ORIGINAL PAPER

ON THE GENERATING FUNCTION FOR BERNSTEIN POLYNOMIALS OF TRIPLE SEQUENCES

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Abstract. *The aim of this paper is to give main properties of the generating function of the Bernstein polynomials of triple sequence spaces. It was proved the recurrence relations and derivative formula for Bernstein polynomials of triple sequences. Further more, some new results are obtained by using this generating function of these polynomials.*

Keywords: *Natural density; triple sequences; generating function; Bernstein polynomials.*

1. INTRODUCTION

In this paper, we study on generating function of Bernstein polynomials and their properties. Bernstein polynomials play an important role in the area of approximation theory and the other areas of mathematics. They also play an important role in physics. Thus now, we give the definition and important properties of these polynomials

(1) They are non negative overthe interval $[a, b]$.

(2) They are symmetric. It can be written as

$$B_{rst}(f,x) = B_{r-m,s-n,t-k}\big(a+b-a(f,x)\big)$$

(3) Being continuous each polynomial has only one maximum over the interval $[a, b]$ at $x = a + \frac{(b-a)m}{r} + \frac{(b-a)n}{s} + \frac{(b-a)k}{t}$ by the interval value theorem.

(4) The set of these polynomials of degree (r, s, t) forms a partition of unity as

$$\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} B_{rst}(f,x) = 1.$$

(5) A Bernstein polynomaials can always be written as a linear combination of polynomials of higher order as

$$B_{(r-1)(s-1)(t-1)}(f,x) = \left(\frac{r-m}{r}\right)\left(\frac{s-n}{s}\right)\left(\frac{t-k}{t}\right)B_{rst}(f,x) + \left(\frac{m+1}{r}\right)\left(\frac{n+1}{s}\right)\left(\frac{k+1}{t}\right)B_{rst}(f,x).$$

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There are $(rst)^{th}$ degree Bernstein polynomials. For mathematical convention, we usually set $B_{rst}=0$ if $m,n,k<0$ or $m>r,n>s,k>t$. Moreover, the Bernstein polynomials can be defined interms of forward differences as follows

$$(f,x)=\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\Delta^{mnk}f(0)\binom{r}{m}\binom{s}{n}\binom{t}{k}x^{m+n+k}(1-x)^{(m-r)+(n-s)+(k-t)}.$$

Let K be a subset of the set of positive integers $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$, and let us denote the set $\{(m,n,k)\in K: m\leq u, n\leq v, k\leq w\}$ by K_{uvw}. Then the natural density of K is given by $\delta(K)=lim_{u,v,w\to\infty}\frac{|K_{uvw}|}{uvw}$, where $|K_{uvw}|$ denotes the number of elements in K_{uvw}. Clearly, a finite subset has natural density zero, and we have $\delta(K^c)=1-\delta(K)$ where $K^c=\mathbb{N}\backslash K$ is the complement of K. If $K_1\subseteq K_2$, then $\delta(K_1)\leq\delta(K_2)$.

The Bernstein operator of order rst is given by

$$B_{rst}(f,x)=\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}f\left(\frac{mnk}{rst}\right)\binom{r}{m}\binom{s}{n}\binom{t}{k}x^{m+n+k}(1-x)^{(m-r)+(n-s)+(k-t)}$$

where f is a continuous (real or complex valued) function defined on $[0,1]$.

Throughout the paper, $\mathbb{R}$ denotes the real of three dimensional space with metric (X,d). Consider a triple sequence $x=(x_{mnk})$ such that $x_{mnk}\in\mathbb{R}, m,n,k\in\mathbb{N}$.

Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials $\left(B_{rst}(f,x)\right)$ is said to be statistically convergent to $0\in\mathbb{R}$, written as $st-lim\quad x=0$, provided that the set

$$K_\varepsilon:=\{(m,n,k)\in\mathbb{N}^3: |B_{mnk}(f,x)-f(x)|\geq\varepsilon\}$$

has natural density zero for any $\varepsilon>0$. In this case, 0 is called the statistical limit of the triple sequence of Bernstein polynomials. i.e., $\delta(K_\varepsilon)=0$. That is,

$$lim_{r,s,t\to\infty}\frac{1}{r,s,t}|\{\text{m≤r, n≤s, k≤t}: |B_{mnk}(f,x)-f(x)|\geq\varepsilon\}|=0.$$

In this case, we write $\delta-limB_{mnk}(f,x)=f(x)$ or $B_{mnk}(f,x)\to^{S_B}f(x)$.

Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials $\left(B_{rst}(f,x)\right)$ is said to be statistically analytic if there exists a positive number M such that

$$\delta\left(\{(m,n,k)\in\mathbb{N}^3: |B_{mnk}(f,x)-f(x)|^{1/m+n+k}\geq M\}\right)=0$$

That is,

$$lim_{r,s,t\to\infty}\frac{1}{rst}\left|\{\text{m≤r, n≤s, k≤t}: |B_{mnk}(f,x)-f(x)|^{1/m+n+k}\geq M\}\right|=0.$$

A triple sequence (real or complex) can be defined as a function $x:\mathbb{N}\times\mathbb{N}\times\mathbb{N}\to\mathbb{R}(\mathbb{C})$, where $\mathbb{N},\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by Aiyub et al. [1], Esi et al. [2-5], Dutta et al. [6], Debnath et al. [7], Sahiner et al. [8-9], Sharma et al. [10], Subramanian et al. [11-20] and many others.

2. ANALYSIS OF METHOD

The Bernstein polynomials of degree rst can be defined by blending together two Bernstein polynomials of degree $n = 1$. That is, the $(\mathrm{mnk})^{th}(rst)^{th}$ degree Bernstein polynomials can be written as

$$B_{rst}(x) = (1 - x)B_{mnk,r-1,s-1,t-1}(x) + xB_{(m-1,n-1,k-1),(r-1,s-1,t-1)}(x).$$

for $\mathrm{m} \leq \mathrm{r}, \mathrm{n} \leq \mathrm{s}, \mathrm{k} \leq \mathrm{t}$ and the derivative of the $(rst)^{th}$ degree Bernstein polynomials are also polynomials of degree $(r - 1, s - 1, t - 1)$ and they are defined as follows:

$$\frac{d}{dx}B_{mnk,rst}(x) = rst\left[B_{(m-1,n-1,k-1),(r-1,s-1,k-1)}(x) - B_{mnk,(r-1,s-1,k-1)}(x)\right]$$

The Bernstein polynomials of degree rst are defined by

$$B_{mnk,rst}(f,x) = \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} f\left(\frac{mnk}{rst}\right)\binom{r}{m}\binom{s}{n}\binom{t}{k} x^{m+n+k}(1 - x)^{(r-m)+(s-n)+(t-k)}$$

for

$$m = 0,1,2,\cdots,\mathrm{r};\ n = 0,1,2,\cdots,\mathrm{s}; k = 0,1,2,\cdots,\mathrm{t}$$

where

$$\binom{r}{m}\binom{s}{n}\binom{t}{k} = \frac{r!}{m!\,(r-m)!}\frac{s!}{n!\,(s-n)!}\frac{r!}{t!\,(t-k)!}.$$

There are $(r + 1)(s + 1)(t + 1), (r, s, t)^{th}$ degree Bernstein polynomials. for mathematical convenience we usually set $B_{mnk,rst} = 0$ for $m, n, k < 0$ or $m > r, n > s, k > t$. These polynomials are quite easy to write down the coefficients $\binom{r}{m}\binom{s}{n}\binom{t}{k}$ can be obtained from Pascal's triangle, the exponents on the x term increase by on as m, n, k increases, and the exponents on the $(1 - x)$ term decrease by one as m, n, k increases. In the simple cases, we obtain:

