv + \left[b_{22}g_{12} - b_{12}g_{22}\right]dv^2 = 0 \tag{82}$$

Then, the solution spaces of the equations (81) and (82) are same. In other words, the inversion map preserves the curvature lines.

Theorem 12: Let $\overline{M}$ be inverse of the regular surface M in E^3. Then the Gauss curvature of $\overline{M}$ is

$$\overline{K} = 4h^2 + 4h\rho^2 H + \rho^4 K \tag{83}$$

where H and K are the mean and Gauss curvatures of M **,** respectively.

Proof: From (42) we may write

$$\overline{K} = \frac{\overline{b}_{11}\overline{b}_{22} - \overline{b}_{12}^{\,2}}{\overline{g}_{11}\overline{g}_{22} - \overline{g}_{12}^{\,2}}. \tag{84}$$

(61) and (72) are written in their places in (84) and if the obtained equation is arranged then we get

$$\overline{K} = 4h^2 + 4h\rho^2 H + \rho^4 K.$$

This is completed the proof.

Theorem 13: Let $\overline{M}$ be inverse of the regular surface M in E^3. Then the mean curvature of $\overline{M}$ is

$$\overline{H} = -\left(2h + \rho^2 H\right) \tag{85}$$

where H is the mean curvature of M **.**

Proof: From (43) we may write

$$\overline{H} = \frac{\overline{b}_{11}\overline{g}_{22} - 2\,\overline{b}_{12}\overline{g}_{12} + \overline{b}_{22}\overline{g}_{11}}{2\left(\overline{g}_{11}\overline{g}_{22} - \overline{g}_{12}^{\,2}\right)}. \tag{86}$$

If (61) and (72) are written in their places in (86) and the obtained equation is arranged, then we get

$$\overline{H} = -\left(2h + \rho^2 H\right).$$

This is completed the proof.

Theorem 14: Let $\overline{M}$ be inverse of the regular surface M in E^3. Then the principal curvatures of $\overline{M}$ is

$$\begin{cases} \overline{k}_1 = -\left[2h + \rho^2 k_1\right] \\ \overline{k}_2 = -\left[2h + \rho^2 k_2\right] \end{cases} \tag{87}$$

where k_1 and k_2 are the principal curvatures of M, respectively.

Proof: From (41) we may write

$$\begin{cases} \overline{k}_1 = \overline{H} + \sqrt{\overline{H}^2 - \overline{K}} \\ \overline{k}_2 = \overline{H} - \sqrt{\overline{H}^2 - \overline{K}} \end{cases} \tag{88}$$

(83) and (85) are written in their places in (88) and if the obtained equation is arranged then we get

$$\begin{aligned} \overline{k}_1 &= -\left(2h + \rho^2 H\right) - \rho^2 \sqrt{H^2 - K} \\ &= -\left[2h + \rho^2\left(H + \sqrt{H^2 - K}\right)\right] \\ &= -\left[2h + \rho^2 k_1\right] \end{aligned} \tag{89}$$

and

$$\begin{aligned} \overline{k}_2 &= -\left(2h + \rho^2 H\right) + \rho^2 \sqrt{H^2 - K} \\ &= -\left[2h + \rho^2\left(H - \sqrt{H^2 - K}\right)\right] \\ &= -\left[2h + \rho^2 k_2\right]. \end{aligned} \tag{90}$$

From (89) and (90) we have

$$\begin{cases} \overline{k}_1 = -\left[2h + \rho^2 k_1\right] \\ \overline{k}_2 = -\left[2h + \rho^2 k_2\right] \end{cases}$$

This is completed the proof.

Alternative Proof: Let the lines of curvature of the M are the parameter curves the we may write $g_{12} = b_{12} = 0$. From (36) and (72) we get

$$\begin{cases} \overline{g}_{11} = \dfrac{1}{\rho^4} g_{11} \\ \overline{g}_{12} = 0 \\ \overline{g}_{22} = \dfrac{1}{\rho^4} g_{22} \end{cases}.$$

and

$$\begin{cases} \overline{b}_{11} = -\dfrac{1}{\rho^4}\left(2hg_{11} + \rho^2 b_{11}\right) \\ \overline{b}_{12} = 0 \\ \overline{b}_{22} = -\dfrac{1}{\rho^4}\left(2hg_{22} + \rho^2 b_{22}\right) \end{cases}.$$

Using the Olin de Rodrigues curvature formula we get

$$\begin{cases} \overline{k}_1 = -\left[2h + \rho^2 k_1\right] \\ \overline{k}_2 = -\left[2h + \rho^2 k_2\right] \end{cases}.$$

Then we can give the corollary.

