*ORIGINAL PAPER*

# GENERATING FUNCTIONS OF EVEN AND ODD GAUSSIAN NUMBERS AND POLYNOMIALS

NABIHA SABA[1], ALI BOUSSAYOUD[1], MOHAMED KERADA[1]

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***Abstract.*** *In this study, we introduce a new class of generating functions of odd and even Gaussian* (*p*,*q*)*-Fibonacci numbers, Gaussian* (*p*,*q*)*-Lucas numbers, Gaussian* (*p*,*q*)*-Pell numbers, Gaussian* (*p*,*q*)*-Pell Lucas numbers, Gaussian Jacobsthal numbers and Gaussian Jacobsthal Lucas numbers and we will recover the new generating functions of some Gaussian polynomials at odd and even terms. The technique used her is based on the theory of the so called symmetric functions.*

***Keywords:*** *Symmetric functions; generating functions; odd and even Gaussian* (*p*,*q*)*-numbers; odd and even Gaussian polynomials.*

***2010 Mathematics Subject Classifications:*** *05E05; 11B39.*

## 1. INTRODUCTION AND PRELIMINARY RESULTS

For $p$ and $q$ positive real numbers, the Gaussian ($p$,$q$)-Fibonacci numbers $\{GF_{p,q,n}\}_{n\geq 0}$ and Gaussian ($p$,$q$)-Lucas numbers $\{GL_{p,q,n}\}_{n\geq 0}$ are defined by the same second-order homogeneous linear recurrence relation:

$$GU_{p,q,n} = pGU_{p,q,n-1} + qGU_{p,q,n-2},\ \forall n \geq 2,$$

but with different conditions $GF_{p,q,0} = i$, $GF_{p,q,1} = 1$ and $GL_{p,q,0} = 2 - ip$, $GL_{p,q,1} = p + 2iq$, (see [1]). These Gaussian ($p$,$q$)-numbers have been introduced as a generalizations of usuel Gaussian Fibonacci numbers $\{GF_n\}_{n\geq 0}$ and Gaussian Lucas numbers $\{GL_n\}_{n\geq 0}$ which were defined in [2, 3]. Obviously, $GF_n = GF_{1,1,n}$ and $GL_n = GL_{1,1,n}$.

The Gaussian ($p$,$q$)-Pell numbers $\{GP_{p,q,n}\}_{n\geq 0}$ [4] is defined recursively as follows, for $p$ and $q$ positive real numbers:

$$GP_{p,q,n} = 2pGP_{p,q,n-1} + qGP_{p,q,n-2},\ \forall n \geq 2,$$

with initial terms $GP_{p,q,0} = i$ and $GP_{p,q,1} = 1$. The Gaussian ($p$,$q$)-Pell Lucas numbers $\{GQ_{p,q,n}\}_{n\geq 0}$ is defined by the same manner but with the initial terms $GQ_{p,q,0} = 2 - 2ip$ and $GQ_{p,q,1} = 2p + 2iq$. The Gaussian ($p$,$q$)-Pell numbers and Gaussian ($p$,$q$)-Pell Lucas numbers

[1]Mohamed Seddik Ben Yahia University, LMAM Laboratory and Department of Mathematics,
Jijel, Algeria.E-mail: sabarnhf1994@gmail.com; aboussayoud@yahoo.fr; mkerad9@gmail.com.

are the natural extensions of Gaussian Pell numbers $\{GP_n\}_{n\geq 0}$ and Gaussian Pell Lucas numbers $\{GQ_n\}_{n\geq 0}$, i.e., $GP_n = GP_{1,1,n}$ and $GQ_n = GQ_{1,1,n}$, these Gaussian numbers ($GP_n$ and $GQ_n$) are defined in the paper [5].

The authors in [6] defined and studied the Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers, they gave some properties and results of them such as generating function, Binet's formula, explicit formula and *Q*-matrix. They also presented explicit combinatorial and determinantal expressions of these numbers. The recurrence relations of these Gaussian numbers given in the following definitions:

**Definition 1.1.** [6] *The Gaussian Jacobsthal numbers, denoted by* $\{GJ_n\}_{n\geq 0}$ *is defined recurrently by:*

$$\begin{cases} GJ_n = GJ_{n-1} + 2GJ_{n-2},\ \forall n \geq 2 \\ GJ_0 = \frac{i}{2},\ GJ_1 = 1 \end{cases}.$$

**Definition 1.2.** [6] *The Gaussian Jacobsthal Lucas numbers, denoted by* $\{Gj_n\}_{n\geq 0}$ *is defined as:*

$$\begin{cases} Gj_n = Gj_{n-1} + 2Gj_{n-2},\ \forall n \geq 2 \\ Gj_0 = 2 - \frac{i}{2},\ Gj_1 = 1 + 2i \end{cases}.$$

On the other hand, many kinds of generalizations of second order recurrence relations of numbers and polynomials have been presented in the literature. For example, Horadam in [7] defined generalized Fibonacci sequence $\{H_n\}_{n\geq 0}$ by:

$$H_n = H_{n-1} + H_{n-2},\ \forall n \geq 3,$$

with $H_1 = p$ and $H_2 = p + q$, where *p* and *q* are arbitrary integers. The same author in [8, 9], presented another generalized Fibonacci sequence. And in [10], the authors made the generalized polynomials of second order sequences as follow:

$$G_n(x) = (p_0 + p_1 x) G_{n-1}(x) + (q_0 + q_1 x) G_{n-2}(x),\ \forall n \geq 2,$$

with $G_0(x) = \alpha_0$ and $G_1(x) = \beta_0 + \beta_1 x$, where $\{p_0, p_1, q_0, q_1, \alpha_0, \beta_0, \beta_1\} \in \mathbb{C}$. The special cases of this sequence are listed as follows:

- If $p_1 = q_0 = \beta_1 = 0,\ p_0 = \beta_0 = 1,\ q_1 = 2$ and $\alpha_0 = \frac{i}{2}$, we get the Gaussian Jacobsthal polynomials [11], known as:

$$\begin{cases} GJ_0(x) = \frac{i}{2}, GJ_1(x) = 1 \\ GJ_n(x) = GJ_{n-1}(x) + 2xGJ_{n-2}(x),\ \forall n \geq 2 \end{cases},$$

or

$$\{GJ_n(x)\}_{n\in\mathbb{N}} = \{\tfrac{i}{2},\ 1,\ 1 + xi,\ 2x + 1 + ix,\ 4x + 1 + ix(2x+1), \ldots\}.$$