The triple sequence of Bernstein polynomials of degree 1 is

$$B_{000,111}(x) = (1 - x)^3,$$

$$B_{111,111}(x) = x^3$$

and figure as follow:

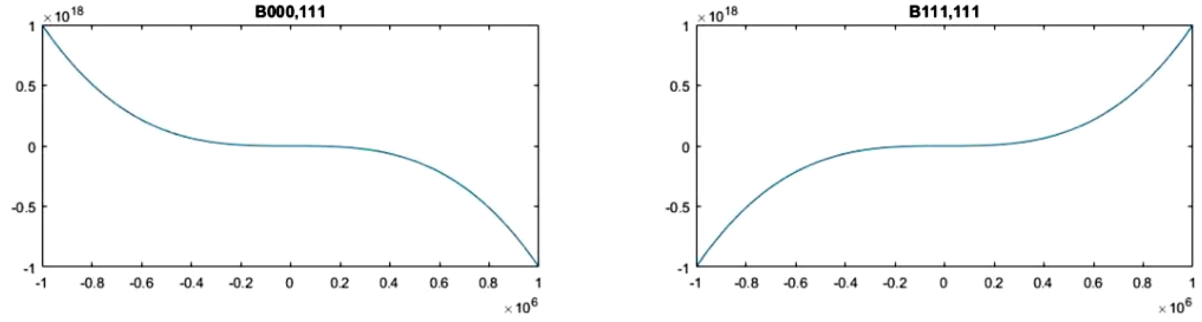

Figure 1. The triple sequence of Bernstein polynomials of degree 1.

The triple sequence of Bernstein polynomials of degree 2 are

$$B_{000,222}(x) = (1-x)^6,$$

$$B_{111,222}(x) = 6x^3(1-x)^3,$$

$$B_{222,222}(x) = x^6$$

and figure as follow:

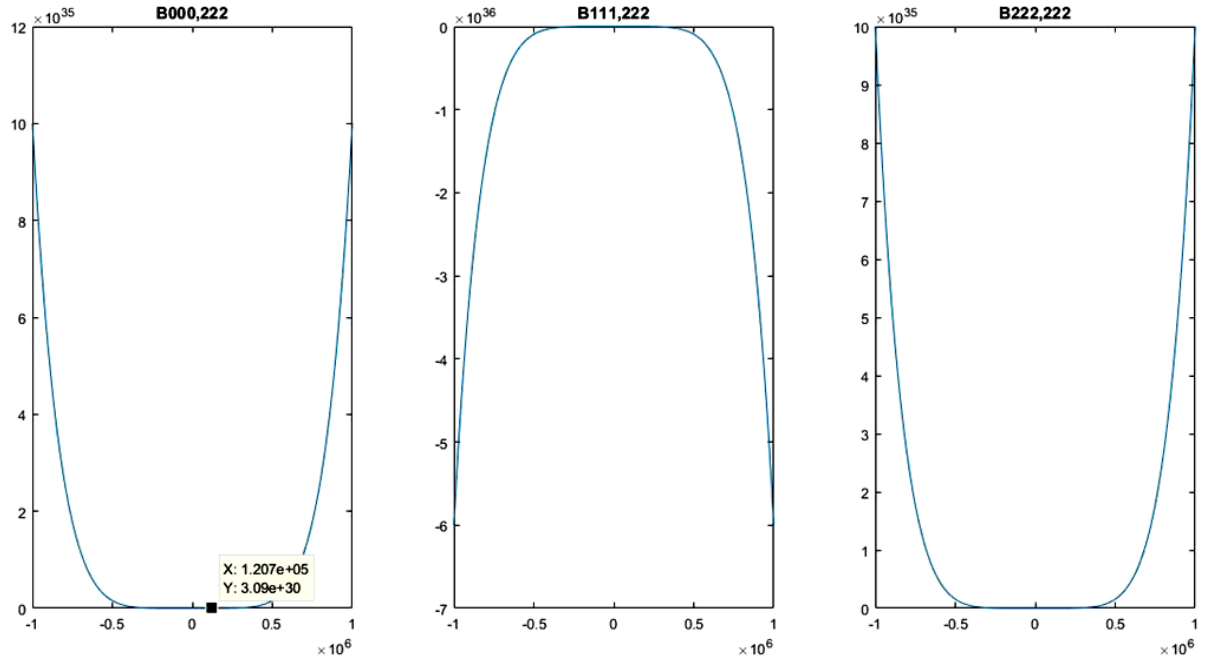

Figure 2. The triple sequence of Bernstein polynomials of degree 2.

The triple sequence of Bernstein polynomials of degree 3 are

$$B_{000,333}(x) = (1-x)^9,$$

$$B_{111,333}(x) = 9x^3(1-x)^6,$$

$$B_{222,333}(x) = 9x^6(1-x)^3,$$

$$B_{333,333}(x) = x^9$$

and figures as follows:

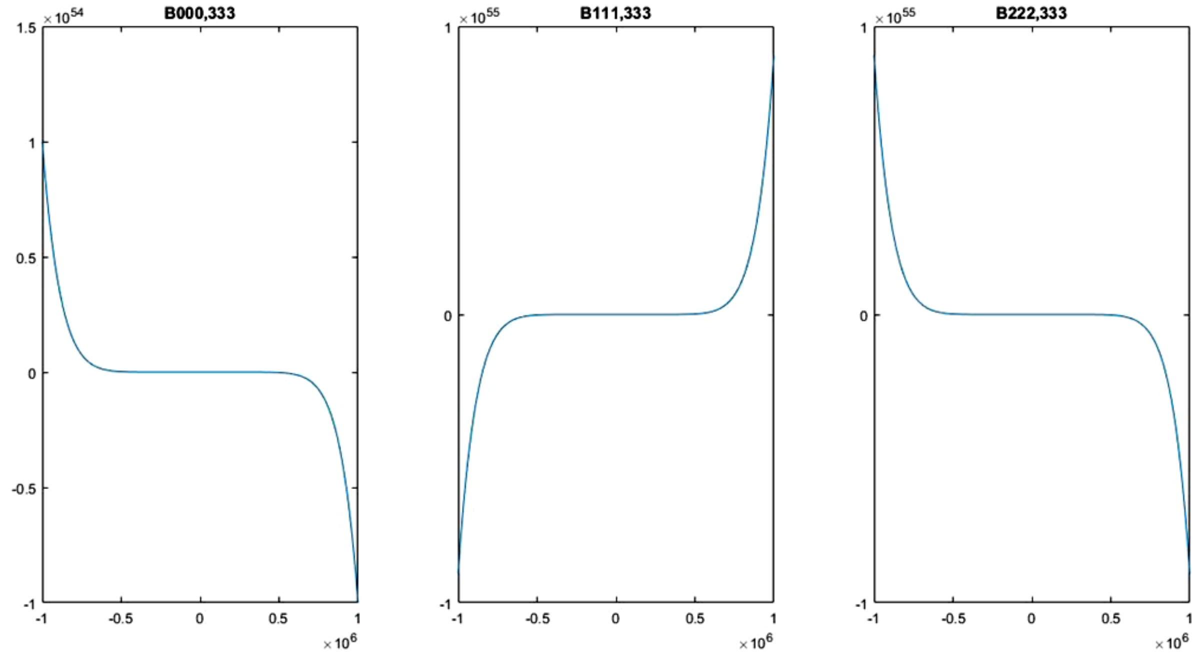

Figure 3. The triple sequence of Bernstein polynomials of degree 3.

3. MAIN RESULTS

Bernoulli polynomials can be defined by their generating functions.

Now, we shall study on the generating function of Bernstein polynomials of triple sequence in the form.

