ORIGINAL PAPER

A MATRIX PRESENTATION OF HIGHER ORDER DERIVATIVES OF BÉZIER CURVE AND SURFACE

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Abstract. *In this study, the Bézier curves and surfaces, which have an important place in interactive design applications, are expressed in matrix form using a special matrix that gives the coefficients of the Bernstein base polynomial. The matrix forms of higher order derivatives of the Bézier curves and surfaces are obtained. It is demonstrated by numerical examples that the bidirectional transition between the control points and parametric equations of the Bézier curves and surfaces can be easily achieved using these matrix forms. In addition, it is demonstrated that this type of curve and surface, whose control points are known, its higher order derivatives can be calculated without it's parametric equations. In this study, the Bézier curves and surfaces are presented in a more easily understandable and easy to use format in algebraic form for designers.*

Keywords: *Bézier curves; Bézier surfaces; the matrix form; the derivative for Bézier.*

1. INTRODUCTION AND PRELIMINARY INFORMATION

A curve generation is very important for computer aided geometric design. It is not easy to transfer a curve that can simply be drawn on a paper to a computer using a mathematical function that a computer can understand. Even more difficult is to be able to design a formula that represents a curve. The Bézier curves, which are introduced to theoretical mathematics by famous mathematician Bernstein long before computers, are used as a technique of generating curves in computer environment. The works of Pierre Bézier and Paul de Casteljau make these curves indispensable for graphic professionals. Although these two engineers got the same results separately from each other for automobile design based on Bernstein polynomials since the first article on the subject is published by Bézier this curve is known as the Bézier curve. Recently, this type of curve has attracted the attention of mathematicians and many studies have been done on the geometric properties of these curves [1-3]. Even different types of these curves have been studied in spaces with different metrics such as Minkowski space [4-6]. In his study, Floater tried to obtain on the basis of the equation with the help of Casteljau's algorithm high order derivatives of a Bézier curve [7].

A Bézier curve is defined with the control points and a polynomial function that combines them. The first and last control point selected is the start and end point of the curve, the curve passes through these points. The other points in between are used to determine the structure of the curve, and the curve moves towards these points, but usually it does not pass through these points. The number of control points is 1 more than the degree of the parametric equation of the curve. The shape that formed by sequential joining of control points is called descriptive shape and the Bézier curve is formed on the convex side of this shape. A Bézier

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curve can be expressed as a combination of the Bézier curves where the last control point of one is equal to the first control point of the other, or it can be divided into pieces by the same idea.

The Bézier surfaces are defined with the control points and a two-parameter polynomial function that connects them. Some features of the Bézier curve also apply to the Bézier surface. The control points are connected consecutively using the line segments. Thus, the resulting shape can be called the control point grid. The Bézier surface generally does not touch the control points outside the corners of this grid; it is located in the convex body of the grid. Also, the Bézier surface matches the surface creadted by a Bézier curve passing through the control points on that edge along the edges of the grid patch. The closed surfaces can be created if the first and last control points of a Bézier surface are taken equal. Bernstein polynomials are used also in solutions of differential equations. The approximate solution of the linear differential equation with variable coefficient, which characterizes a special curve relative to Frenet frame, is found by the matrix sorting method developed by Bernstein polynomial approach. In this solution, a special matrix is obtained to express Bernstein polynomial coefficients [8, 9]. In this study, first, the Bézier curves are defined in matrix form with the help of this special matrix used in the solution method we developed. It is shown that the parametric equation of a Bézier curve given with the control points can be found with this matrix form. Then, by proving that this special matrix is a reversible matrix, a matrix equality that can be used to find the control points of a Bézier curve given with the parametric equation is obtained. Secondly, a matrix form is obtained for the higher order derivatives of a Bézier curve. It is shown that the k*th* order derivative of a Bézier curve given with the control points can be calculated without its own parametric equation with the help of this matrix form. In previous studies, the Bézier curves are expressed in the matrix form with different method, but there is no similar study for higher order derivatives.

Finally, a matrix form is created for the Bézier surfaces and their derivatives. It is shown that the bidirectional transition between the parametric equation and the control points of the Bézier surfaces can be achieved with this form. Also, matrix relations are given for higher order partial derivatives of the Bézier surfaces. It is shown that its higher order partial derivatives without calculating the parametric equation of a surface given with the control points can be found using this relation.

The purpose of this study is to contribute to the algebraic direction of this type of curve and surface.

2. THE MATRIX FORM OF THE BÉZIER CURVES

Definition. The Bézier curve $P(t)$ of n*th* degree with the control point of (n + 1) is a parametric function and it is defined with

$$P(t) = \sum_{i=0}^{n} B_{i,n}(t)\, P_i, \qquad 0 \le t \le 1 \tag{1}$$

where $B_{i,n}(t) = \binom{n}{i} t^i (1-t)^{n-i}$ symbolizes Bernstein base polynomial in the range $[0,1]$ and $P_i \in R^{\alpha}, \alpha \le n$ are the control points of the Bézier curve. Here i is the index and t is the parameter [10]. It is obvious that a Bézier curve is expressed as a linear combination of Bernstein base polynomial and the vector-valued coefficients called poles or the control points. The Bézier curve defined by (1) be expressed in the matrix form as follows:

$$\boldsymbol{P}(t)=\boldsymbol{B_n}(t)\boldsymbol{P} \tag{2}$$

for $\boldsymbol{B_n}(t)=[B_{0,n}(t) \quad B_{1,n}(t) \quad \cdots \quad B_{n,n}(t)]$ and $\boldsymbol{P}=[P_0 \quad P_1 \quad \cdots \quad P_n]^T$.