- For $p_1 = q_0 = 0,\ p_0 = \beta_0 = 1,\ q_1 = 2,\ \alpha_0 = 2 - \frac{i}{2}$ and $\beta_1 = 2i$ we reduce the Gaussian Jacobsthal Lucas polynomials [11], known as:

$$\begin{cases} Gj_0(x) = 2 - \frac{i}{2}, Gj_1(x) = 1 + 2ix \\ Gj_n(x) = Gj_{n-1}(x) + 2xGj_{n-2}(x),\ \forall n \geq 2 \end{cases},$$

or

$$\{Gj_n(x)\}_{n\in\mathbb{N}} = \{2 - \tfrac{i}{2},\ 1 + 2xi,\ 4x + 1 + ix,\ 6x + 1 + xi(4x + 1),\ 8x^2 + 8x + 1 + xi(6x + 1), ...\}.$$

- In the case when $p_0 = q_1 = \beta_1 = 0,\ q_0 = \beta_0 = 1,\ p_1 = 2$ and $\alpha_0 = i$, we get the Gaussian Pell polynomials [12], known as:

$$\begin{cases} GP_0(x) = i, GP_1(x) = 1 \\ GP_n(x) = 2xGP_{n-1}(x) + GP_{n-2}(x),\ \forall n \geq 2 \end{cases},$$

or

$$\{GP_n(x)\}_{n\in\mathbb{N}} = \{i,\ 1,\ 2x + i,\ 4x^2 + 1 + 2ix,\ 8x^3 + 4x + i(4x^2 + 1), ...\}.$$

In [13], the author defined Gaussian Pell Lucas polynomials as follows:

$$\begin{cases} GQ_0(x) = 2 - 2xi, GQ_1(x) = 2x + 2i \\ GQ_n(x) = 2xGQ_{n-1}(x) + GQ_{n-2}(x),\ \forall n \geq 2 \end{cases},$$

or

$$\{GQ_n(x)\}_{n\in\mathbb{N}} = \{2 - 2ix,\ 2x + 2i,\ 4x^2 + 2 + 2xi,\ 8x^3 + 6x + 2i(2x^2 + 1),$$
$$16x^4 + 16x^2 + 2 + 2xi(4x^2 + 3), ...\}.$$

Next, we recall some backgrounds about the symmetric functions.

**Definition 1.3.** [14] *Let k and n be two positive integers and* $\{e_1, e_2, ..., e_n\}$ *are set of given variables the k-th complete homogeneous symmetric function* $h_k(a_1, a_2, ..., a_n)$ *is defined by:*

$$h_k(e_1, e_2, ..., e_n) = \sum_{i_1 + i_2 + ... + i_n = k} e_1^{i_1} e_2^{i_2} ... e_n^{i_n} \qquad (k \geq 0),$$

*with* $i_1, i_2, ..., i_n \geq 0$.

**Remark 1.1.** *Set* $h_0(e_1, e_2, ..., e_n) = 1$, *by usual convention. For* $k < 0$, *we set* $h_k(e_1, e_2, ..., e_n) = 0$.

**Definition 1.4.** [15, 16] *Let A and E be any two alphabets. We define* $S_n(A - E)$ *by the following form:*

$$\frac{\prod_{e\in E}(1 - ez)}{\prod_{a\in A}(1 - az)} = \sum_{n=0}^{\infty} S_n(A - E) z^n, \tag{1.1}$$

*with the condition* $S_n(A-E)=0$ *for* $n<0$.

Equation (1.1) can be rewritten in the following form:

$$\sum_{n=0}^{\infty} S_n(A-E)z^n = \left(\sum_{n=0}^{\infty} S_n(A)z^n\right) \times \left(\sum_{n=0}^{\infty} S_n(-E)z^n\right),$$

where

$$S_n(A-E) = \sum_{j=0}^{n} S_{n-j}(-E)S_j(A).$$

**Definition 1.5.** [17, 18] *Let n be positive integer and* $E=\{e_1,e_2\}$ *are set of given variables. Then, the n-th symmetric function* $S_n(e_1+e_2)$ *is defined by:*

$$S_n(E)=S_n(e_1+e_2)=\frac{e_1^{n+1}-e_2^{n+1}}{e_1-e_2},$$

*with*

$$\begin{gathered} S_0(E)=S_0(e_1+e_2)=1, \\ S_1(E)=S_1(e_1+e_2)=e_1+e_2, \\ S_2(E)=S_2(e_1+e_2)=e_1^2+e_1e_2+e_2^2, \\ \vdots \end{gathered}$$

**Definition 1.6.** [19, 20] *Given a function f on* $\mathbb{R}^n$, *the divided difference operator is defined as follows:*

$$\partial_{e_i e_{i+1}}(f)=\frac{f(e_1,\cdots,e_i,e_{i+1},\cdots,e_n)-f(e_1,\cdots,e_{i-1},e_{i+1},e_i,e_{i+2},\cdots,e_n)}{e_i-e_{i+1}}.$$

**Definition 1.7.** [21] *Given an alphabet* $E=\{e_1,e_2\}$, *the symmetrizing operator* $\delta_{e_1e_2}^k$ *is defined by:*

$$\delta_{e_1e_2}^k(f)=\frac{e_1^k f(e_1)-e_2^k f(e_2)}{e_1-e_2},\ k\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}=\{0,1,2,3,...\}. \tag{1.2}$$

If $k=0$, the operator (1.2) gives us

$$\begin{aligned} \delta_{e_1e_2}^0(f) &= \frac{f(e_1)-f(e_2)}{e_1-e_2} \\ &= \partial_{e_1e_2}(f). \end{aligned}$$

## 2. MAIN RESULTS

In this part, we are now in a position to provide some new theorems by using the symmetrizing operator $\delta_{e_1e_2}^{k}$ for $k \in \{0,1,2\}$. We now begin with the following theorem.

**Theorem 2.1.** *Given an alphabet* $E = \{e_1, e_2\}$, *we have*

$$\sum_{n=0}^{\infty} S_{2n-1}(E) z^n = \frac{(e_1+e_2)z}{(1-e_1^2 z)(1-e_2^2 z)}. \tag{2.1}$$

*Proof*: By applying the divided difference operator $\partial_{e_1e_2}$ to the series $\sum_{n=0}^{\infty} e_1^{2n} z^n = \frac{1}{1-e_1^2 z}$, we get

$$\partial_{e_1e_2} \sum_{n=0}^{\infty} e_1^{2n} z^n = \partial_{e_1e_2} \frac{1}{1-e_1^2 z}$$

$$\Leftrightarrow \frac{\sum_{n=0}^{\infty} e_1^{2n} z^n - \sum_{n=0}^{\infty} e_2^{2n} z^n}{e_1 - e_2} = \frac{\frac{1}{1-e_1^2 z} - \frac{1}{1-e_2^2 z}}{e_1 - e_2}$$

$$\Leftrightarrow \sum_{n=0}^{\infty} \frac{e_1^{2n} - e_2^{2n}}{e_1 - e_2} z^n = \frac{(1-e_2^2 z) - (1-e_1^2 z)}{(e_1-e_2)(1-e_1^2 z)(1-e_2^2 z)}$$

$$\Leftrightarrow \sum_{n=0}^{\infty} S_{2n-1}(E) z^n = \frac{(e_1+e_2)z}{(1-e_1^2 z)(1-e_2^2 z)}.$$

Hence, we obtain the desired result.