Corollary 8: Let $\overline{M}$ be inverse of the regular surface M in E^3. Then we have

$$\overline{k}_1 - \overline{k}_2 = -\rho^2\left(k_1 - k_2\right) \tag{91}$$

Proof: The proof is clear.

Theorem 15: Let $\overline{M}$ be inverse of the regular surface M in E^3. Then the Christoffel symbols of the second kind of $\overline{M}$ are

$$\begin{cases} \overline{\Gamma}_{11}^{1} = \dfrac{1}{\rho^4}\left[\Gamma_{11}^{1} - \dfrac{2}{\rho}\left(\rho_u - \dfrac{\alpha}{W^2} g_{12}\right)\right] \\ \overline{\Gamma}_{12}^{1} = \overline{\Gamma}_{21}^{1} = \dfrac{1}{\rho^4}\left[\Gamma_{12}^{1} - \dfrac{2}{\rho}\left(\rho_v + \dfrac{\beta}{W^2} g_{12}\right)\right] \\ \overline{\Gamma}_{22}^{1} = \dfrac{1}{\rho^4}\left[\Gamma_{22}^{1} - \dfrac{2\beta}{\rho W^2} g_{22}\right] \\ \overline{\Gamma}_{11}^{2} = \dfrac{1}{\rho^4}\left[\Gamma_{11}^{2} - \dfrac{2\alpha}{\rho W^2} g_{11}\right] \\ \overline{\Gamma}_{12}^{2} = \overline{\Gamma}_{21}^{2} = \dfrac{1}{\rho^4}\left[\Gamma_{12}^{2} - \dfrac{2}{\rho}\left(\rho_u + \dfrac{\alpha}{W^2} g_{12}\right)\right] \\ \overline{\Gamma}_{22}^{2} = \dfrac{1}{\rho^4}\left[\Gamma_{22}^{2} - \dfrac{2}{\rho}\left(\rho_v - \dfrac{\beta}{W^2} g_{12}\right)\right] \end{cases} \tag{92}$$

where

$$\begin{cases}\alpha = \rho_u g_{12} - \rho_v g_{11} \\ \beta = \rho_v g_{12} - \rho_u g_{22}\end{cases}. \tag{93}$$

and

$$W^2 = g_{11}g_{22} - (g_{12})^2.$$

Proof: By (26) we may write

$$\begin{cases}
\overline{\Gamma}_{11}^1 = \dfrac{\overline{g}_{22}\left(\overline{g}_{11}\right)_u - 2\overline{g}_{12}\left(\overline{g}_{12}\right)_u + \overline{g}_{12}\left(\overline{g}_{11}\right)_v}{2\overline{W}^2} \\
\overline{\Gamma}_{12}^1 = \overline{\Gamma}_{21}^1 = \dfrac{\overline{g}_{22}\left(\overline{g}_{11}\right)_v - \overline{g}_{12}\left(\overline{g}_{22}\right)_u}{2\overline{W}^2} \\
\overline{\Gamma}_{22}^1 = -\dfrac{\overline{g}_{22}\left(\overline{g}_{22}\right)_u - 2\overline{g}_{22}\left(\overline{g}_{12}\right)_v + \overline{g}_{12}\left(\overline{g}_{22}\right)_v}{2\overline{W}^2} \\
\overline{\Gamma}_{11}^2 = -\dfrac{\overline{g}_{11}\left(\overline{g}_{11}\right)_v - 2\overline{g}_{11}\left(\overline{g}_{12}\right)_u + \overline{g}_{12}\left(\overline{g}_{11}\right)_u}{2\overline{W}^2} \\
\overline{\Gamma}_{12}^2 = \overline{\Gamma}_{21}^2 = \dfrac{\overline{g}_{11}\left(\overline{g}_{22}\right)_u - \overline{g}_{12}\left(\overline{g}_{11}\right)_v}{2\overline{W}^2} \\
\overline{\Gamma}_{22}^2 = \dfrac{\overline{g}_{11}\left(\overline{g}_{22}\right)_v - 2\overline{g}_{12}\left(\overline{g}_{12}\right)_v + \overline{g}_{12}\left(\overline{g}_{22}\right)_u}{2\overline{W}^2}
\end{cases} \tag{94}$$