Let the following matrix definitions be made in order to calculate this derivative with the method we developed. It is expressed primarily in the format of

$$\boldsymbol{B_n}(t)=\boldsymbol{T}(t)\boldsymbol{D}^T, \tag{3}$$

where the matrix $\boldsymbol{D}$ is defined as

$$\boldsymbol{D}=\begin{bmatrix} d_{00} & d_{01} & \cdots & d_{0n} \\ d_{10} & d_{11} & \cdots & d_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n0} & d_{n1} & \cdots & d_{nn} \end{bmatrix}, d_{ij}=\begin{cases} (-1)^{j-i}\binom{n}{i}\binom{n-i}{j-i} & ; \quad i \le j \\ 0 & ; \quad i > j \end{cases}$$

and $\boldsymbol{T}(t)=[1 \quad t \quad \cdots \quad t^n]$. So the equality (2) can be rewritten as follows:

$$\boldsymbol{P}(t)=\boldsymbol{T}(t)\boldsymbol{D}^T\boldsymbol{P}. \tag{4}$$

The matrix $\boldsymbol{D}$ defined and used here can be used in equations and characterizations of different curve types defined with Bernstein polynomials in different ranges. For a curve defined with Bernstein base polynomials in any a, b range, the elements of the matrix $\boldsymbol{D}$ are defined in the form of

$$d_{ij}=\begin{cases} \dfrac{(-1)^{j-i}}{R^j}\dbinom{n}{i}\dbinom{n-i}{j-i} & ; \quad i \le j \\ 0 & ; \quad i > j \end{cases}$$

where the R being the maximum breadth of the range. Since a Bézier curve is defined at $t \in [0,1]$, R = 1 is used in the study.

It is clear that the defined matrix $\boldsymbol{D}$ is the upper triangular square matrix, and the matrix $\boldsymbol{D}^T$ is the lower triangular square matrix. Since $\prod_{i=0}^{n} d_{ii} \neq 0,$ for $\forall i \in R$, $Det\ (\boldsymbol{D}) \neq 0$ and $Det\ (\boldsymbol{D}^T) \neq 0$ can be obtained. From this it is seen that matrices $\boldsymbol{D}$ and $\boldsymbol{D}^T$ are invertible matrices.

Lemma: The transition between a Bézier curve expressed in the form of $\boldsymbol{P}(t)=\boldsymbol{T}(t)\boldsymbol{C}$ and its control points is achieved by means of the matrix equality $\boldsymbol{C}=\boldsymbol{D}^T\boldsymbol{P}$.

Proof: The Bézier curve $P(t)$ can be written in format $\boldsymbol{P}(t)=\boldsymbol{T}(t)\boldsymbol{C}$ with the help of (4). Here, $\boldsymbol{C}$ is coefficients matrix and $\boldsymbol{T}(t)$ can be called force matrix. In this case; the equality

$$\boldsymbol{C}=\boldsymbol{D}^T\boldsymbol{P}$$

can be obtained. It is clear that this obtained matrix equality provide the transition between a Bézier curve given with t^i force-based and its control points. In other words, using this equality, by calculating the force coefficients of a Bézier curve given with the control points, it's parametric equation can be obtained. On the other hand;

$$\boldsymbol{P}=(\boldsymbol{D^T})^{-1}\boldsymbol{C}$$

can be written. This is an indication that the control points of a Bézier curve of n*th* degrees can be calculated with the help of this matrix equality.

Example. Let’s take the Bézier curve $P(t) = \big(P_1(t), P_2(t)\big)$ in R^2 for

$P_1(t) = -91t^9 + 518t^8 - 1296t^7 + 1932t^6 - 1764t^5 + 126t^4 - 336t^3 + 36t^2 + 27t - 10P_2(t) = (-159t^9 + 1877t^8 - 2124t^7 + 2184t^6 - 1008t^5 + 378t^4 + 18t$.

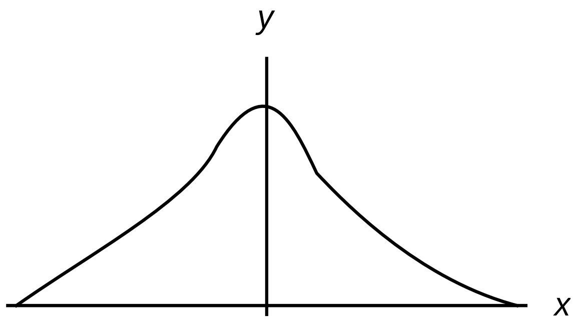

Figure 1. The graph of the given curve

Let's calculate the control points of the curve $P(t)$ whose graph is given in Fig. 1. The Mathematica program is used for calculation throughout the study. The coefficients matrix of the curve $P(t) = (P_1(t), P_2(t))$ is as follows:

$$\boldsymbol{C} = \begin{bmatrix} -10 & 0 \\ 27 & 18 \\ 36 & 0 \\ -336 & 0 \\ 126 & 378 \\ -1764 & -1008 \\ 1932 & 2184 \\ -1296 & -2124 \\ 518 & 1877 \\ -91 & -159 \end{bmatrix}.$$

