ORIGINAL PAPER

ON THE EXPANSION FORMULA FOR A SINGULAR STURM-LIOUVILLE OPERATOR

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Abstract. *In this study, on the semi-axis, Sturm - Liouville problem under boundary condition depending on spectral parameter is considered. In what follows scattering data is defined and its properties are given for the problem. The kernel of resolvent operator which is Green function is constructed. Using Titchmarsh method, expansion is obtained according to eigenfunctions and expansion formula is expressed with the scattering data.*

Keywords: *expansion formula; resolvent operator; scattering data; contour integration method.*

1. INTRODUCTION

The boundary value problems for Sturm-Liouville equation with spectral parameter in the boundary condition have many interesting applications in mathematical physics and geophysics (see [1-3]) and the references therein). The application of boundary value problems can be seen in several field such as in wave theory of this fields. The solution of the a wave equation can be reduced to a boundary value problem as follows:

$$\rho(x)\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}-q(x)u,$$

with the boundary condition

$$\left.\frac{\partial u}{\partial x}\right|_{x=0}+\left.\left(\alpha_0 u+\alpha_1\frac{\partial u}{\partial t}-\alpha_2\frac{\partial^2 u}{\partial t^2}\right)\right|_{x=0}=0.$$

Here, the coefficients are not depend on x. Several special cases of this problem were investigated by many authors such as: in the case of the potential $q(x)\equiv 0$ and $\alpha_0=0, \alpha_1=0,$ the boundary value problem associated with a diffusion case was handled in [1]. Eventually, during the solution of the equation, we consider the boundary value problem, namely, the Sturm Liouville problem as follows:

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$$-u'' + q(x)u = \mu^2 \rho(x)u, \quad \left(0 < x < +\infty\right), \tag{1.1}$$

under boundary condition

$$u'(0) = -(\alpha_0 + i\alpha_1\mu + \alpha_2\mu^2)u(0), \tag{1.2}$$

where $\rho(x)$ is a positive piecewise continous function, $\alpha_i\ (i = 0,1,2)$ are real numbers such that α_1, α_2 are real positive numbers, μ is a complex parameter. Also, $q(x)$ that is integrable is a real valued function in $[0,+\infty)$ and satisfies the property as:

$$\int_0^{+\infty} x\left|q(x)\right| dx < \infty. \tag{1.3}$$

For classic case, namely, when boundary condition does not involve spectral parameter, the direct and inverse problems of the boundary value problem have been investigated in [4-8]. The Sturm Liouville problem including spectral parameter with the boundary condition related to scattering theory was studied by T. Regge in [9, 10]. These problems are known as Regge problem in the literature. Questions dealing with the expansion of functions in series, in connection with the Regge problem on the semi axis, were studied in the paper [11, 12]. The spectrum and questions connected with the completeness of the system of eigenfunctions were investigated in [13]. Direct and inverse problems of scattering theory for differential operators with the spectral parameters on the boundary condition were considered in [14-17].

Different from the classic case, (1.1) equation includes discontinuous $\rho(x)$ function. The direct and inverse scattering problems for equation (1.1) is completely solved in [18-20]. For the case $\alpha_1 = 0$, spectral properties of the problem on the semi-axis and an expansion formula for the boundary problem were given in [21].

The purpose of this work is to obtain expansion formula according to the eigenfunctions for boundary value problems (1.1)- (1.3). Firstly, we assume that in (1.1) the function $\rho(x)$ has a discontinuity point :

$$\rho(x) = \begin{cases} b^2, & 0 \le x < x_1, \\ 1, & x \ge x_1, \end{cases}$$

where $0 < x_1 \ne 1$. When the coefficient has discontinuity at the point $x = x_1$, the discontinuity strongly influencess the structure of representation of the Jost solution and the main equation of scattering problem.

The rest of the paper is organized as follows: in Section 2, some special solutions of (1.1) equation and scattering data are defined. Moreover, resolvent operator as an integral operator is shown and Green function of this operator is obtained. In Section 3, the expansion formula according to the eigenfunctions is obtained by applying contour integration method or Titchmarsh method [22].