Differentiating (56) with respect to the parameters u and v we get

$$\begin{cases}
\left(\overline{g}_{11}\right)_u = \dfrac{1}{\rho^4}\left[-4\rho_u g_{11} + \rho\left(g_{11}\right)_u\right] \\
\left(\overline{g}_{11}\right)_v = \dfrac{1}{\rho^4}\left[-4\rho_v g_{11} + \rho\left(g_{11}\right)_v\right] \\
\left(\overline{g}_{12}\right)_u = \dfrac{1}{\rho^4}\left[-4\rho_u g_{12} + \rho\left(g_{12}\right)_u\right] \\
\left(\overline{g}_{12}\right)_v = \dfrac{1}{\rho^4}\left[-4\rho_v g_{12} + \rho\left(g_{12}\right)_v\right] \\
\left(\overline{g}_{22}\right)_u = \dfrac{1}{\rho^4}\left[-4\rho_u g_{22} + \rho\left(g_{22}\right)_u\right] \\
\left(\overline{g}_{22}\right)_v = \dfrac{1}{\rho^4}\left[-4\rho_v g_{22} + \rho\left(g_{22}\right)_v\right]
\end{cases}. \tag{95}$$

Substituting (59) and (95) into (94) and rearranging we get

$$\begin{cases} \overline{\Gamma}_{11}^{1} = \dfrac{1}{\rho^{4}}\left[\Gamma_{11}^{1} - \dfrac{2}{\rho}\left(\rho_u - \dfrac{\alpha}{W^2} g_{12}\right)\right] \\ \overline{\Gamma}_{12}^{1} = \overline{\Gamma}_{21}^{1} = \dfrac{1}{\rho^{4}}\left[\Gamma_{12}^{1} - \dfrac{2}{\rho}\left(\rho_v + \dfrac{\beta}{W^2} g_{12}\right)\right] \\ \overline{\Gamma}_{22}^{1} = \dfrac{1}{\rho^{4}}\left[\Gamma_{22}^{1} - \dfrac{2\beta}{\rho W^2} g_{22}\right] \\ \overline{\Gamma}_{11}^{2} = \dfrac{1}{\rho^{4}}\left[\Gamma_{11}^{2} - \dfrac{2\alpha}{\rho W^2} g_{11}\right] \\ \overline{\Gamma}_{12}^{2} = \overline{\Gamma}_{21}^{2} = \dfrac{1}{\rho^{4}}\left[\Gamma_{12}^{2} - \dfrac{2}{\rho}\left(\rho_u + \dfrac{\alpha}{W^2} g_{12}\right)\right] \\ \overline{\Gamma}_{22}^{2} = \dfrac{1}{\rho^{4}}\left[\Gamma_{22}^{2} - \dfrac{2}{\rho}\left(\rho_v - \dfrac{\beta}{W^2} g_{12}\right)\right] \end{cases}$$

where

$$\begin{cases} \alpha = \rho_u g_{12} - \rho_v g_{11} \\ \beta = \rho_v g_{12} - \rho_u g_{22} \end{cases}$$

and

$$W^2 = g_{11} g_{22} - \left(g_{12}\right)^2 .$$

This is completed the proof.