Since the curve is 9*th* degree, for $n = 9$ the matrix $(\boldsymbol{D}^{\boldsymbol{T}})^{-1}$ is calculated and the matrix

$$\boldsymbol{P} = \begin{bmatrix} -10 & 0 \\ -7 & 2 \\ -3 & 4 \\ -2 & 6 \\ 0 & 9 \\ 1 & 7 \\ 2 & 5 \\ 3 & 4 \\ 6 & 1 \\ 10 & 0 \end{bmatrix}$$

is found by using the equality $\boldsymbol{P} = (\boldsymbol{D}^{\boldsymbol{T}})^{-1}\boldsymbol{C}$. So the control points of the given curve $P(t)$ are obtained as follows:

$$P_0 = (-10,0), P_1 = (-7,2), P_2 = (-3,4), P_3 = (-2,6), P_4 = (0,9), P_5 = (1,7), P_6 = (2,5), P_7 = (3,4), P_8 = (6,1), P_9 = (10,0)$$

Example. Let's find the parametric expression of the Bézier curve $P(t)$ given with the control points $P_0 = (18,0,0), P_1 = (18,18,0), P_2 = (20,20,4), P_3 = (10,10,25), P_4 = (5,25,30)$, $P_5 = (10,10,35), P_6 = (-20,10,35), P_7 = (-10,-10,20), P_8 = (0,8,18), P_9 = (10,18,25)$.

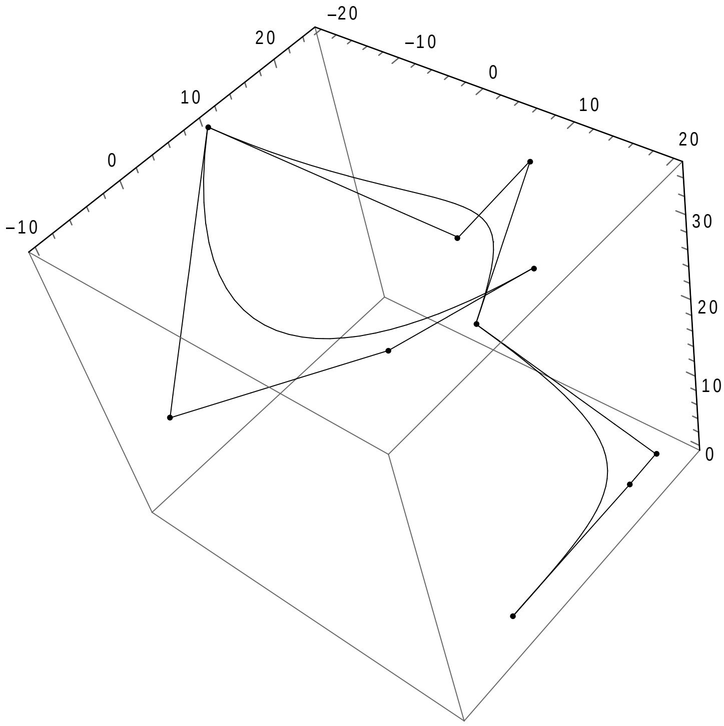

Figure 2. The graph of the curve given with the control points

The graph of the curve $P(t)$ given with the control points is plotted in Fig. 2. Since the curve is given with 10 control points, its degree is $n = 9$ and since the control points have 3 components, it is $P(t) \in R^3$ and expressed as $P(t) = (P_1(t), P_2(t), P_3(t))$.

The transpose of the matrix D is obtained as follows:

$$D^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -9 & 9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 36 & -72 & 36 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -84 & 252 & -252 & 84 & 0 & 0 & 0 & 0 & 0 & 0 \\ 126 & -378 & 756 & -504 & 126 & 0 & 0 & 0 & 0 & 0 \\ -126 & 630 & -1260 & 1260 & -630 & 126 & 0 & 0 & 0 & 0 \\ 84 & -504 & 1260 & -1680 & 1260 & -504 & 84 & 0 & 0 & 0 \\ -36 & 252 & -756 & 1260 & -1260 & 756 & -252 & 36 & 0 & 0 \\ 9 & -56 & 252 & -504 & 630 & -378 & 252 & -72 & 9 & 0 \\ -1 & 9 & -36 & 84 & -126 & 126 & -84 & 36 & -9 & 1 \end{bmatrix}$$

for $n = 9$. The matrix C is calculated as

$$\boldsymbol{C} = \begin{bmatrix} 18 & 0 & 0 \\ 0 & 162 & 0 \\ 72 & -576 & 144 \\ -1176 & 336 & 1092 \\ 6174 & 6426 & -5796 \\ -5418 & -15750 & 11970 \\ 420 & 26628 & -13860 \\ 7308 & -24804 & 9036 \\ -5796 & 14274 & 1620 \\ 2224 & -2862 & 229 \end{bmatrix}$$

by using the $\boldsymbol{C} = \boldsymbol{D}^T\boldsymbol{P}$. The parametric equation of this curve $P(t)$ given with the control points in R^3 is obtained as

$$P(t) = (2224t^9 - 5796t^8 + 7308t^7 + 420t^6 - 5418t^5 + 6174t^4 - 1176t^3 + 72t^2 + 18, -2862t^9 + 14274t^8 - 24804t^7 + 26628t^6 - 15750t^5 + 6426t^4 + 336t^3 - 576t^2 + 162t, 229t^9 + 1620t^8 + 9036t^7 - 13860t^6 + 11970t^5 - 5796t^4 + 1092t^3 + 144t^2\ ).$$

