ORIGINAL PAPER

BICOMPLEX BALANCING AND BICOMPLEX LUCAS-BALANCING QUATERNIONS

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Abstract. *In this paper, bicomplex balancing quaternions and bicomplex Lucas-balancing quaternions was defined. Moreover we give some properties involving these sequences. Finally, a different way to obtained the n th terms of these sequence is stated using the determinant of tridiagonal matrix whose entries are bicomplex balancing quaternions and bicomplex Lucas-balancing quaternions.*

Keywords: *bicomplex balancing; bicomplex balancing quaternions; bicomplex Lucas-Balancing quaternions.*

1. INTRODUCTION

The bicomplex numbers are defined by the basis $\{1, i, j, ij\}$, where i, j and ij satisfy the following properties:

$$i^2 = -1, \qquad j^2 = -1, \quad ij = ji$$

A bicomplex number q can be expressed as follows:

$$q = q_1 + iq_2 + jq_3 + ijq_4$$

or

$$q = (q_1 + iq_2) + j(q_3 + iq_4)$$

where q_1, q_2, q_3 and q_4 are reel numbers.

Also the set of bicomplex numbers can be expressed by a basis $\{1, i, j, ij\}$ as

$$K = \{q = q_1 + iq_2 + jq_3 + ijq_4 : q_1, q_2, q_3, q_4 \in IR\}.$$

A set of bicomplex numbers K is a real vector space with the addition and scalar multiplication operations. In addition, the vector space with the properties of scalar multiplication and the product of the bicomplex numbers is a commutative algebra. Again the bicomplex numbers form a commutative ring with unity which contain the complex numbers. For more details about this type numbers [1-2].

Quaternions were formally introduced by W.R. Hamilton in 1843. In [3] D. Tasci and N.F. Yalcin studied Fibonacci p-quaternions. Again in [4], D. Tasci studied k-Jacobsthal and k-Jacobsthal Lucas quaternions. In [5] F.T. Aydın studied bicomplex k- Fibonacci quaternions. Bicomplex numbers, just like quaternions, are a generalization of complex

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numbers of entities specified by four real numbers. But quaternions are non-commutative, whereas, bicomplex numbers and bicomplex quaternions are commutative. An other difference is that, quaternions form a division algebra, but bicomplex quaternions do not form a division algebra.

Bicomplex numbers were introduced by C. Segre in 1892 [6]. Further G. B. Price, the bicomplex numbers gave in his book on multicomplex spaces and functions [7].

In recent years it has been stated that several properties of quantum mechanics can be generalized to the bicomplex numbers; the Schrödinger equation for a particle in one dimension was generalized [8]. Further the fractal structures of these numbers are studied [9]. In [10], Horadam introduced the Fibonacci quaternions sequence.

Behera and Panda introduced the concept of balancing numbers [11]. They defined a balancing number n as a solution of Diophantine equation. A positive integer n is called balancing number if

$$1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r)$$

for some natural number r. Here r is called the balancer corresponding to the balancing number n. For example 6 and 35 are balancing numbers with balancers 2 and 14, respectively. Again some authors proved that the balancing numbers fulfil the following recurrence relation:

$$B_{n+1} = 6B_n - B_{n-1} ,\ n\geq 1$$

where $B_0=0$ and $B_1=1$.

Panda [12] studied several fascinating properties of balancing numbers calling the positive square root of $8x^2+1$, a Lucas- balancing number for each balancing number x. All balancing numbers x and corresponding Lucas-balancing numbers y are positive integer solutions of Diophantine equation $8x^2 + 1 = y^2$. Balancing and Lucas-balancing numbers share the same linear recurrence $x_{n+1} = 6x_n - x_{n-1}$,while initial values of balancing numbers are $x_0=0$, $x_1=1$ and for Lucas-balancing numbers $x_0=1$, $x_1=3$. Ray studied some Diophantine equations involving balancing and Lucas- balancing numbers [13-16].