**Theorem 2.2.** *Given an alphabet* $E = \{e_1, e_2\}$, *we have*

$$\sum_{n=0}^{\infty} S_{2n}(E) z^n = \frac{1+e_1e_2 z}{(1-e_1^2 z)(1-e_2^2 z)}. \tag{2.2}$$

*Proof*: By applying the operator $\delta_{e_1e_2}$ to the series $\sum_{n=0}^{\infty} e_1^{2n} z^n = \frac{1}{1-e_1^2 z}$, we get

$$\delta_{e_1e_2} \sum_{n=0}^{\infty} e_1^{2n} z^n = \delta_{e_1e_2} \frac{1}{1-e_1^2 z}$$

$$\Leftrightarrow \frac{e_1\sum_{n=0}^{\infty} e_1^{2n} z^n - e_2\sum_{n=0}^{\infty} e_2^{2n} z^n}{e_1 - e_2} = \frac{\frac{e_1}{1-e_1^2 z} - \frac{e_2}{1-e_2^2 z}}{e_1 - e_2}$$

$$\Leftrightarrow \sum_{n=0}^{\infty} \frac{e_1^{2n+1} - e_2^{2n+1}}{e_1 - e_2} z^n = \frac{e_1(1-e_2^2 z) - e_2(1-e_1^2 z)}{(e_1-e_2)(1-e_1^2 z)(1-e_2^2 z)}$$

$$\Leftrightarrow \sum_{n=0}^{\infty} S_{2n}(E) z^n = \frac{1+e_1e_2 z}{(1-e_1^2 z)(1-e_2^2 z)}.$$

Thus, this completes the proof.

**Theorem 2.3.** *Given an alphabet* $E=\{e_1,e_2\}$, *we have*

$$\sum_{n=0}^{\infty} S_{2n+1}(E)z^n=\frac{e_1+e_2}{(1-e_1^2z)(1-e_2^2z)}. \tag{2.3}$$

*Proof*: By applying the operator $\delta^2_{e_1e_2}$ to the series $\sum_{n=0}^{\infty} e_1^{2n}z^n=\frac{1}{1-e_1^2z}$, we get

$$\delta^2_{e_1e_2}\sum_{n=0}^{\infty} e_1^{2n}z^n=\delta^2_{e_1e_2}\frac{1}{1-e_1^2z}$$

$$\Leftrightarrow \frac{e_1^2\sum_{n=0}^{\infty}e_1^{2n}z^n-e_2^2\sum_{n=0}^{\infty}e_2^{2n}z^n}{e_1-e_2}=\frac{\frac{e_1^2}{1-e_1^2z}-\frac{e_2^2}{1-e_2^2z}}{e_1-e_2}$$

$$\Leftrightarrow \sum_{n=0}^{\infty}\frac{e_1^{2n+2}-e_2^{2n+2}}{e_1-e_2}z^n=\frac{e_1^2(1-e_2^2z)-e_2^2(1-e_1^2z)}{(e_1-e_2)(1-e_1^2z)(1-e_2^2z)}$$

$$\Leftrightarrow \sum_{n=0}^{\infty} S_{2n+1}(E)z^n=\frac{e_1+e_2}{(1-e_1^2z)(1-e_2^2z)}.$$

As required.

## 3. APPLICATIONS OF THEOREMS

In this part, we now consider the previous theorems in order to derive a new generating functions of odd and even Gaussian (*p*,*q*)-Fibonacci numbers, Gaussian (*p*,*q*)-Lucas numbers, Gaussian (*p*,*q*)-Pell numbers, Gaussian (*p*,*q*)-Pell Lucas numbers, Gaussian Jacobsthal numbers and Gaussian Jacobsthal Lucas numbers, and we calculate the new generating functions of Gaussian Pell polynomials, Gaussian Pell Lucas polynomials, Gaussian Jacobsthal polynomials and Gaussian Jacobsthal Lucas polynomials at odd and even terms.

- For the case $E=\{e_1,-e_2\}$ with replacing $e_2$ by $(-e_2)$ in Theorems 2.1, 2.2 and 2.3, we have:

$$\sum_{n=0}^{\infty} S_{2n-1}(e_1+[-e_2])z^n=\frac{(e_1-e_2)z}{1-\left((e_1-e_2)^2+2e_1e_2\right)z+e_1^2e_2^2z^2}, \tag{3.1}$$

$$\sum_{n=0}^{\infty} S_{2n}(e_1+[-e_2])z^n=\frac{1-e_1e_2z}{1-\left((e_1-e_2)^2+2e_1e_2\right)z+e_1^2e_2^2z^2}, \tag{3.2}$$

$$\sum_{n=0}^{\infty} S_{2n+1}(e_1+[-e_2])z^n=\frac{e_1-e_2}{1-\left((e_1-e_2)^2+2e_1e_2\right)z+e_1^2e_2^2z^2}, \tag{3.3}$$

respectively.

*3.1. ORDINARY GENERATING FUNCTIONS OF ODD AND EVEN GAUSSIAN (p,q)-NUMBERS*

This part consists of three cases.

**Case 1.** The substitution of $\begin{cases} e_1 - e_2 = p \\ e_1 e_2 = q \end{cases}$ in Eqs. (3.1), (3.2) and (3.3), we obtain:

$$\sum_{n=0}^{\infty} S_{2n-1}\left(e_1 + \left[-e_2\right]\right) z^n = \frac{pz}{1-\left(p^2+2q\right)z + q^2 z^2}, \tag{3.4}$$

$$\sum_{n=0}^{\infty} S_{2n}\left(e_1 + \left[-e_2\right]\right) z^n = \frac{1-qz}{1-\left(p^2+2q\right)z + q^2 z^2}, \tag{3.5}$$

$$\sum_{n=0}^{\infty} S_{2n+1}\left(e_1 + \left[-e_2\right]\right) z^n = \frac{p}{1-\left(p^2+2q\right)z + q^2 z^2}, \tag{3.6}$$

respectively, and we have the following theorems.