If the parameter curves of M are the lines of curvature, then we have $g_{12} = 0$. From here we get

$$\begin{cases} \alpha = -\rho_v g_{11} \\ \beta = -\rho_u g_{22} \\ W^2 = g_{11} g_{22} \end{cases} . \tag{96}$$

Substituting (85) into (80) and rearranging we get

$$\begin{cases} \overline{\Gamma}_{11}^{1} = \dfrac{1}{\rho^{4}}\left[\Gamma_{11}^{1} - \dfrac{2\rho_u}{\rho}\right] \\ \overline{\Gamma}_{12}^{1} = \overline{\Gamma}_{21}^{1} = \dfrac{1}{\rho^{4}}\left[\Gamma_{12}^{1} - \dfrac{2\rho_v}{\rho}\right] \\ \overline{\Gamma}_{22}^{1} = \dfrac{1}{\rho^{4}}\left[\Gamma_{22}^{1} + \dfrac{2\rho_u g_{22}}{\rho g_{11}}\right] \\ \overline{\Gamma}_{11}^{2} = \dfrac{1}{\rho^{4}}\left[\Gamma_{11}^{2} + \dfrac{2\rho_v g_{11}}{\rho g_{22}}\right] \\ \overline{\Gamma}_{12}^{2} = \overline{\Gamma}_{21}^{2} = \dfrac{1}{\rho^{4}}\left[\Gamma_{12}^{2} - \dfrac{2\rho_u}{\rho}\right] \\ \overline{\Gamma}_{22}^{2} = \dfrac{1}{\rho^{4}}\left[\Gamma_{22}^{2} - \dfrac{2\rho_v}{\rho}\right] \end{cases} \tag{97}$$

Here, $\Gamma_{ij}^{k}, 1 \le i, j, k \le 2,$ are the Christoffel symbols of the second kind in (28).

4. EXAMPLES

In this section, we will find the equations of the inverse surface of some surfaces and draw the graphics of theirs by the Maple software program.

Example 4. 1: Let us consider the surface M

$$z = (x-1)^2 + (y+3)^2. \tag{98}$$

Then the parametric equation of this surface is

$$X(u,v) = (u+1, v-3, u^2+v^2) \tag{99}$$

Substituting (98) into (54) we obtain

$$\rho^2 = (u+1)^2 + (v-3)^2 + (u^2+v^2). \tag{100}$$

Then, by (53) the equation of the inverse surface $\overline{M}$ of M is

$$\overline{X} = \frac{1}{(u+1)^2 + (v-3)^2 + (u^2+v^2)} \left(u+1, v-3, u^2+v^2\right). \tag{101}$$

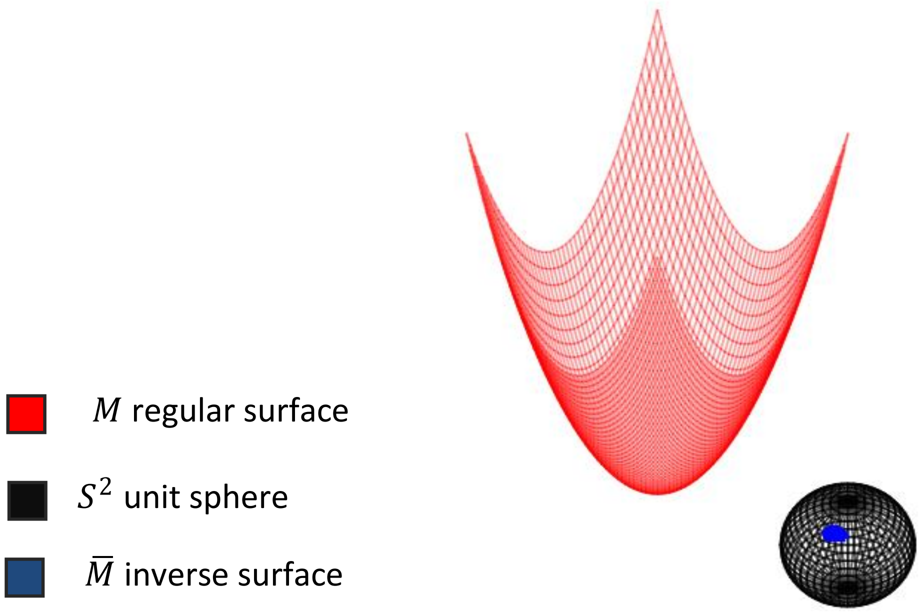

Figure 4.1. The inverse surface of M with respect to s^2.