We denote the n *th* balancing and Lucas-balancing numbers by B_n and C_n, respectively.

The sequences $\{B_n\}_{n=0}^{\infty}$ and $\{C_n\}_{n=0}^{\infty}$ satisfy the following recurrence relations:

$$B_{n+1} = 6B_n - B_{n-1},\ (n\geq 1),\ B_0=0,\ B_1=1 \tag{1}$$

$$C_{n+1} = 6C_n - C_{n-1},\ (n\geq 1),\ C_0=1,\ C_1=3 \tag{2}$$

We note that the Binet-like formulas for balancing and Lucas- balancing numbers are

$$B_n = \frac{\alpha^n-\beta^n}{\alpha-\beta)} \text{ and } C_n = \frac{\alpha^n+\beta^n}{\alpha-\beta)} \tag{3}$$

respectively. We remark that $\alpha=3+\sqrt{8}$ and $\beta=3-\sqrt{8}$.

2. BICOMPLEX BALANCING QUATERNIONS

Definition 1. The bicomplex balancing quaternions are defined by

$$QB_n = B_n + iB_{n+1} + jB_{n+2} + ijB_{n+3} \tag{4}$$

where $B_n$is the *n* th balancing number.

Lemma 2. For n≥1, we have

$$QB_{n+1} + QB_{n-1} = 6QB_n.$$

Proof: By the Definition 1, we write

$$QB_{n+1} + QB_{n-1} = (B_{n+1} + B_{n-1}) + i(B_{n+2} + B_n) + j(B_{n+3}+B_n) + ij((B_{n+4}+B_n)$$

Now, considering (1), we have

$$B_{n+1} + B_{n-1} = 6B_n.$$

So we write

$$\begin{aligned} QB_{n+1} + QB_{n-1} &= 6B_n + i(6B_{n+1}) + j(B_{n+2}) + ij((B_{n+3}) \\ &= 6(B_n + iB_{n+1} + jB_{n+2} + ijB_{n+3}) \\ &= 6QB_n. \end{aligned}$$

Thus the proof is complete.
Considering the Lemma 2, we obtain the following recurrence relation:

$$QB_{n+1} = 6QB_n - QB_{n-1}, (\mathrm{n} \geq 1) \tag{5}$$

with the initial values, $\mathrm{QB}_0 = \mathrm{i} + 6\mathrm{j} + 35\mathrm{ij}$ and $\mathrm{QB}_1 = 1 + 6\mathrm{i} + 35\mathrm{j} + 204\mathrm{ij}$.

Theorem 3. (The Binet-like formula for bicomplex balancing quaternions) For n≥0

$$\mathrm{QB}_n = \frac{\mathrm{a}\alpha^{\mathrm{n}} - \mathrm{b}\beta^{\mathrm{n}}}{\alpha - \beta}$$

where a and b are bicomplex quaternions defined by $a=1+i\alpha+j\alpha^2+ij\alpha^3$ and $b=1+i\beta+j\beta^2+ij\beta^3$, respectively. Moreover α and β are the roots of the equation $r^2- 6r +1 = 0$, i.e., $\alpha = 3+\sqrt{8}$ and $\beta = 3-\sqrt{8}$.

Proof: Using the Binet formula for balancing numbers, i.e.,

$$\mathrm{B}_n = \frac{\alpha^{\mathrm{n}} - \beta^{\mathrm{n}}}{\alpha - \beta}$$

we find

$$
\begin{aligned}
QB_n &= B_n + iB_{n+1} + jB_{n+2} + ijB_{n+3} \\
&= \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) + i\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) + j\left(\frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta}\right) + ij\left(\frac{\alpha^{n+3} - \beta^{n+3}}{\alpha - \beta}\right) \\
&= \frac{(1 + i\alpha + j\alpha^2 + ij\alpha^3)\alpha^n - (1 + i\beta + j\beta^2 + ij\beta^3)\beta^n}{\alpha - \beta} \\
&= \frac{a\alpha^n - b\beta^n}{\alpha - \beta}.
\end{aligned}
$$

Theorem 4. The generating function of the bicomplex balancing quaternions is

$$\sum_{n=0}^{\infty} QB_n x^n = \frac{QB_0 + (QB_1 - 6QB_0)x}{1 - 6x + x^2},$$

where $QB_0 = i + 6j + 35ij$ and $QB_1 = 1 + 6i + 35j + 204ij$.