2. JOST FUNCTION AND SCATTERING DATA

Denote by $f(x,\mu)$ the solution of (1.1) which coincides with the function $e^{i\mu x}$ as $x \to \infty$ for the μ from the closed upper plane in the form below:

$$\lim_{x\to+\infty} f(x,\mu)e^{-i\mu x} = 1.$$

This solution is called Jost solution of (1.1). The function

$$\varphi(\mu) = f'(0,\mu) + (\alpha_0 + i\alpha_1\mu + \alpha_2\mu^2) f(0,\mu),$$

is called Jost function of (1.1)-(1.2). When $q(x) \equiv 0$ the function

$$f_0(x,\mu) = \frac{1}{2}\left(1+\rho(x)^{-\frac{1}{2}}\right)e^{i\mu\eta^+(x)} + \frac{1}{2}\left(1-\rho(x)^{-\frac{1}{2}}\right)e^{i\mu\eta^-(x)},$$

is the solution of (1.1), where

$$\eta^{\pm}(x) = \pm x\sqrt{\rho(x)} + x_1(1 \mp \sqrt{\rho(x)}).$$

Thus, the Jost solution $f(x,\mu)$ which can be represented in the form

$$f(x,\mu) = f_0(x,\mu) + \int_{\eta^+(x)}^{+\infty} L(x,t)e^{i\mu t}dt,$$

where $L(x,.) \in L_1(\eta^+(x),+\infty)$. For real $\mu \neq 0$, the functions $f(x,\mu)$ and $\overline{f(x,\mu)}$ form the fundamental system of solutions of equation (1.1) and Wronskian of this system is equal to $2i\mu$ (see [8]).

By using $\chi(x,\mu)$, we denote the solution of (1.1) satisfying initial data

$$\chi(0,\mu) = 1, \quad \chi'(0,\mu) = -(\alpha_0 + i\alpha_1\mu + \alpha_2\mu^2).$$

The following propositions can be proved analogously to [15].

Proposition 2.1. For real values of $\mu \neq 0$, the identity is valid as follow:

$$\frac{2i\mu\chi(x,\mu)}{\varphi(\mu)} = \overline{f(x,\mu)} - S(\mu) f(x,\mu), \tag{2.1}$$

where

$$S(\mu) = \frac{\overline{f'(0,\mu)} + (\alpha_0 + i\alpha_1\mu + \alpha_2\mu^2)\overline{f(0,\mu)}}{f'(0,\mu) + (\alpha_0 + i\alpha_1\mu + \alpha_2\mu^2) f'(0,\mu)},$$

$$|S(\mu)| = 1,$$

$$\varphi(\mu) = f'(0,\mu) + (\alpha_0 + i\alpha_1\mu + \alpha_2\mu^2) f'(0,\mu).$$

The function $S(\mu)$ is called the sacttering function of the boundary value problem (1.1) -(1.3) and $\varphi(\mu)$ is called the Jost function of (1.1) -(1.3).

Proposition 2.2. The function $\varphi(\mu)$ may have only a finite number of zeros in the upper half plane $Im\mu > 0.$ These zeros are all simple and lie on the imaginary axis.

Let the numbers $\mu = i\mu_j (j = 1, 2, ..., n)$ are zeros of the function $\varphi(\mu).$ The numbers

$$m_j^{-2} \equiv \int_0^\infty \left| f(x, i\mu_j) \right|^2 dx + \frac{\alpha_1 + 2\alpha_2\mu_j}{2\mu_j} \left| f(0, i\mu_j) \right|^2 = \frac{-i\varphi(i\mu_j) f(0, i\mu_j)}{2\mu_j}, \tag{2.2}$$

$(j = 0, 1, 2..., n)$ are defined as norming numbers for the boundary value problem (1.1)-(1.3). The functions

$$V(x,\mu) = \overline{f(x,\mu)} - S(\mu) f(x,\mu), \quad (0 < \mu < \infty), \tag{2.3}$$

and

$$V(x, i\mu_j) = m_j f(x, i\mu_j), \quad (j = 0, 1, 2..., n), \tag{2.4}$$

are bounded solutions of the problem (1.1)-(1.3).

Let us define the inner product in the weighted Hilbert space

$$L_2(0, +\infty; \rho) = \left\{ f(x) \left| \int_0^\infty |f(x)|^2 \rho(x) dx < \infty \right. \right\},$$

as

$$\langle f, g \rangle = \int_0^\infty f(x) \overline{g(x)} \rho(x) dx.$$

For every fixed $b > 0$ in $[0, b],$ the functions u, u' are absolutely continuous. Assume that, $\ell(u)$ is an operator which represented as:

$$\ell(u) \equiv \frac{1}{\rho(x)} \left(u'' + q(x) u \right) \in L_2(0, +\infty; \rho).$$

Let $D(\mu)$ be set of consisting of the functions $u \in L_2(0, +\infty; \rho)$ that satisfies the boundary (1.2). By the L_μ is denoted operator with domain $D_\mu = D(L_\mu)$ such that for $u \in D_\mu$ satisfies $L_\mu u = \ell(u).$ If the parameter μ takes values from all points of $\mu-$plane, a family of singular operators L_μ depends on the parameter μ is defined [23].