3. THE MATRIX FORM FOR HIGHER ORDER DERIVATIVES OF THE BÉZIER CURVE

The k*th* order derivative of a Bézier curve $P(t)$ represented by (2) can be written in the form

$$\boldsymbol{P}^{(\boldsymbol{k})}(t) = \boldsymbol{B}_{\boldsymbol{n}}^{(\boldsymbol{k})}(t)\boldsymbol{P}. \tag{5}$$

Here,

$$\boldsymbol{B}_n^{(k)}(t) = \boldsymbol{T}^{(\boldsymbol{k})}(t)\boldsymbol{D}^T \tag{6}$$

and for

$$\boldsymbol{T}^{(k)}(t) = \boldsymbol{T}(t)\boldsymbol{B}^k \tag{7}$$

the matrix $\boldsymbol{B}$ is defined as follows:

$$\boldsymbol{B} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Thus, since $\boldsymbol{B}_n^{(k)}(t) = \boldsymbol{T}(t)\boldsymbol{B}^k\boldsymbol{D}^T$, the equation (5) is obtained as

$$\boldsymbol{P}^{(k)}(t) = \boldsymbol{T}(t)\boldsymbol{B}^k\boldsymbol{D}^T\boldsymbol{P}. \tag{8}$$

This is the expression in matrix form of the k*th* order derivative of a Bézier curve.

Let the k*th* derivative of the Bézier curve $P(t)$ is written as $\boldsymbol{P}^{(k)}(t) = \boldsymbol{T}(t)\boldsymbol{W}$ in equality (8). The matrix $\boldsymbol{T}(t)$ is called the force matrix, and the matrix $\boldsymbol{W}$ is called the coefficient matrix of the k*th* derivative of this curve. In this case,

$$\boldsymbol{W} = \boldsymbol{B}^k \boldsymbol{D}^T \boldsymbol{P} .$$

Using this matrix equation developed by the method used here, the force coefficients of the k*th* derivative of a Bézier curve $P(t)$ given with the control points can be obtained.

Example. Let's find the 3*rd* derivative of the Bézier curve $P(t)$ given with the control points $P_0 = (15,0,0), P_1 = (15,15,0), P_2 = (25,25,4), P_3 = (12,12,20), P_4 = (-5,25,10)$.

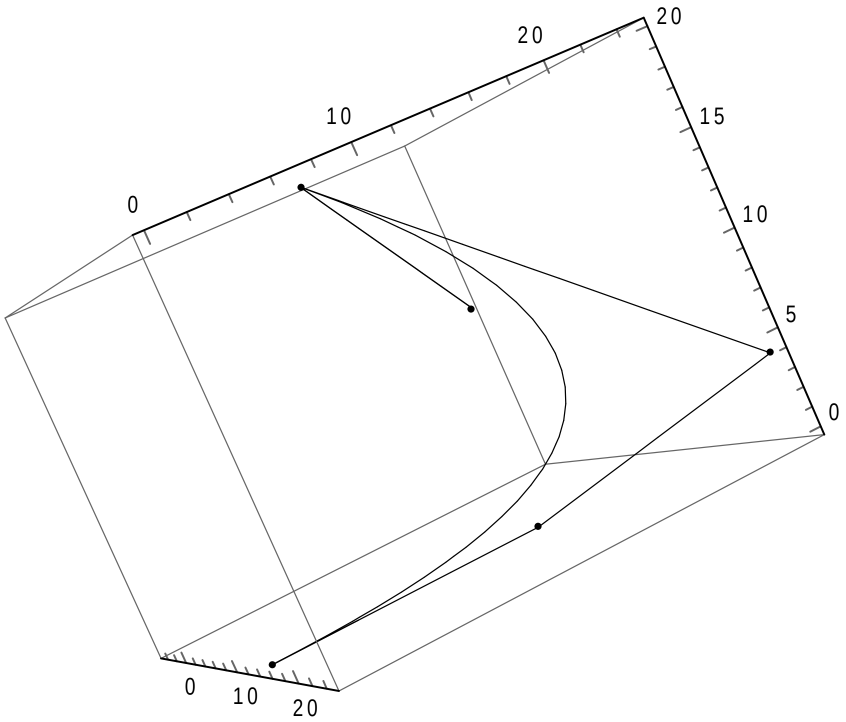

Figure 3. The graph of the given curve

The graph of the curve $P(t)$ given the control points is plotted in Fig. 3. Since the curve is given with 5 control points, its degree is n = 4 and since the control points have 3 components, it becomes $P(t) \in R^3$ and is expressed as $P(t) = (P_1(t), P_2(t), P_3(t))$. The coefficient matrix of the 3*rd* derivative of the curve $P(t)$ becomes $\boldsymbol{W} = \boldsymbol{B}^3 \boldsymbol{D}^T \boldsymbol{P}$. We find the matrices

$$\boldsymbol{D}^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 12 & -12 & 4 & 0 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix}, \qquad \boldsymbol{B}^3 = \begin{bmatrix} 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 24 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

for n = 4 and calculate the coefficient matrix $\boldsymbol{W}$ as follows:

$$\boldsymbol{W}=\begin{bmatrix}-1512 & -432 & 192\\ 2208 & 1608 & -1104\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

The equation

$$\boldsymbol{P}^{(3)}(t)=(2208t-1512,1608t-432,-1104t+192)$$

is obtained with the help of the equality $\boldsymbol{P}^{(3)}(t)=\boldsymbol{T}(t)\boldsymbol{W}$. As can be seen, the $3rd$ derivative of the $4th$ degree curve is obtained from the $1st$ degree as it should be. It has been demonstrated that a desired curve can be found without finding its own equation with this method.