Proof: Let

$$f(x) = \sum_{n=0}^{\infty} QB_n x^n = QB_0 + QB_1 x + QB_2 x^2 + \cdots + QB_n x^n + \cdots$$

be the generating function of the bicomplex balancing quaternions. Since

$$6xf(x) = 6QB_0 x + 6QB_1 x^2 + 6QB^2 x^3 + \cdots + QB_{n-1} x^n + \cdots$$

and

$$x^2 f(x) = QB_0 x^2 + QB_1 x^3 + QB^2 x^4 + \cdots + QB_{n-2} x^n + \cdots$$

and considering the recurrence relation for bicomplex balancing quaternions we obtain

$$(1 - 6x + x^2)f(x) = QB_0 + (QB_1 - 6QB_0)x$$

or

$$f(x) = \frac{QB_0 + (QB_1 - 6QB_0)x}{(1 - 6x + x)}$$

So the theorem is proved.

Theorem 5. (The Catalan-like identity for bicomplex balancing quaternions)

$$QB_{n-r} QB_{n+r} - (QB_n)^2 = -(ab)B_r^2$$

where $n, r \in \mathbb{Z}^+$, $n > r$ and B_r is the *r* th balancing number.

Proof: Using the Binet-like formula for the bicomplex balancing quaternions and taking into account that $\alpha\beta = 1$, we have

$$QB_{n-r}QB_{n+r}-(QB_n)^2 = \left(\frac{a\alpha^{n-r}-b\beta^{n-r}}{\alpha-\beta}\right)\left(\frac{a\alpha^{n+r}-b\beta^{n+r}}{\alpha-\beta}\right)-\left(\frac{a\alpha^{n}-b\beta^{n}}{\alpha-\beta}\right)^2$$
$$= -\frac{ab}{(\alpha-\beta)^2}(\alpha\beta)^n\left[\left(\frac{\beta^r}{\alpha^r}+\frac{\alpha^r}{\beta^r}\right)-2\right]$$
$$= -\frac{ab}{(\alpha-\beta)^2}\left(\frac{\alpha^{2r}+\beta^{2r}-2}{(\alpha\beta)^r}\right)$$
$$= -\frac{ab}{(\alpha-\beta)^2}(\alpha^r-\beta^r)^2$$
$$= -(ab)\left(\frac{\alpha^r-\beta^r}{\alpha-\beta}\right)^2$$
$$= -(ab)B_r^2$$

Corollary 6. (Cassini's-like identity for bicomplex balancing quaternions)

$$QB_{n-r}QB_{n+r}-(QB_n)^2 = -(ab).$$

Proof: Note that for r = 1, the equality (6) gives Cassini's identity. Further we remark that $B_1^2 = 1$.

Theorem 7. (D'ocagne's-like identity for bicomplex balancing quaternions) If m > n and m,n∈ $\mathbb{Z}^+$, then

$$QB_mQB_{n+1} - QB_{m+1}QB_n = -(ab)\,B_{m-n}.$$

where B_{m-n} is (m-n) th balancing number.