Theorem. The triple sequence of Bernstein polynomials, $B_{mnk,rst}(x)$, have the following generating function $F_{mnk}(x,t) = \frac{(tx)^{m+n+k}e^{t}}{m!n!k!e^{tx}}$

$$F_{mnk}(x,t) = \sum_{r=m}^{\infty}\sum_{s=n}^{\infty}\sum_{t=k}^{\infty} B_{mnk,rst}(x)\frac{t^{r+s+t}}{r!\,s!\,t!}, m,n,k \in N_0, t \in \mathbb{C} \tag{3.1}$$

Proof: By using the definition of generating function we have

$$\sum\nolimits_{r=m}^{\infty}\sum\nolimits_{s=n}^{\infty}\sum\nolimits_{t=k}^{\infty} B_{mnk,rst}(x)\frac{t^{r+s+t}}{r!\,s!\,t!} =$$

$$= \sum_{r=m}^{\infty}\sum_{s=n}^{\infty}\sum_{t=k}^{\infty}\binom{r}{m}\binom{s}{n}\binom{t}{k}x^{m+n+k}(1-x)^{(r-m)+(s-n)+(t-k)}\frac{t^{r+s+t}}{(rst)!}$$
$$= \sum_{r=m}^{\infty}\sum_{s=n}^{\infty}\sum_{t=k}^{\infty}\frac{r!}{m!\,(r-m)!}\frac{s!}{n!\,(s-n)!}\frac{t!}{k!\,(t-k)!}x^{m+n+k}$$
$$(1-x)^{(r-m)+(s-n)+(t-k)}\frac{t^{m+n+k}t^{(r-m)+(s-n)+(t-k)}}{r!\,s!\,t!}$$
$$= \sum_{r=m}^{\infty}\sum_{s=n}^{\infty}\sum_{t=k}^{\infty}\frac{(tx)^{m+n+k}(t-tx)^{(r-m)+(s-n)+(t-k)}}{m!\,n!\,k!\,(r-m)!\,(s-n)!\,(t-k)!}$$
$$= \sum_{r=m}^{\infty}\sum_{s=n}^{\infty}\sum_{t=k}^{\infty}\frac{(tx)^{m+n+k}(t-tx)^{(r-m)+(s-n)+(t-k)}}{(mnk)!\,(r-m)!\,(s-n)!\,(t-k)!}$$
$$= \sum_{r=m}^{\infty}\sum_{s=n}^{\infty}\sum_{t=k}^{\infty}\frac{(tx)^{m+n+k}}{m!\,n!\,k!}\left(\frac{e^{t}}{e^{tx}}\right).$$

By using the generating function we find the recurrence relation and derivative of the generating function of Bernstein polynomials. Now, we can obtain all Bernstein polynomials as follows: Here the first five such polynomials respectively, are

$$B_{000,000}(x) = 1, B_{000,111}(x) = (1-x)^3$$

and

$$B_{111,111}(x) = x^3, B_{000,222}(x) = (1-x)^6, B_{111,222}(x) = 6x^3(1-x)^3$$

and

$$B_{222,222}(x) = x^6, B_{000,333}(x) = (1-x)^9, B_{111,333}(x) = 9x^3(1-x)^6, B_{222,333}(x) = 9x^6(1-x)^3$$

and

$$B_{333,333}(x) = x^9, B_{000,555}(x) = (1-x)^{15}, B_{111,555}(x) = 125x^3(1-x)^{12}, B_{222,555}(x) = 1000x^6(1-x)^9, B_{333,555}(x) = 1000x^9(1-x)^6, B_{444,555}(x) = 125x^{12}(1-x)^3$$

and

$$B_{555,555}(x) = x^{15}.$$

If we differentiate the generating function of Bernstein polynomials given by (3.1) with respect to x, we obtain the relation as follows:

$$\frac{\partial}{\partial x}F_{mnk}(x,t) = \frac{(tx)^{m+n+k}e^{t}}{m!\,n!\,k!\,e^{xt}}$$
$$= \frac{t^{m+n+k}e^{t}}{m!\,n!\,k!}\frac{x^{m+n+k}}{e^{xt}}$$
$$= \frac{t^{m+n+k}e^{t}}{m!\,n!\,k!}\left[\frac{e^{xt}(m+n+k)x^{(m-1)+(n-1)+(k-1)} - x^{m+n+k}e^{xt}t}{(e^{xt})^2}\right]$$
$$= \frac{t^{m+n+k}e^{t}}{m!\,n!\,k!}\left[\frac{e^{xt}}{(e^{xt})^2}\left[(m+n+k)x^{(m-1)+(n-1)+(k-1)} - x^{m+n+k}t\right]\right]$$

$$= \frac{t^{m+n+k}e^t}{m!\,n!\,k!}\left[\frac{1}{e^{xt}}\left[(m+n+k)x^{(m-1)+(n-1)+(k-1)} - x^{m+n+k}t\right]\right]$$
$$= \frac{(m+n+k)t^{m+n+k}e^t x^{(m-1)+(n-1)+(k-1)}}{m!\,n!\,k!\,e^{xt}} - \frac{t^{m+n+k}e^t t x^{m+n+k}}{m!\,n!\,k!\,e^{xt}}$$
$$= t\left[\frac{(m+n+k)(xt)^{(m-1)+(n-1)+(k-1)}e^t}{(mnk)!\,e^{xt}} - \frac{(tx)^{m+n+k}e^t}{(mnk)!\,e^{xt}}\right]$$
$$= t\left[\frac{3(xt)^{(m-1)+(n-1)+(k-1)}e^t}{(m-1)!\,(n-1)!\,(k-1)!\,e^{xt}} - \frac{(tx)^{m+n+k}e^t}{m!\,n!\,k!\,e^{xt}}\right]$$

$$\frac{\partial}{\partial x}F_{mnk}(x,t) = t\big[3F_{m-1,n-1,k-1}(x,t) - F_{mnk}(x,t)\big].$$

or

$$\sum\nolimits_{r=m}^{\infty}\sum\nolimits_{s=n}^{\infty}\sum\nolimits_{t=k}^{\infty} B_{mnk,rst}(x)\frac{t^{r+s+t}}{r!\,s!\,t!}$$
$$= \sum\nolimits_{r=m}^{\infty}\sum\nolimits_{s=n}^{\infty}\sum\nolimits_{t=k}^{\infty}(r+s$$
$$+t)\left[\frac{\left[B_{m-1n-1k-1,r-1s-1t-1}(x) - B_{mnk,r-1s-1t-1}\right]t^{r+s+t}}{(rst)(r-1)!\,(s-1)!\,(t-1)!}\right]$$

$$\sum\nolimits_{r=m}^{\infty}\sum\nolimits_{s=n}^{\infty}\sum\nolimits_{t=k}^{\infty} B_{mnk,rst}(x)\frac{t^{r+s+t}}{r!\,s!\,t!} = t\big[3F_{m-1,n-1,k-1}(x,t) - F_{mnk}(x,t)\big].$$

Thus we have

$$\sum\nolimits_{r=m}^{\infty}\sum\nolimits_{s=n}^{\infty}\sum\nolimits_{t=k}^{\infty} B_{mnk,rst}(x)\frac{t^{r+s+t}}{r!\,s!\,t!} = t\big[3F_{m-1,n-1,k-1}(x,t) - F_{mnk}(x,t)\big]$$

which is a derivative property of the generating function of Bernstein polynomials. So we can write the following recurrence relation $t\frac{\partial}{\partial x}F_{mnk}(x,t) = \big[3F_{m-1,n-1,k-1}(x,t) - F_{mnk}(x,t)\big]$.

4. CONCLUSION

In this paper, we give main properties of the generating function of the Bernstein polynomials of triple sequence spaces and proved recurrence relations and derivative formula for Bernstein polynomials of triple sequences. In addition some new results are obtained by using this generating function of these polynomials.

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