**Theorem 3.1.** *For* $n \in \mathbb{N}$, *the new generating function of even Gaussian* (*p,q*)*-Fibonacci numbers is given by*

$$\sum_{n=0}^{\infty} GF_{p,q,2n} z^n = \frac{i + \left(p - i\left(p^2+q\right)\right)z}{1-\left(p^2+2q\right)z + q^2 z^2}. \tag{3.7}$$

*Proof*: By [1], we have

$$GF_{p,q,n} = iS_n\left(e_1 + \left[-e_2\right]\right) + (1-ip)S_{n-1}\left(e_1 + \left[-e_2\right]\right).$$

Writing $(2n)$ instead of $(n)$, we get

$$GF_{p,q,2n} = iS_{2n}\left(e_1 + \left[-e_2\right]\right) + (1-ip)S_{2n-1}\left(e_1 + \left[-e_2\right]\right).$$

Then,

$$\begin{aligned} \sum_{n=0}^{\infty} GF_{p,q,2n} z^n &= \sum_{n=0}^{\infty}\left(iS_{2n}\left(e_1 + \left[-e_2\right]\right) + (1-ip)S_{2n-1}\left(e_1 + \left[-e_2\right]\right)\right) z^n \\ &= i\sum_{n=0}^{\infty} S_{2n}\left(e_1 + \left[-e_2\right]\right) z^n + (1-ip)\sum_{n=0}^{\infty} S_{2n-1}\left(e_1 + \left[-e_2\right]\right) z^n. \end{aligned}$$

Multiplying the equation (3.5) by $(i)$ and adding it to the equation obtained by (3.4) multiplying by $(1-ip),$ then we obtain the following equality:

$$\sum_{n=0}^{\infty} GF_{p,q,2n} z^n = \frac{i(1-qz)}{1-\left(p^2+2q\right)z+q^2z^2} + \frac{(1-ip)pz}{1-\left(p^2+2q\right)z+q^2z^2}$$
$$= \frac{i+\left(p-i\left(p^2+q\right)\right)z}{1-\left(p^2+2q\right)z+q^2z^2}.$$

Hence, we obtain the desired result.

**Theorem 3.2.** *For* $n \in \mathbb{N}$, *the new generating function of even Gaussian* (*p,q*)*-Lucas numbers is given by*

$$\sum_{n=0}^{\infty} GL_{p,q,2n} z^n = \frac{2-ip+\left(ip\left(p^2+3q\right)-\left(p^2+2q\right)\right)z}{1-\left(p^2+2q\right)z+q^2z^2}. \tag{3.8}$$

*Proof*: In [1], we have $GL_{p,q,n} = (2-ip)S_n\left(e_1+[-e_2]\right)+\left(i\left(p^2+2q\right)-p\right)S_{n-1}\left(e_1+[-e_2]\right).$

By the same method given in Theorem 3.1, the proof can be easily made. So we omit the proof.

**Theorem 3.3.** *For* $n \in \mathbb{N}$, *the new generating function of odd Gaussian* (*p,q*)*-Fibonacci numbers is given by*

$$\sum_{n=0}^{\infty} GF_{p,q,2n+1} z^n = \frac{1+q(ip-1)z}{1-\left(p^2+2q\right)z+q^2z^2}. \tag{3.9}$$

*Proof*: By referred to [1], we have

$$GF_{p,q,n} = iS_n\left(e_1+[-e_2]\right)+(1-ip)S_{n-1}\left(e_1+[-e_2]\right).$$

Substituting $n$ by $(2n+1)$, we obtain

$$GF_{p,q,2n+1} = iS_{2n+1}\left(e_1+[-e_2]\right)+(1-ip)S_{2n}\left(e_1+[-e_2]\right).$$

Then,

$$\sum_{n=0}^{\infty} GF_{p,q,2n+1} z^n = \sum_{n=0}^{\infty}\left(iS_{2n+1}\left(e_1+[-e_2]\right)+(1-ip)S_{2n}\left(e_1+[-e_2]\right)\right)z^n$$
$$= i\sum_{n=0}^{\infty} S_{2n+1}\left(e_1+[-e_2]\right)z^n+(1-ip)\sum_{n=0}^{\infty} S_{2n}\left(e_1+[-e_2]\right)z^n.$$

Multiplying the equation (3.6) by $(i)$ and adding it to the equation obtained by (3.5) multiplying by $(1-ip)$, then we obtain the following equality:

$$\sum_{n=0}^{\infty} GF_{p,q,2n+1} z^n = \frac{ip}{1-\left(p^2+2q\right)z+q^2z^2} + \frac{(1-ip)(1-qz)}{1-\left(p^2+2q\right)z+q^2z^2}$$
$$= \frac{1+q\left(ip-1\right)z}{1-\left(p^2+2q\right)z+q^2z^2}.$$

As required.

**Theorem 3.4.** *For* $n \in \mathbb{N}$, *the new generating function of odd Gaussian* (*p,q*)*-Lucas numbers is given by*

$$\sum_{n=0}^{\infty} GL_{p,q,2n+1} z^n = \frac{p+2iq+q\left(p-i\left(p^2+2q\right)\right)z}{1-\left(p^2+2q\right)z+q^2z^2}. \tag{3.10}$$

*Proof*: Recall from [1] that $GL_{p,q,n} = (2-ip)S_n\left(e_1+[-e_2]\right)+\left(i\left(p^2+2q\right)-p\right)S_{n-1}\left(e_1+[-e_2]\right)$.

By the same method given in Theorem 3.3, the proof can be easily made.

- If we take $p=q=1$ in the Eqs. (3.7), (3.8), (3.9) and (3.10), we get the new generating functions of even and odd Gaussian Fibonacci and Gaussian Lucas numbers. The calculation and results are listed in the Tab.1.

**Table 1. Generating functions for even and odd Gaussian Fibonacci and Gaussian Lucas numbers.**

| Coefficient of $z^n$ | Generating function |
|---|---|
| $GF_{2n}$ | $\frac{i+(1-2i)z}{1-3z+z^2}$ |
| $GL_{2n}$ | $\frac{2-i+(4i-3)z}{1-3z+z^2}$ |
| $GF_{2n+1}$ | $\frac{1+(i-1)z}{1-3z+z^2}$ |
| $GL_{2n+1}$ | $\frac{1+2i+(1-3i)z}{1-3z+z^2}$ |

**Case 2.** The substitution of $\begin{cases} e_1-e_2=2p \\ e_1e_2=q \end{cases}$ in Eqs. (3.1), (3.2) and (3.3), we obtain:

$$\sum_{n=0}^{\infty} S_{2n-1}\left(e_1+[-e_2]\right)z^n = \frac{2pz}{1-2\left(2p^2+q\right)z+q^2z^2}, \tag{3.11}$$

$$\sum_{n=0}^{\infty} S_{2n}\left(e_1+[-e_2]\right)z^n = \frac{1-qz}{1-2\left(2p^2+q\right)z+q^2z^2}, \tag{3.12}$$

$$\sum_{n=0}^{\infty} S_{2n+1}\left(e_1+\left[-e_2\right]\right)z^n = \frac{2p}{1-2\left(2p^2+q\right)z+q^2z^2}, \tag{3.13}$$

respectively, and we have the following theorems.