3. THE RESOLVENT OPERATOR AND EXPANSION FORMULA

Suppose that R_μ is a resolvent operator of (1.1)-(1.3) and $\varphi(\mu) \neq 0$.

Theorem 3.1. The resolvent R_μ is the integral operator

$$R_\mu g = \int_0^{+\infty} R(x,t,\mu)\rho(t)g(t)dt, \tag{3.1}$$

with the kernel

$$R(x,t;\mu) = -\frac{1}{\varphi(\mu)}\begin{cases} f(x,\mu)\chi(t,\mu), & \mathrm{x} \le \mathrm{t}, \\ \chi(x,\mu)f(t,\mu), & \mathrm{t} \le \mathrm{x}, \end{cases} \tag{3.2}$$

where $g(x) \in D_\mu$ is a finite function at infinity.

Proof: To construct the resolvent operator of boundary value problem (1.1)-(1.2), our aim is to solve the following boundary value problem

$$-u'' + q(x)u = \mu^2\rho(x)u + g(x)\rho(x), \quad (0 < x < +\infty), \tag{3.3}$$

$$u'(0) + \left(\alpha_0 + i\alpha_1\mu + \alpha_2\mu^2\right)u(0) = 0. \tag{3.4}$$

We can write the solution of (3.3), (3.4) in the form

$$u(x,\mu) = d_1(x,\mu)\chi(x,\mu) + d_2(x,\mu)f(x,\mu). \tag{3.5}$$

Using the method of variation of parameters, we get

$$\begin{aligned} d_1(x,\mu) &= -\frac{1}{\varphi(\mu)}\int_x^{+\infty} f(t,\mu)g(t)\rho(t)dt, \\ d_2(t,\mu) &= d_2(0,\mu) + \int_0^x \chi(t,\mu)g(t)\rho(t)dt. \end{aligned} \tag{3.6}$$

Since $u(x,\mu) \in L_{2,\rho}(0,+\infty)$ then $d_1(0,\infty) = 0$. Using the condition (3.4), we have $d_2(0,\mu) = 0$ and substituting (3.6) into the (3.5), we obtain the kernel of resolvent operator which is Green function in the form (3.2). Theorem 3.1. is completed.

By using the properties of the Jost solution $f(x,\mu)$ and the special function $\chi(t,\mu)$, it is shown that by standart method (see[11],p.302-304), the solution $u(x,\mu) = \int_0^{+\infty} R(x,t,\mu)g(t)\rho(t)dt$ of the equation (3.3) for $\forall g \in L_{2,\rho}(0,+\infty;\rho)$ satisfies the condition (3.4). Therefore, since $\varphi(\mu)\neq 0$ for all μ on the upper half line, the integral operator is bounded and defined by formula (3.1).

Lemma 3.2. Let $g(x)$ denote a twice continuosly differentiable a finite function at infinity. Then as $|\mu|\to\infty$, $Im\mu>0$, the following holds:

$$\int_0^\infty R(x,t,\mu)g(t)\rho(t)dt=-\frac{g(x)}{\mu^2}+\frac{R_1(x,\mu)}{\mu^2}, \tag{3.7}$$

where $R_1(x,\mu)=\int_0^{+\infty}R(x,t,\mu)\,\widetilde{g(t)}\,\rho(t)dt$ as $\widetilde{g(t)}=-g''(t)+q(t)g(t)$.

Proof: Applying the Theorem 3.1. and integrating by parts, the proof can be easily done.

Theorem 3.3. The expansion formula is given by

$$\delta(x-t)=\sum_{j=1}^{n}V(x,i\mu_j)V(t,i\mu_j)\rho(t)+\frac{\alpha_1}{2\pi}\int_0^\infty V(x,\mu)V(t,\mu)\rho(t)d\mu, \tag{3.8}$$

where $\delta(x)$ is the Dirac delta function.