Example. Let's find the $2nd$ derivative of the Bézier curve $P(t)$ given with the control points $P_0=(-20,0), P_1=(-16,-5), P_2=(-10,-8), P_3=(0,8), P_4=(5,10)$.

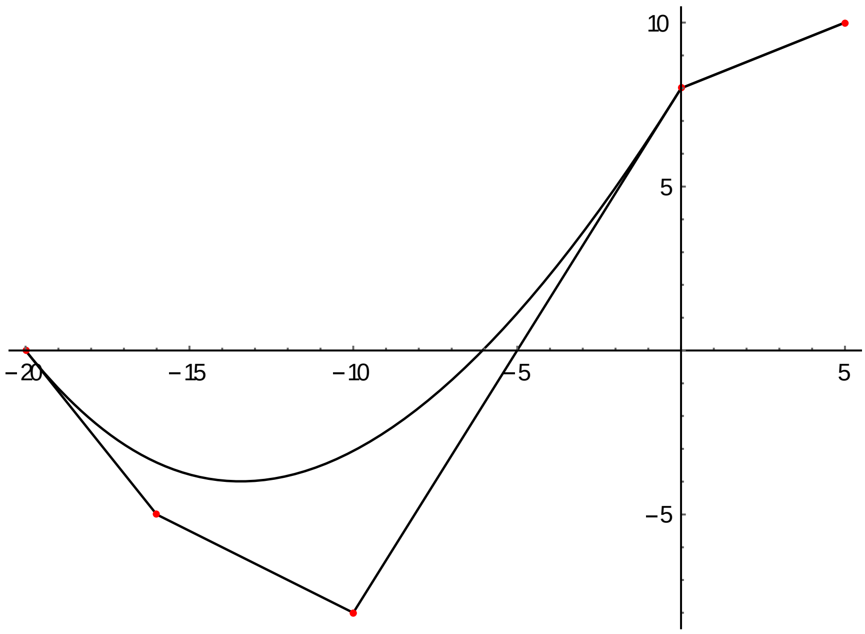

Figure 4. The graph of the given curve

The graph of the curve $P(t)$ given with the control points is plotted in Fig. 4. Since the curve is given with 5 control points, its degree is $n=4$ and since the control points have 2 components, it becomes $P(t)\in R^2$ and is expressed as $P(t)=(P_1(t),P_2(t))$. The coefficient matrix of the second derivative of the curve $P(t)$ becomes $\boldsymbol{W}=\boldsymbol{B}^2\boldsymbol{D}^T\boldsymbol{P}$. We find the matrices

$$\boldsymbol{D}^T=\begin{bmatrix}1 & 0 & 0 & 0 & 0\\ -4 & 4 & 0 & 0 & 0\\ 6 & -12 & 6 & 0 & 0\\ -4 & 12 & -12 & 4 & 0\\ 1 & -4 & 6 & -4 & 1\end{bmatrix}, \boldsymbol{B}^2=\begin{bmatrix}0 & 0 & 2 & 0 & 0\\ 0 & 0 & 0 & 6 & 0\\ 0 & 0 & 0 & 0 & 12\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix},$$

for n = 4 and k = 2, and calculate the coefficient matrix $\boldsymbol{W}$ as follows:

$$\boldsymbol{W} = \begin{bmatrix} 24 & 24 \\ 48 & 408 \\ -132 & -600 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The second derivative of the given curve is obtained as $P^{(2)}(t) = (-132t^2 + 48t + 24, -600t2+400t+24$ with the help of the equality $\boldsymbol{P}(2)t=\boldsymbol{T}t\boldsymbol{W}$.

4. THE MATRIX FORM OF THE BÉZIER SURFACES

Definition. A Bézier surface, that its degree is (n, m), with $(n+1).(m+1)$ grain control point is expressed parametrically in the form of

$$P(t_1, t_2) = \sum_{i=0}^{n} \sum_{j=0}^{m} P_{ij} B_{i,n}(t_1) B_{j,m}(t_2), \qquad 0 \leq t_1, t_2 \leq 1. \tag{9}$$

$P_{ij} \in R^{\alpha}$ are the control points of the Bézier surface [11]. The value of α determines the belonging of the Bézier surface. For example, for $\alpha = 4$, the surface lies in a 4-dimensional space. The Bézier surfaces defined by (9) can be expressed in the form

$$\boldsymbol{P}(t_1, t_2) = \boldsymbol{B}_n \boldsymbol{Q}_m \boldsymbol{P} \tag{10}$$

for the matrices defined as

$$\boldsymbol{B_n} = [B_{0,n}(t_1) \quad B_{1,n}(t_1) \quad \cdots \quad B_{n,n}(t_1)], \boldsymbol{Q_m} = \begin{bmatrix} \boldsymbol{B}_m & 0 & \cdots & 0 \\ 0 & \boldsymbol{B}_m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{B}_m \end{bmatrix},$$

$$\boldsymbol{P} = [P_{00} \quad P_{01} \quad \cdots \quad P_{0m} \quad P_{10} \quad \cdots \quad P_{1m} \quad \cdots \quad P_{nm}]^T.$$