**Theorem 3.5.** *For* $n \in \mathbb{N}$, *the new generating function of even Gaussian* (*p*,*q*)*-Pell numbers is given by*

$$\sum_{n=0}^{\infty} GP_{p,q,2n} z^n = \frac{i+\left(2p-i\left(4p^2+q\right)\right)z}{1-2\left(2p^2+q\right)z+q^2z^2}. \tag{3.14}$$

*Proof*: Recall that, we have

$$GP_{p,q,n} = iS_n\left(e_1+\left[-e_2\right]\right)+\left(1-2ip\right)S_{n-1}\left(e_1+\left[-e_2\right]\right), \ \left(\text{see } [1]\right).$$

By setting $n=2n$, we get

$$GP_{p,q,2n} = iS_{2n}\left(e_1+\left[-e_2\right]\right)+\left(1-2ip\right)S_{2n-1}\left(e_1+\left[-e_2\right]\right).$$

Then,

$$\begin{aligned}\sum_{n=0}^{\infty} GP_{p,q,2n} z^n &= \sum_{n=0}^{\infty}\left(iS_{2n}\left(e_1+\left[-e_2\right]\right)+\left(1-2ip\right)S_{2n-1}\left(e_1+\left[-e_2\right]\right)\right)z^n \\ &= i\sum_{n=0}^{\infty} S_{2n}\left(e_1+\left[-e_2\right]\right)z^n+\left(1-2ip\right)\sum_{n=0}^{\infty} S_{2n-1}\left(e_1+\left[-e_2\right]\right)z^n.\end{aligned}$$

Multiplying the equation (3.12) by $\left(i\right)$ and adding it to the equation obtained by (3.11) multiplying by $\left(1-2ip\right)$, then we obtain the following equality:

$$\begin{aligned}\sum_{n=0}^{\infty} GP_{p,q,2n} z^n &= \frac{i\left(1-qz\right)}{1-2\left(2p^2+q\right)z+q^2z^2}+\frac{\left(1-2ip\right)2pz}{1-2\left(2p^2+q\right)z+q^2z^2} \\ &= \frac{i+\left(2p-i\left(4p^2+q\right)\right)z}{1-2\left(2p^2+q\right)z+q^2z^2}.\end{aligned}$$

Hence, we obtain the desired result.

**Theorem 3.6.** *For* $n \in \mathbb{N}$, *the new generating function of even Gaussian* (*p*,*q*)*-Pell Lucas numbers is given by*

$$\sum_{n=0}^{\infty} GQ_{p,q,2n} z^n = \frac{2-2ip+\left(2ip\left(4p^2+3q\right)-2\left(2p^2+q\right)\right)z}{1-2\left(2p^2+q\right)z+q^2z^2}. \tag{3.15}$$

*Proof*: By referred to [1], we have

$$GQ_{p,q,n} = (2-2ip)S_n\left(e_1+[-e_2]\right)+\left(i\left(4p^2+2q\right)-2p\right)S_{n-1}\left(e_1+[-e_2]\right).$$

By the same method given in Theorem 3.5, the proof can be easily made.

**Theorem 3.7.** *For* $n\in\mathbb{N}$, *the new generating function of odd Gaussian* (*p,q*)*-Pell numbers is given by*

$$\sum_{n=0}^{\infty} GP_{p,q,2n+1}z^n = \frac{1+q(2ip-1)z}{1-2\left(2p^2+q\right)z+q^2z^2}. \tag{3.16}$$

*Proof*: By referred to [1], we have

$$GP_{p,q,n} = iS_n\left(e_1+[-e_2]\right)+(1-2ip)S_{n-1}\left(e_1+[-e_2]\right), \text{ (see [1]).}$$

By putting $n=2n+1$, we get

$$GP_{p,q,2n+1} = iS_{2n+1}\left(e_1+[-e_2]\right)+(1-2ip)S_{2n}\left(e_1+[-e_2]\right).$$

Then,

$$\begin{aligned}\sum_{n=0}^{\infty} GP_{p,q,2n+1}z^n &= \sum_{n=0}^{\infty}\left(iS_{2n+1}\left(e_1+[-e_2]\right)+(1-2ip)S_{2n}\left(e_1+[-e_2]\right)\right)z^n \\ &= i\sum_{n=0}^{\infty} S_{2n+1}\left(e_1+[-e_2]\right)z^n+(1-2ip)\sum_{n=0}^{\infty} S_{2n}\left(e_1+[-e_2]\right)z^n.\end{aligned}$$

Multiplying the equation (3.13) by (*i*) and adding it to the equation obtained by (3.12) multiplying by $(1-2ip)$, then we obtain the following equality:

$$\begin{aligned}\sum_{n=0}^{\infty} GP_{p,q,2n+1}z^n &= \frac{2ip}{1-2\left(2p^2+q\right)z+q^2z^2}+\frac{(1-2ip)(1-qz)}{1-2\left(2p^2+q\right)z+q^2z^2} \\ &= \frac{1+q(2ip-1)z}{1-2\left(2p^2+q\right)z+q^2z^2}.\end{aligned}$$

This completes the proof.

**Theorem 3.8.** *For* $n\in\mathbb{N}$, *the new generating function of odd Gaussian* (*p,q*)*-Pell Lucas numbers is given by*

$$\sum_{n=0}^{\infty} GQ_{p,q,2n+1}z^n = \frac{2p+2iq+2q\left(p-i\left(2p^2+q\right)\right)z}{1-2\left(2p^2+q\right)z+q^2z^2}. \tag{3.17}$$

*Proof*: It is well-known from [1] that

$$GQ_{p,q,n} = (2-2ip)S_n\left(e_1+[-e_2]\right)+\left(i\left(4p^2+2q\right)-2p\right)S_{n-1}\left(e_1+[-e_2]\right).$$

By the same method given in Theorem 3.7, the proof can be easily made.

- If we take $p=q=1$ in the relationships (3.14), (3.15), (3.16) and (3.17), we get the new generating functions of even and odd Gaussian Pell and Gaussian Pell Lucas numbers. The calculation and results are listed in the Tab.2.