Proof: Let Γ_R denotes the circle of radius R and center zero which boundary contour is positive oriented. Suppose $D=\{z\,\big|\,|z|\le R,|Imz|\ge\varepsilon\}$ denote that the positive oriented boundary contour of D as $\Gamma_{R,\varepsilon}$. $\Gamma^{(1)}_{R,\varepsilon}$ denotes the contour of half arc of Γ_R denotes that include points z satisfying the condition $Imz>\varepsilon$ and $\Gamma^{(2)}_{R,\varepsilon}$ denotes be a half-arc that include $Imz<-\varepsilon$ points of Γ_R. It is clear that $\Gamma_{R,\varepsilon}=\Gamma^{(1)}_{R,\varepsilon}\cup\Gamma^{(2)}_{R,\varepsilon}$. $\Gamma^{(3)}_{R,\varepsilon}$ denotes a negative oriented curve formed with $Imz=\mp\varepsilon$ lines and be arcs including points z satisfying the conditions $|Imz|\le\varepsilon$. According to this, $\Gamma_{R,\varepsilon}=\Gamma_R\cup\Gamma^{(3)}_{R,\varepsilon}$. Therefore, we can use property of the integration

$$\int_{\Gamma_{R,\varepsilon}}=\int_{\Gamma_R}+\int_{\Gamma^{(3)}_{R,\varepsilon}}. \tag{3.9}$$

Put

$$F(x,\mu)=\int_0^\infty R(x,t,\mu)g(t)dt.$$

By multiplying both sides of (3.7) by $\frac{1}{2\pi i}\mu$, also integrating it with respect to μ the contour the contour $\Gamma_{R,\varepsilon}$, we obtain

$$\frac{1}{2\pi i}\int_{\Gamma_{R,\varepsilon}}\mu F(x,\mu)d\mu=-\frac{1}{2\pi i}\int_{\Gamma_{R,\varepsilon}}\frac{g(x)}{\mu}d\mu+\frac{1}{2\pi i}\int_{\Gamma_{R,\varepsilon}}O(\frac{1}{\mu})d\mu. \tag{3.10}$$

Utilizing (3.9), we get the following equation

$$\frac{1}{2\pi i}\int_{\Gamma_{R,\varepsilon}}\mu F(x,\mu)d\mu=\frac{1}{2\pi i}\int_{\Gamma_{R}}\mu F(x,\mu)d\mu+\frac{1}{2\pi i}\int_{\Gamma^{(3)}_{R,\varepsilon}}\mu F(x,\mu)d\mu. \tag{3.11}$$

By virtue of (3.10), the integral on the right hand side of (3.11)

$$\begin{aligned}\frac{1}{2\pi i}\int_{\Gamma_{R}}\mu F(x,\mu)d\mu&=-g(x)+\frac{1}{2\pi i}\int_{\Gamma_{R}}O(\frac{1}{\mu})d\mu\\&=g(x),\end{aligned} \tag{3.12}$$

and

$$\lim_{\substack{R\to\infty\\ \varepsilon\to\infty}}\frac{1}{2\pi i}\int_{\Gamma^{(3)}_{R,\varepsilon}}\mu F(x,\mu)d\mu=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\mu\left[F(x,\mu+i0)-F(x,\mu-i0)\right]d\mu. \tag{3.13}$$

are calculated. Substituting (3.12) and (3.13) in (3.11), we have

$$\lim_{\substack{R\to\infty\\ \varepsilon\to\infty}}\frac{1}{2\pi i}\int_{\Gamma_{R,\varepsilon}}\mu F(x,\mu)d\mu=-g(x)+\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\mu\left[F(x,\mu+i0)-F(x,\mu-i0)\right]d\mu. \tag{3.14}$$

Let us introduce the residue calculus, we reach

$$\frac{1}{2\pi i}\int_{\Gamma_{R,\varepsilon}}\mu F(x,\mu)d\mu=\sum_{j=1}^{n}\operatorname*{Res}_{\mu=i\mu_j}\left[\mu F(x,\mu)\right]+\sum_{j=1}^{n}\operatorname*{Res}_{\mu=-i\mu_j}\left[\mu F(x,\mu)\right]. \tag{3.15}$$

We use the relations (3.14), (3.15) to imply that

$$\begin{aligned}g(x)=&-\sum_{j=1}^{n}\operatorname*{Res}_{\mu=i\mu_j}[\mu F(x,\mu)]-\sum_{j=1}^{n}\operatorname*{Res}_{\mu=-i\mu_j}[\mu F(x,\mu)]\\&+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\mu[F(x,\mu+i0)-F(x,\mu-i0)]d\mu.\end{aligned} \tag{3.16}$$