Example 4.2: Let us consider the surfsce

$$M \ldots X(u,v)=(2\cos v, 2\sin v, u) \ . \tag{102}$$

Substituting (102) into (54) we obtain

$$\rho^2 = 4+u^2. \tag{103}$$

Then, by (53) the equation of the inverse surface $\overline{M}$ of M is

$$\overline{M} \ldots \overline{X}(u,v)=\left(\frac{2}{4+u^2}\cos v, \frac{2}{4+u^2}\sin v, \frac{u}{4+u^2}\right). \tag{104}$$

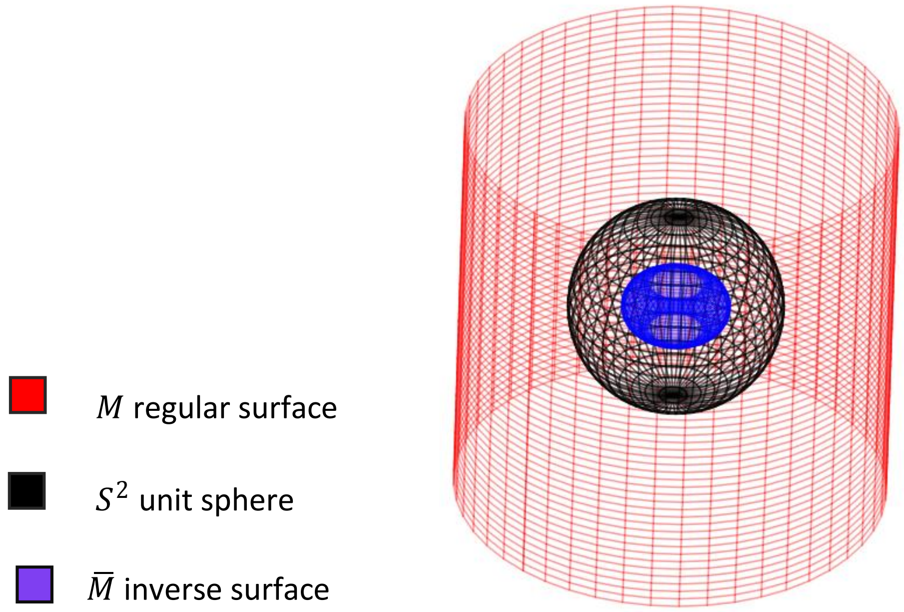

Figure 4.2. The inverse surface of *M* with respect to s^2.

Example 4.3: Let us consider the surface M

$$\begin{cases} x = 3\cos u \sin v \\ y = 3\sin u \sin v \\ z = 3\cos v \end{cases} \tag{105}$$

Substituting (105) into (54) we obtain

$$\rho^2 = 9 \tag{106}$$

Then, by (53) the equation of the inverse surface $\overline{M}$ of M is

$$\overline{X} = \left(\frac{1}{3}\cos u \sin v, \frac{1}{3}\sin u \sin v, \frac{1}{3}\cos v\right).$$