**Table 2. Generating functions for even and odd Gaussian Pell and Gaussian Pell Lucas numbers.**

| **Coefficient of $z^n$** | **Generating function** |
|---|---|
| $GP_{2n}$ | $\dfrac{i+(2-5i)z}{1-6z+z^2}$ |
| $GQ_{2n}$ | $\dfrac{2-2i+2(7i-3)z}{1-6z+z^2}$ |
| $GP_{2n+1}$ | $\dfrac{1+(2i-1)z}{1-6z+z^2}$ |
| $GQ_{2n+1}$ | $\dfrac{2+2i+2(1-3i)z}{1-6z+z^2}$ |

**Case 3.** The substitution of $\begin{cases} e_1-e_2=1 \\ e_1e_2=2 \end{cases}$ in Eqs. (3.1), (3.2) and (3.3), we obtain:

$$\sum_{n=0}^{\infty} S_{2n-1}\left(e_1+[-e_2]\right)z^n = \frac{z}{1-5z+4z^2}, \tag{3.18}$$

$$\sum_{n=0}^{\infty} S_{2n}\left(e_1+[-e_2]\right)z^n = \frac{1-2z}{1-5z+4z^2}, \tag{3.19}$$

$$\sum_{n=0}^{\infty} S_{2n+1}\left(e_1+[-e_2]\right)z^n = \frac{1}{1-5z+4z^2}, \tag{3.20}$$

respectively, and we have the following theorems.

**Theorem 3.9.** *For $n\in\mathbb{N}$, the new generating function of even Gaussian Jacobsthal numbers is given by*

$$\sum_{n=0}^{\infty} GJ_{2n}z^n = \frac{i+(2-3i)z}{2-10z+8z^2}. \tag{3.21}$$

*Proof*: We have

$$GJ_n = \frac{i}{2}S_n\left(e_1+[-e_2]\right)+\left(1-\frac{i}{2}\right)S_{n-1}\left(e_1+[-e_2]\right).$$

Writing $(2n)$ instead of $(n)$, we get

$$GJ_{2n} = \frac{i}{2} S_{2n}\left(e_1 + [-e_2]\right) + \left(1 - \frac{i}{2}\right) S_{2n-1}\left(e_1 + [-e_2]\right).$$

Then,

$$\begin{aligned}\sum_{n=0}^{\infty} GJ_{2n} z^n &= \sum_{n=0}^{\infty} \left( \frac{i}{2} S_{2n}\left(e_1 + [-e_2]\right) + \left(1 - \frac{i}{2}\right) S_{2n-1}\left(e_1 + [-e_2]\right) \right) z^n \\ &= \frac{i}{2} \sum_{n=0}^{\infty} S_{2n}\left(e_1 + [-e_2]\right) z^n + \left(1 - \frac{i}{2}\right) \sum_{n=0}^{\infty} S_{2n-1}\left(e_1 + [-e_2]\right) z^n.\end{aligned}$$

Multiplying the equation (3.19) by $\left(\frac{i}{2}\right)$ and adding it to the equation obtained by (3.18) multiplying by $\left(1 - \frac{i}{2}\right)$, then we obtain the following equality:

$$\begin{aligned}\sum_{n=0}^{\infty} GJ_{2n} z^n &= \frac{i\,(1-2z)}{2\left(1-5z+4z^2\right)} + \frac{(2-i)\,z}{2\left(1-5z+4z^2\right)} \\ &= \frac{i + (2-3i)\,z}{2-10z+8z^2}.\end{aligned}$$

Thus, this completes the proof.

**Theorem 3.10.** *For* $n \in \mathbb{N}$, *the new generating function of even Gaussian Jacobsthal Lucas numbers is given by*

$$\sum_{n=0}^{\infty} Gj_{2n} z^n = \frac{4 - i + (7i - 10)z}{2-10z+8z^2}. \tag{3.22}$$

*Proof*: We know that

$$Gj_n = \left(2 - \frac{i}{2}\right) S_n\left(e_1 + [-e_2]\right) + \left(\frac{5i}{2} - 1\right) S_{n-1}\left(e_1 + [-e_2]\right).$$

By the same method given in Theorem 3.9, the proof can be easily made.

**Theorem 3.11.** *For* $n \in \mathbb{N}$, *the new generating function of odd Gaussian Jacobsthal numbers is given by*

$$\sum_{n=0}^{\infty} GJ_{2n+1} z^n = \frac{2 + 2(i-2)z}{2-10z+8z^2}. \tag{3.23}$$

*Proof*: We have

$$GJ_n = \frac{i}{2} S_n\left(e_1 + [-e_2]\right) + \left(1 - \frac{i}{2}\right) S_{n-1}\left(e_1 + [-e_2]\right).$$

Substituting $n$ by $(2n+1)$, we obtain

$$GJ_{2n+1} = \frac{i}{2} S_{2n+1}\left(e_1 + [-e_2]\right) + \left(1 - \frac{i}{2}\right) S_{2n}\left(e_1 + [-e_2]\right).$$

Then,

$$\begin{aligned}\sum_{n=0}^{\infty} GJ_{2n+1} z^n &= \sum_{n=0}^{\infty} \left( \frac{i}{2} S_{2n+1}\left(e_1 + [-e_2]\right) + \left(1 - \frac{i}{2}\right) S_{2n}\left(e_1 + [-e_2]\right) \right) z^n \\ &= \frac{i}{2} \sum_{n=0}^{\infty} S_{2n+1}\left(e_1 + [-e_2]\right) z^n + \left(1 - \frac{i}{2}\right) \sum_{n=0}^{\infty} S_{2n}\left(e_1 + [-e_2]\right) z^n.\end{aligned}$$

Multiplying the equation (3.20) by $\left(\frac{i}{2}\right)$ and adding it to the equation obtained by (3.19) multiplying by $\left(1 - \frac{i}{2}\right)$, then we obtain the following equality:

$$\begin{aligned}\sum_{n=0}^{\infty} GJ_{2n+1} z^n &= \frac{i}{2\left(1 - 5z + 4z^2\right)} + \frac{(2-i)(1-2z)}{2\left(1 - 5z + 4z^2\right)} \\ &= \frac{2 + 2(i-2)z}{2 - 10z + 8z^2}.\end{aligned}$$

As required.

**Theorem 3.12.** *For $n \in \mathbb{N}$, the new generating function of odd Gaussian Jacobsthal Lucas numbers is given by*

$$\sum_{n=0}^{\infty} Gj_{2n+1} z^n = \frac{2 + 4i + 2(2-5i)z}{2 - 10z + 8z^2}. \tag{3.24}$$

*Proof*: Since

$$Gj_n = \left(2 - \frac{i}{2}\right) S_n\left(e_1 + [-e_2]\right) + \left(\frac{5i}{2} - 1\right) S_{n-1}\left(e_1 + [-e_2]\right).$$

By the same method given in Theorem 3.11, the proof can be easily made.