We define by $\psi(x,\mu)$ the solution of the equation (1.1) by initial conditions

$$\psi(0,\mu)=0,\quad \psi'(0,\mu)=-\alpha_1.$$

It is clear that, we get the Wronksian as:

$$W\left\{\chi(x,\mu),\psi(x,\mu)\right\}=\alpha_1\neq 0.$$

For the solution of the equation (1.1)

$$f(x,\mu)=f(0,\mu)\chi(x,\mu)-\frac{\varphi(\mu)}{\alpha_1}\psi(x,\mu),$$

is obtained. Considering this equation in (3.2), we get

$$R(x,t,\mu)=-\frac{1}{\varphi(\mu)}f(0,\mu)\chi(x,\mu)\int_0^\infty\chi(t,\mu)d\mu + \begin{cases}\psi(x,\mu)\chi(t,\mu), & \mathrm{x}\leq \mathrm{t},\\ \psi(t,\mu)\chi(x,\mu), & \mathrm{t}\leq \mathrm{x}.\end{cases}$$

Here, it can be seen easily

$$F(x,\mu)=-\frac{1}{\varphi(\mu)}f(0,\mu)\chi(x,\mu)\int_0^{+\infty}\chi(t,\mu)g(t)\rho(t)dt + \psi(x,\mu)\int_0^x\chi(t,\mu)g(t)\rho(t)dt+\chi(x,\mu)\int_x^{+\infty}\psi(t,\mu)f(t)\rho(t)dt. \tag{3.17}$$

Obviously,

$$\operatorname*{Res}_{\mu=i\mu_j}\left[\mu F(x,\mu)\right]=-\frac{i\mu_j}{\dot{\varphi}(i\mu_j)}f(0,i\mu_j)\chi(x,i\mu_j)\int_0^{+\infty}\chi(x,i\mu_j)f(t)\rho(t)dt.$$

By the help of the last equation and relations (2.2), (2.3) we obtain

$$\sum_{j=1}^n\operatorname*{Res}_{\mu=i\mu_j}\left[\mu F(x,\mu)\right]+\sum_{j=1}^n\operatorname*{Res}_{\mu=-i\mu_j}\left[\mu F(x,\mu)\right]=\sum_{j=1}^n V(x,i\mu_j)\int_0^{+\infty}V(t,i\mu_j)g(t)\rho(t)dt. \tag{3.18}$$

According to (3.17) and $F(x,\mu-i0)=\overline{F(x,\mu+i0)}$, then we get

$$F(x,\mu+i0)-F(x,\mu-i0)=\left[-\frac{f(0,\mu)}{\varphi(\mu)}+\frac{f\overline{(0,\mu)}}{\overline{\varphi(\mu)}}\right]\chi(x,\mu)\int_0^{+\infty}\chi(t,\mu)g(t)\rho(t)dt$$

$$=\frac{2i\mu}{|\varphi(\mu)|^2}|f(0,\mu)|^2\chi(x,\mu)\int_0^{+\infty}\chi(t,\mu)g(t)\rho(t)dt.$$

It is obtained from here

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\mu\left[F(x,\mu+i0)-F(x,\mu-i0)\right]d\mu=-\frac{2\alpha_1}{\pi}\int_0^{+\infty}\frac{\mu^2}{|\varphi(\mu)|^2}\chi(x,\mu)\int_0^{+\infty}\chi(t,\mu)g(t)\rho(t)dtd\mu$$

$$=\frac{\alpha_1}{2\pi}\int_0^{+\infty}V(x,\mu)\int_0^{+\infty}V(t,\mu)g(t)\rho(t)dtd\mu. \tag{3.19}$$

Taking (3.18) and (3.19) into (3.16), we obtain what is known as an expansion formula

$$g(x)=\sum_{j=1}^{n}V(x,i\mu_j)\int_0^{+\infty}V(t,i\mu_j)g(t)\rho(t)dt+\frac{\alpha_1}{2\pi}\int_0^{+\infty}V(x,\mu)\int_0^{+\infty}V(t,\mu)g(t)\rho(t)dtd\mu. \tag{3.20}$$

It can be shown that the integrals on the right side converge in the metric of the space $L_2(0,\infty)$.

4. CONCLUSION

In this article, it is obtained expansion formula according to the eigenfunctions for a second order equation with discontinuous coefficient containing quadratic spectral parameter. In this study, the expansion formula is obtained by spectral data different from other studies. The techniques and methods can be applied to the other different equations with different boundary conditions and new expansion formulas can be obtained.

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