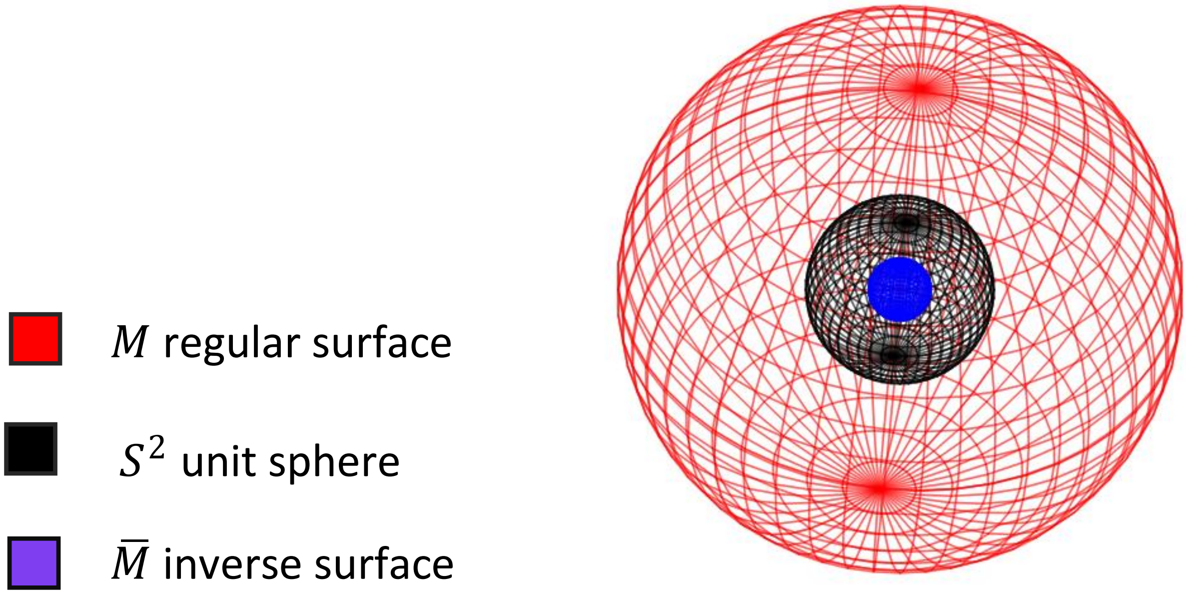

Figure 4.3. The inverse surface of *M* with respect to *s*^{2}.

Example 4. 4: Let us consider the surface M

$$\begin{cases} x = (3+\cos u)\cos v \\ y = (3+\cos u)\sin v \\ z = \sin u \end{cases} \tag{107}$$

This is a torus surface.
Substituting (107) into (54) we obtain

$$\rho^2 = 2(5+3\cos u) \tag{108}$$

Combining (53), (107) and (108) the equation of the inverse surface $\overline{M}$ of M is

$$\overline{X} = \left(\frac{(3+\cos u)\cos v}{2(5+3\cos u)}, \frac{(3+\cos u)\sin v}{2(5+3\cos u)}, \frac{\sin u}{2(5+3\cos u)} \right).$$

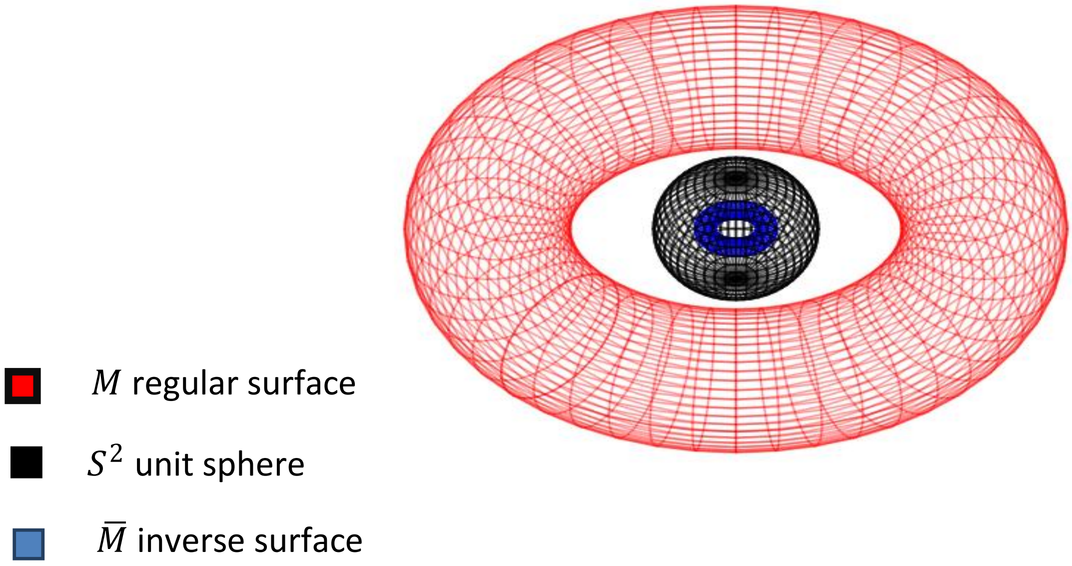

Figure 4.4. The inverse surface of *M* with respect to s^2.