### *3.2. ORDINARY GENERATING FUNCTIONS OF ODD AND EVEN GAUSSIAN POLYNOMIALS*

This part consists of two cases.

**Case 1.** The substitution of $\begin{cases} e_1 - e_2 = 1 \\ e_1 e_2 = 2x \end{cases}$ in Eqs. (3.1), (3.2) and (3.3), we obtain:

$$\sum_{n=0}^{\infty} S_{2n-1}\left(e_1 + [-e_2]\right) z^n = \frac{z}{1-(4x+1)z+4x^2z^2}, \tag{3.25}$$

$$\sum_{n=0}^{\infty} S_{2n}\left(e_1 + [-e_2]\right) z^n = \frac{1-2xz}{1-(4x+1)z+4x^2z^2}, \tag{3.26}$$

$$\sum_{n=0}^{\infty} S_{2n+1}\left(e_1 + [-e_2]\right) z^n = \frac{1}{1-(4x+1)z+4x^2z^2}, \tag{3.27}$$

respectively, and we have the following theorems.

**Theorem 3.13.** *For $n \in \mathbb{N}$, the new generating function of even Gaussian Jacobsthal polynomials is given by*

$$\sum_{n=0}^{\infty} GJ_{2n}(x) z^n = \frac{i + (2 - i(2x+1))z}{2 - 2(4x+1)z + 8x^2z^2}. \tag{3.28}$$

*Proof*: By [10], we have

$$GJ_n(x) = \frac{i}{2} S_n\left(e_1 + [-e_2]\right) + \left(1 - \frac{i}{2}\right) S_{n-1}\left(e_1 + [-e_2]\right).$$

By setting $n = 2n$, we get

$$GJ_{2n}(x) = \frac{i}{2} S_{2n}\left(e_1 + [-e_2]\right) + \left(1 - \frac{i}{2}\right) S_{2n-1}\left(e_1 + [-e_2]\right).$$

Then,

$$\begin{aligned}\sum_{n=0}^{\infty} GJ_{2n}(x) z^n &= \sum_{n=0}^{\infty} \left( \frac{i}{2} S_{2n}\left(e_1 + [-e_2]\right) + \left(1 - \frac{i}{2}\right) S_{2n-1}\left(e_1 + [-e_2]\right) \right) z^n \\ &= \frac{i}{2} \sum_{n=0}^{\infty} S_{2n}\left(e_1 + [-e_2]\right) z^n + \left(1 - \frac{i}{2}\right) \sum_{n=0}^{\infty} S_{2n-1}\left(e_1 + [-e_2]\right) z^n.\end{aligned}$$

Multiplying the equation (3.26) by $\left(\frac{i}{2}\right)$ and adding it to the equation obtained by (3.25) multiplying by $\left(1-\frac{i}{2}\right)$, then we obtain the following equality:

$$\sum_{n=0}^{\infty} GJ_{2n}(x)z^n = \frac{i(1-2xz)}{2\left(1-(4x+1)z+4x^2z^2\right)} + \frac{(2-i)z}{2\left(1-(4x+1)z+4x^2z^2\right)}$$
$$= \frac{i+\left(2-i(2x+1)\right)z}{2-2(4x+1)z+8x^2z^2}.$$

Hence, we obtain the desired result.

**Theorem 3.14.** *For $n \in \mathbb{N}$, the new generating function of even Gaussian Jacobsthal Lucas polynomials is given by*

$$\sum_{n=0}^{\infty} Gj_{2n}(x)z^n = \frac{4-i+\left(i(6x+1)-2(4x+1)\right)z}{2-2(4x+1)z+8x^2z^2}. \tag{3.29}$$

*Proof*: We know that

$$Gj_n(x) = \left(2-\frac{i}{2}\right)S_n(e_1+[-e_2]) + \left(i\left(2x+\frac{1}{2}\right)-1\right)S_{n-1}(e_1+[-e_2]), \text{ (see [10]).}$$

By the same method given in Theorem 3.13, the proof can be easily made.

**Theorem 3.15.** *For $n \in \mathbb{N}$, the new generating function of odd Gaussian Jacobsthal polynomials is given by*

$$\sum_{n=0}^{\infty} GJ_{2n+1}(x)z^n = \frac{2+2x(i-2)z}{2-2(4x+1)z+8x^2z^2}. \tag{3.30}$$

*Proof*: By referred to [10], we have

$$GJ_n(x) = \frac{i}{2}S_n(e_1+[-e_2]) + \left(1-\frac{i}{2}\right)S_{n-1}(e_1+[-e_2]).$$

By putting $n = 2n+1$, we get

$$GJ_{2n+1}(x) = \frac{i}{2}S_{2n+1}(e_1+[-e_2]) + \left(1-\frac{i}{2}\right)S_{2n}(e_1+[-e_2]).$$

Then,

$$\sum_{n=0}^{\infty} GJ_{2n+1}(x)z^n = \sum_{n=0}^{\infty}\left(\frac{i}{2}S_{2n+1}(e_1+[-e_2]) + \left(1-\frac{i}{2}\right)S_{2n}(e_1+[-e_2])\right)z^n$$
$$= \frac{i}{2}\sum_{n=0}^{\infty} S_{2n+1}(e_1+[-e_2])z^n + \left(1-\frac{i}{2}\right)\sum_{n=0}^{\infty} S_{2n}(e_1+[-e_2])z^n.$$

Multiplying the equation (3.27) by $\left(\frac{i}{2}\right)$ and adding it to the equation obtained by (3.26) multiplying by $\left(1-\frac{i}{2}\right)$, then we obtain the following equality:

$$\sum_{n=0}^{\infty} GJ_{2n+1}(x)z^n = \frac{i}{2\left(1-(4x+1)z+4x^2z^2\right)} + \frac{(2-i)(1-2xz)}{2\left(1-(4x+1)z+4x^2z^2\right)}$$
$$= \frac{2+2x(i-2)z}{2-2(4x+1)z+8x^2z^2}.$$

As required.

**Theorem 3.16.** *For* $n \in \mathbb{N}$, *the new generating function of odd Gaussian Jacobsthal Lucas polynomials is given by*

$$\sum_{n=0}^{\infty} Gj_{2n+1}(x)z^n = \frac{2+4ix+2x\left(2-i(4x+1)\right)z}{2-2(4x+1)z+8x^2z^2}. \tag{3.31}$$

*Proof*: Once more, by [10] we have

$$Gj_n(x) = \left(2-\frac{i}{2}\right)S_n\left(e_1+[-e_2]\right) + \left(i\left(2x+\frac{1}{2}\right)-1\right)S_{n-1}\left(e_1+[-e_2]\right).$$

By the same method given in Theorem 3.15, the proof can be easily made. So we omit the proof.