For the absence of confusion, the matrix $\boldsymbol{D}$ calculated for the n-indexed Bernstein polynomial is represented by $\boldsymbol{D}_n$ and the matrix $\boldsymbol{D}$ is calculated for the m-indexed Bernstein polynomial $\boldsymbol{D}_m$. The expressions

$$\boldsymbol{B_n} = \boldsymbol{T_1} \boldsymbol{D_n}^T \text{and } \boldsymbol{Q_m} = \overline{\boldsymbol{T}}_2 \overline{\boldsymbol{D}}$$

are obtained with the help of

$$\boldsymbol{T_1} = \boldsymbol{T}(t_1) = [1 \quad t_1 \quad \cdots \quad t_1^n], \overline{\boldsymbol{T}}_2 = \begin{bmatrix} \boldsymbol{T_2} & 0 & \cdots & 0 \\ 0 & \boldsymbol{T_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \boldsymbol{0} & \cdots & \boldsymbol{T_2} \end{bmatrix}$$

$$\boldsymbol{T_2} = \boldsymbol{T}(t_2) = [1 \quad t_2 \quad \cdots \quad t_2^m], \overline{\boldsymbol{D}} = \begin{bmatrix} D_m^T & 0 & \cdots & 0 \\ 0 & D_m^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \boldsymbol{0} & \cdots & D_m^T \end{bmatrix}.$$

So the equality (10) can be rewritten as follows:

$$\boldsymbol{P}(t_1, t_2) = \boldsymbol{T_1}(\boldsymbol{D_n})^T \, \overline{\boldsymbol{T}}_2 \overline{\boldsymbol{D}} \boldsymbol{P}. \tag{11}$$

It is clear that this obtained matrix equation provides the transition between the paramatric expression of a Bézier surface and its control points. In other words, with this equation, the parametric equation can be obtained by calculating the force coefficients of a Bézier surface given with the control points.

Example. Let's find the parametric equation of the Bézier surface formed by the control points

$$P_{00} = (-10,-10,10), P_{01} = (50,-10,-45), P_{02} = (80,-10,50), P_{10} = (-10,40,20), P_{11} = (90,50,-35), P_{12} = (110,40,110), P_{20} = (-10,90,-10), P_{21} = (50,110,25), P_{22} = (80,90,50)$$

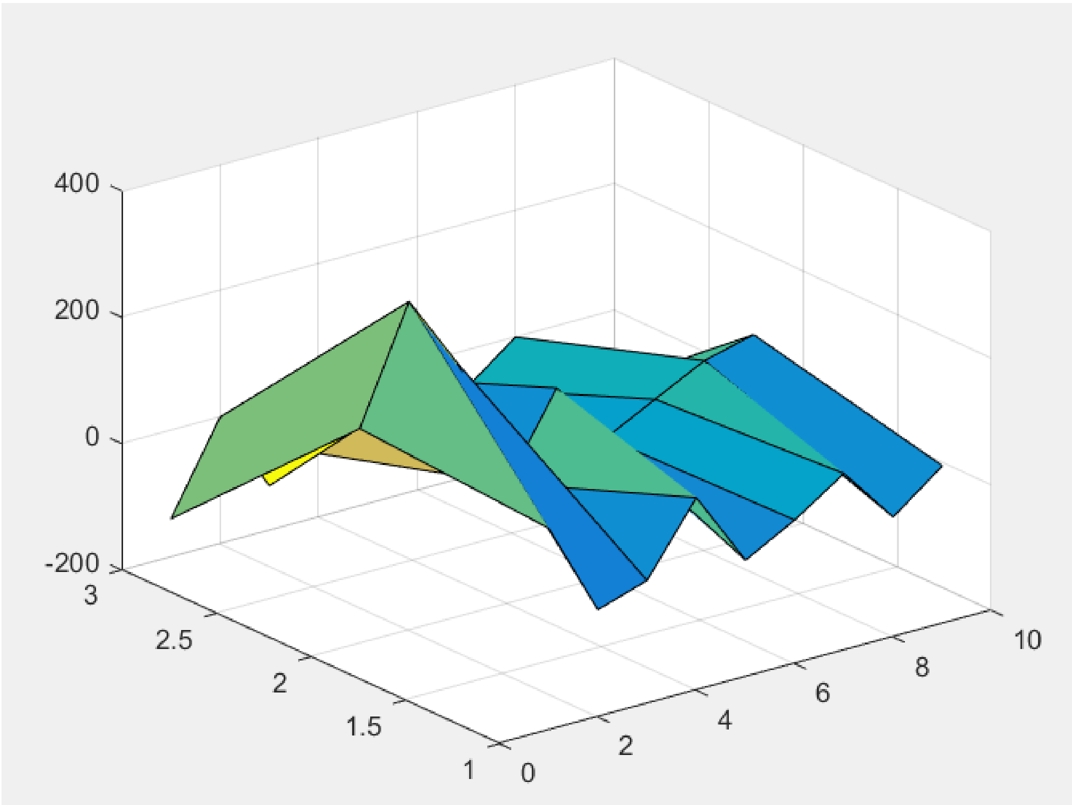

Figure 5. The control point grid of the given surface

The graph of the control point grid of the surface $P(t_1, t_2)$ given with the control points is plotted in Fig. 5. Firstly,

$$(\boldsymbol{D}_n)^T = (\boldsymbol{D}_m)^T = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