Example 4.5: Let us consider the surface

$$M ... X(u,v) = (u+1, v-3, u^2+v^2). \tag{99}$$

Substituting (99) into (54) we obtain

$$\rho^2 = (u+1)^2 + (v-3)^2 + (u^2+v^2). \tag{100}$$

Thus, by (53) the equation of the inverse surface $\overline{M}$ of M is

$$\overline{M} ... \overline{X} = \frac{1}{(u+1)^2 + (v-3)^2 + (u^2+v^2)} \left(u+1, v-3, u^2+v^2\right). \tag{101}$$

Then we can easily seen that

$$\begin{cases} g_{11} = 1+4u^2 \\ g_{12} = 4uv \\ g_{11} = 1+4v^2 \end{cases} \tag{102}$$

and

$$\begin{cases} b_{11} = \dfrac{2}{\sqrt{1+4\left(u^2+v^2\right)}} \\ b_{12} = 0 \\ b_{22} = \dfrac{2}{\sqrt{1+4\left(u^2+v^2\right)}} \end{cases} \tag{103}$$

Substituting (102) and (103) into (30) and rearranging we have

$$-uvdu^2+\left(u^2-v^2\right)dudv+uvdv^2=0$$

or

$$\left(udv-vdu\right)\left(udu+vdv\right)=0\,. \tag{104}$$

So we have to solve the differential equations

$$udv-vdu=0 \tag{105}$$

and

$$udu+vdv=0. \tag{106}$$

If these equations are solved, $v=cu,\ c=const.,$ and $v=\mp\sqrt{k-u^2},\ k=const.,$ are obtained.

Thus, the curvature lines of M are

$$\alpha(u)=\left(u+1,cu-3,\left(1+c^2\right)u^2\right) \tag{107}$$

and

$$\beta(u)=\left(u+1,\mp\sqrt{k-u^2}-3,k\right) \tag{108}$$

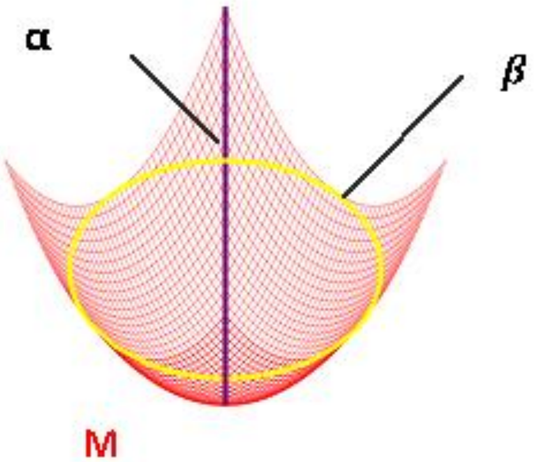

Figure 4.5. (a) The curvature lines on *M*.

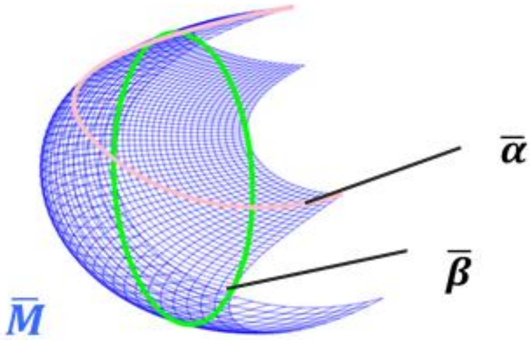

Figure 4.5. (b) The curvature lines on $\overline{M}$.

On the other hand, the curvature lines of $\overline{M}$ are

$$\bar{\alpha}(u)=\frac{1}{(u+1)^2+(cu-3)^2+\left(\left(1+c^2\right)u^2\right)^2}\left(u+1,cu-3,\left(1+c^2\right)u^2\right) \tag{109}$$

and

$$\overline{\beta}(u) = \frac{1}{(u+1)^2 + \left(\mp\sqrt{k-u^2}-3\right)^2 + k^2}\left(u+1, \mp\sqrt{k-u^2}-3, k\right) \tag{110}$$

5. CONCLUSION

In this study, it was discussed geometry of the inverse surface of a regular surface with respect to a unit sphere in E^3. Some characteristic properties of inverse surface were obtained related to the similar characteristic properties of regular surface. Finally, examples supported our assertion are given and their graphs are drawn by Maple 18 software program.

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