**Case 2.** The substitution of $\begin{cases} e_1 - e_2 = 2x \\ e_1 e_2 = 1 \end{cases}$ in Eqs. (3.1), (3.2) and (3.3), we obtain:

$$\sum_{n=0}^{\infty} S_{2n-1}\left(e_1+[-e_2]\right)z^n = \frac{2xz}{1-2\left(2x^2+1\right)z+z^2}, \tag{3.32}$$

$$\sum_{n=0}^{\infty} S_{2n}\left(e_1+[-e_2]\right)z^n = \frac{1-z}{1-2\left(2x^2+1\right)z+z^2}, \tag{3.33}$$

$$\sum_{n=0}^{\infty} S_{2n+1}\left(e_1+[-e_2]\right)z^n = \frac{2x}{1-2\left(2x^2+1\right)z+z^2}, \tag{3.34}$$

respectively, and we have the following theorems.

**Theorem 3.17.** *For* $n \in \mathbb{N}$, *the new generating function of even Gaussian Pell polynomials is given by*

$$\sum_{n=0}^{\infty} GP_{2n}(x)z^n = \frac{i+\left(2x-i\left(4x^2+1\right)\right)z}{1-2\left(2x^2+1\right)z+z^2}. \tag{3.35}$$

*Proof*: Recall that, we have

$$GP_n(x) = iS_n\left(e_1+[-e_2]\right)+(1-2ix)S_{n-1}\left(e_1+[-e_2]\right), \text{ (see [10]).}$$

Writing $(2n)$ instead of $(n)$, we obtain

$$GP_{2n}(x) = iS_{2n}\left(e_1+[-e_2]\right)+(1-2ix)S_{2n-1}\left(e_1+[-e_2]\right).$$

Then,

$$\begin{aligned}\sum_{n=0}^{\infty} GP_{2n}(x)z^n &= \sum_{n=0}^{\infty}\left(iS_{2n}\left(e_1+[-e_2]\right)+(1-2ix)S_{2n-1}\left(e_1+[-e_2]\right)\right)z^n \\ &= i\sum_{n=0}^{\infty} S_{2n}\left(e_1+[-e_2]\right)z^n+(1-2ix)\sum_{n=0}^{\infty} S_{2n-1}\left(e_1+[-e_2]\right)z^n.\end{aligned}$$

Multiplying the equation (3.33) by $(i)$ and adding it to the equation obtained by (3.32) multiplying by $(1-2ix)$, then we obtain the following equality:

$$\begin{aligned}\sum_{n=0}^{\infty} GP_{2n}(x)z^n &= \frac{i(1-z)}{1-2\left(2x^2+1\right)z+z^2}+\frac{(1-2ix)2xz}{1-2\left(2x^2+1\right)z+z^2} \\ &= \frac{i+\left(2x-i\left(4x^2+1\right)\right)z}{1-2\left(2x^2+1\right)z+z^2}.\end{aligned}$$

Hence, we obtain the desired result.

**Theorem 3.18.** *For $n \in \mathbb{N}$, the new generating function of even Gaussian Pell Lucas polynomials is given by*

$$\sum_{n=0}^{\infty} GQ_{2n}(x)z^n = \frac{2-2ix+\left(2ix\left(4x^2+3\right)-2\left(2x^2+1\right)\right)z}{1-2\left(2x^2+1\right)z+z^2}. \tag{3.36}$$

*Proof*: We have

$$GQ_n(x) = (2-2ix)S_n\left(e_1+[-e_2]\right)+\left(i\left(4x^2+2\right)-2x\right)S_{n-1}\left(e_1+[-e_2]\right).$$

By the same method given in Theorem 3.17, the proof can be easily made.

**Theorem 3.19.** *For $n \in \mathbb{N}$, the new generating function of odd Gaussian Pell polynomials is given by*

$$\sum_{n=0}^{\infty} GP_{2n+1}(x) z^n = \frac{1+(2ix-1)z}{1-2(2x^2+1)z+z^2}. \tag{3.37}$$

*Proof*: In [10], we have

$$GP_n(x) = iS_n(e_1+[-e_2]) + (1-2ix) S_{n-1}(e_1+[-e_2]).$$

Substituting $n$ by $(2n+1)$, we obtain

$$GP_{2n+1}(x) = iS_{2n+1}(e_1+[-e_2]) + (1-2ix) S_{2n}(e_1+[-e_2]).$$

Then,

$$\sum_{n=0}^{\infty} GP_{2n+1}(x) z^n = \sum_{n=0}^{\infty} \left( iS_{2n+1}(e_1+[-e_2]) + (1-2ix) S_{2n}(e_1+[-e_2]) \right) z^n$$

$$= i\sum_{n=0}^{\infty} S_{2n+1}(e_1+[-e_2]) z^n + (1-2ix) \sum_{n=0}^{\infty} S_{2n}(e_1+[-e_2]) z^n.$$

Multiplying the equation (3.34) by $(i)$ and adding it to the equation obtained by (3.33) multiplying by $(1-2ix)$, then we obtain the following equality:

$$\sum_{n=0}^{\infty} GP_{2n+1}(x) z^n = \frac{2ix}{1-2(2x^2+1)z+z^2} + \frac{(1-2ix)(1-z)}{1-2(2x^2+1)z+z^2}$$
$$= \frac{1+(2ix-1)z}{1-2(2x^2+1)z+z^2}.$$

Which completes the proof.

**Theorem 3.20.** *For $n \in \mathbb{N}$, the new generating function of odd Gaussian Pell Lucas polynomials is given by*

$$\sum_{n=0}^{\infty} GQ_{2n+1}(x) z^n = \frac{2x+2i+\left(2x-i\left(4x^2+2\right)\right)z}{1-2(2x^2+1)z+z^2}. \tag{3.38}$$

*Proof*: Recall that, we have

$$GQ_n(x) = (2-2ix) S_n(e_1+[-e_2]) + \left(i\left(4x^2+2\right)-2x\right) S_{n-1}(e_1+[-e_2]).$$

By the same method given in Theorem 3.19, the proof can be easily made. So we omit the proof.

## 4. CONCLUSION

In this paper, we have derived three Theorems (2.1, 2.2 and 2.3) by making use of symmetrizing operator given by Definition 1.7. By making use these theorems, we have obtained theorems which are led to generating function for a class of odd and even Gaussian numbers and polynomials.

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