is obtained for $n = 2, m = 2$. Then, parametric equation of this surface $P(t_1, t_2)$ is obtained as follows:

$$P(t_1, t_2) = \big(P_1(t_1, t_2), P_2(t_1, t_2), P_3(t_1, t_2)\big) =$$

$$(-10(1 - 2t_1 + t_1^2)(1 - 2t_2 + t_2^2) + 50(1 - 2t_1 + t_1^2)(2t_2 - 2t_2^2) + 80(1 - 2t_1 + t_1^2)t_2^2$$

$$-10(2t_1 - 2t_1^2)(1 - 2t_2 + t_2^2) + 90(2t_1 - 2t_1^2)(2t_2 - 2t_2^2) + 110(2t_1 - 2t_1^2)t_2^2$$

$$-10t_1^2(1 - 2t_2 + t_2^2) + 50t_1^2(2t_2 - 2t_2^2) + 80t_1^2t_2^2, -10(1 - 2t_1 + t_1^2)(1 - 2t_2 + t_2^2)$$

$$-10(1 - 2t_1 + t_1^2)(2t_2 - 2t_2^2) - 10(1 - 2t_1 + t_1^2)t_2^2 + 40(2t_1 - 2t_1^2)(1 - 2t_2 + t_2^2)$$

$$+50(2t_1 - 2t_1^2)(2t_2 - 2t_2^2) + 40(2t_1 - 2t_1^2)t_2^2 + 90t_1^2(1 - 2t_2 + t_2^2) + 110t_1^2(2t_2 - 2t_2^2)$$

$$+90t_1^2t_2^2, 10(1 - 2t_1 + t_1^2)(1 - 2t_2 + t_2^2) - 45(1 - 2t_1 + t_1^2)(2t_2 - 2t_2^2)$$

$$+50(1 - 2t_1 + t_1^2)t_2^2 + 20(2t_1 - 2t_1^2)(1 - 2t_2 + t_2^2) - 35(2t_1 - 2t_1^2)(2t_2 - 2t_2^2)$$

$$+110(2t_1 - 2t_1^2)t_2^2 - 10t_1^2(1 - 2t_2 + t_2^2) + 25t_1^2(2t_2 - 2t_2^2) + 50t_1^2t_2^2)$$

by using the matrix equality $\boldsymbol{P}(t_1, t_2) = \boldsymbol{T_1}(\boldsymbol{D_n})^T\, \overline{\boldsymbol{T}}_2\overline{\boldsymbol{D}}\boldsymbol{P}$. Because of $P_{ij} \in R^3$, this Bézier surface $P(t_1, t_2)$ given with 9 control points, is located in 3-dimensional space.

5. THE MATRIX FORM FOR DERIVATIVES OF THE BÉZIER SURFACE

The (k_1, k_2) order partial derivative, according to the parameters $(t_{1,}t_2)$, of the Bézier surface $P(t_1, t_2)$ represented by (9) is written as

$$\boldsymbol{P}_{t_{1,}t_2}^{(k_1,k_2)}(t_1, t_2) = \boldsymbol{B}_n^{(k_1)}\boldsymbol{Q}_m^{(k_2)}\boldsymbol{P} \tag{12}$$

and we obtain the following equalities:

$$\boldsymbol{B}_n^{(k_1)} = \boldsymbol{T}_1^{(k_1)}(\boldsymbol{D}_n)^T \text{ and } \boldsymbol{Q}_m^{(k_2)} = \overline{\boldsymbol{T}}_{\boldsymbol{2}}^{\,(k_2)}\overline{\boldsymbol{D}}.$$

Additionally, the expressions

$$\boldsymbol{B}_n^{(k_1)} = \boldsymbol{T}_1\boldsymbol{B}^{k_1}(\boldsymbol{D}_n)^T \text{ and } \boldsymbol{Q}_m^{(k_2)} = \overline{\boldsymbol{T}}_{\boldsymbol{2}}\overline{\boldsymbol{B}}^{k_2}\overline{\boldsymbol{D}}$$

are obtained by using the equalities

$$\boldsymbol{T}_1^{(k_1)} = \boldsymbol{T_1}\boldsymbol{B}^{k_1},\ \overline{\boldsymbol{T}}_{\boldsymbol{2}}^{(\boldsymbol{k_2})} = \overline{\boldsymbol{T}}_{\boldsymbol{2}}\overline{\boldsymbol{B}}^{k_2} \text{ and } \overline{\boldsymbol{B}} = \begin{bmatrix} \boldsymbol{B} & 0 & \cdots & 0 \\ 0 & \boldsymbol{B} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{B} \end{bmatrix}.$$

Thus it is obtained as

$$\boldsymbol{P}_{t_1,t_2}^{(k_1,k_2)}(t_1,t_2) = \boldsymbol{T}_1\boldsymbol{B}^{k_1}(\boldsymbol{D}_n)^T\overline{\boldsymbol{T}}_{\boldsymbol{2}}\overline{\boldsymbol{B}}^{k_2}\overline{\boldsymbol{D}}\boldsymbol{P}. \tag{13}$$

This equality is the equality representing the (k_1,k_2) order partial derivative, according to the parameters $(t_{1,}t_2)$, of the Bézier surface $P(t_1,t_2)$.

The parametric equation of the (k_1,k_2) order partial derivative, according to the parameters $(t_{1,}t_2)$, of the Bézier surface $P(t_1,t_2)$ given with the control points can be obtained using the equality (13).

Example. Let's calculate the (2,1) order partial derivative, according to the parameters $(t_{1,}t_2)$, of the (2,2) order the Bézier surface $P(t_1,t_2)$ given with the control points

$$P_{00} = (100,-50,0), P_{01} = (80,-30,100), P_{02} = (120,80,0), P_{10} = (0,-50,0),$$

$$P_{11} = (60,-20,150), P_{12} = (50,30,0), P_{20} = (-100,-50,0), P_{21} = (110,-40,200),$$

$$P_{22} = (-80,20,0)$$

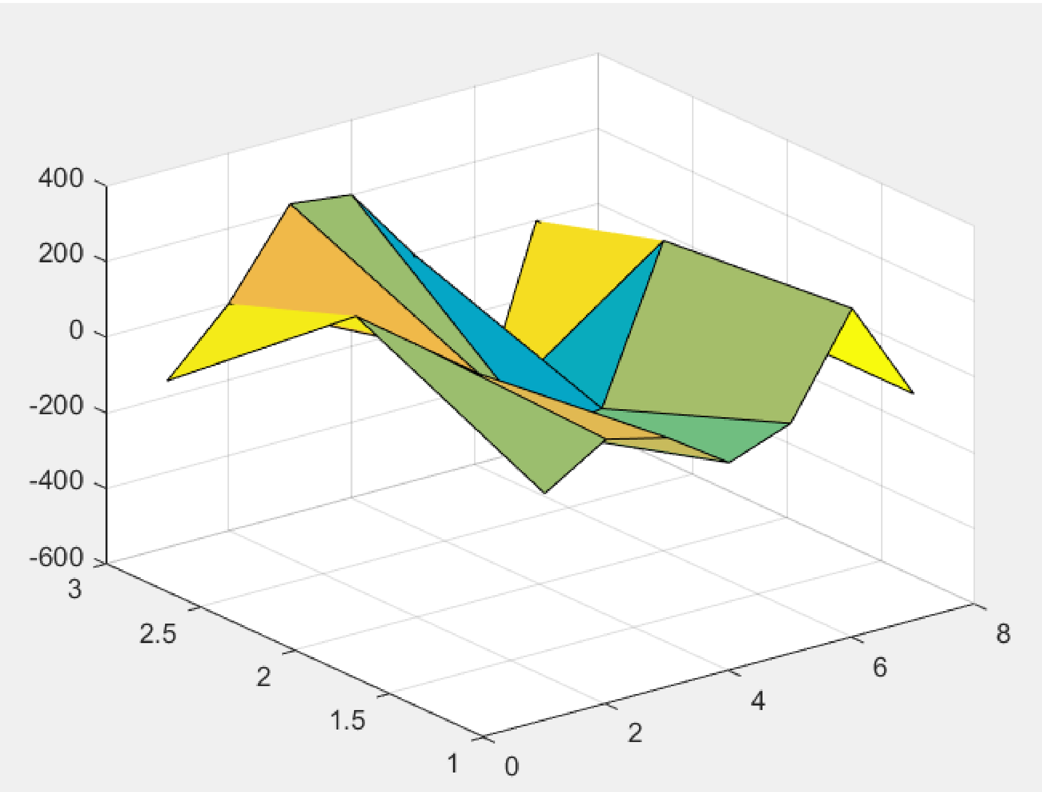

Figure 6. The control point grid of the given surface

The graph of the control point grid of the surface $P(t_1, t_2)$ given the control points is plotted in Fig. 6. Since $P_{ij} \in R^3$, it is clear that this Bézier surface $P(t_1, t_2)$ is located in 3-dimensional space. By using the matrices

$$\boldsymbol{B}^2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, (\boldsymbol{D}_n)^T = (\boldsymbol{D}_m)^T = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix},$$

$$\overline{\boldsymbol{B}}^2 = \begin{bmatrix} \boldsymbol{B}^2 & 0 & \cdots & 0 \\ 0 & \boldsymbol{B}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{B}^2 \end{bmatrix}$$

for $k_1 = 2, k_2 = 1, n = 2, m = 2$, the parametric equation of the $(2,1)$ order partial derivative, according to the parameters $(t_{1,}t_2)$, of the given surface $P(t_1, t_2)$ is obtained as follow:

$$\boldsymbol{P}_{t_{1,}t_2}^{(2,1)} = (140 - 240t_2, -60 + 160t_2, 0).$$

6. CONCLUSIONS

In this study, we put forward that Bézier curves, Bézier surfaces and their derivatives can be expressed in matrix form. We show that can be found of the parametric equation of a Bézier curve given with the control points with the help of the special matrices we have developed. Then, we get matrix equality, where the control points of the Bézier curve, whose parametric equation is known, can be found.

In addition, it is shown that the transition between the control points and the parametric equation can be achieved with the help of this equality by expressing the equation of the Bézier surfaces in matrix form. On the other hand, the higher order derivatives of the Bézier curves and surfaces are expressed in matrix form. It is shown that can be found of the parametric equations of the higher order derivatives, without obtaining their own parametric equations of the Bézier curves and surfaces, known with the control points. The expression of the higher order derivatives of the Bézier curves and surfaces in matrix form is obtained for the first time in this study. It is demonstrated that the matrix forms presented for the Bézier curve, surface and its derivatives are easily usable with explanatory examples.

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