Proof: Using the Binet-like formula for bicomplex balancing quaternions and αβ = 1,we have

$$QB_mQB_{n+1} - QB_{m+1}QB_n = \left(\frac{a\alpha^{m}-b\beta^{m}}{\alpha-\beta}\right)\left(\frac{a\alpha^{n+1}-b\beta^{n+1}}{\alpha-\beta}\right)-\left(\frac{a\alpha^{m+1}-b\beta^{m+1}}{\alpha-\beta}\right)\left(\frac{a\alpha^{m+1}-b\beta^{m+1}}{\alpha-\beta}\right)$$
$$= -(ab)\left[\frac{(\alpha\beta)^n}{(\alpha-\beta)^2}\right](-\alpha^{m-n}\beta-\beta^{m-n}\alpha+\alpha^{m-n}\alpha+\beta^{m-n}\beta)$$
$$= \frac{ab}{(\alpha-\beta)^2}[\,\alpha^{m-n}(-\beta+\alpha)-\beta^{m-n}(\alpha-\beta)]$$
$$= \frac{ab}{(\alpha-\beta)^2}(\alpha-\beta)[\,\alpha^{m-n}-\beta^{m-n}]$$
$$= (ab)\left(\frac{\alpha^{m-n}-\beta^{m-n}}{\alpha-\beta}\right)$$
$$= (ab)B_{m-n}.$$

So the proof is complete.

Now we give the summation formula for bicomplex balancing quaternions.

Theorem 8.

$$\sum_{n=0}^{\infty} QB_i = \frac{1}{4}[5QB_n - QB_{n-1} - (1+i+5j+29ij)].$$

Proof: From Lemma 2, we know that

$$QB_n = 6QB_{n-1} + QB_{n-2}$$

So we write

$\mathrm{QB}_0 = 6\mathrm{QB}_{-1} - \mathrm{QB}_{-2}$
$Q\mathrm{B}_1 = 6\mathrm{QB}_0 - \mathrm{QB}_{-1}$
$\vdots$
$QB_{n-1} = 6QB_{n-2} - QB_{n-3}$
$QB_n = 6QB_{n-1} - QB_{n-2}$

Then summing the above equalities, we obtain

$$\sum_{n=0}^{\infty} \mathrm{QB}_i = \frac{1}{4}[5\mathrm{QB}_n - QB_{n-1} - 5QB_{-1} - QB_{-2}]$$

Now considering $\mathrm{QB}_{-1} = -1 + \mathrm{j} + 6\mathrm{ij}$ and $\mathrm{QB}_{-2} = -6 - \mathrm{i} + \mathrm{ij}$, we have

$$\sum_{n=0}^{\infty} \mathrm{QB}_i = \frac{1}{4}[5\mathrm{QB}_n - QB_{n-1} - (1 + \mathrm{i} + 5\mathrm{j} + 29\mathrm{ij})]$$

Thus the theorem is proved..

Theorem 9. For the integer n≥1, we have

$$\begin{bmatrix} 6 & -1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} Q\mathrm{B}_2 & Q\mathrm{B}_1 \\ Q\mathrm{B}_1 & Q\mathrm{B}_\mathrm{n} \end{bmatrix} = \begin{bmatrix} Q\mathrm{B}_{n+2} & Q\mathrm{B}_{\mathrm{n}+1} \\ Q\mathrm{B}_{n+1} & Q\mathrm{B}_\mathrm{n} \end{bmatrix}.$$

Proof: (By the induction on n). If n = 1, then the result is obvious. We assume that it is true for n-1, i.e.,

$$\begin{bmatrix} 6 & -1 \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} Q\mathrm{B}_2 & Q\mathrm{B}_1 \\ Q\mathrm{B}_1 & Q\mathrm{B}_\mathrm{n} \end{bmatrix} = \begin{bmatrix} Q\mathrm{B}_{n+1} & Q\mathrm{B}_\mathrm{n} \\ Q\mathrm{B}_n & Q\mathrm{B}_{\mathrm{n}-1} \end{bmatrix}.$$

By simple calculation using induction's hypothesis we have

$$\begin{aligned} \begin{bmatrix} 6 & -1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} Q\mathrm{B}_2 & Q\mathrm{B}_1 \\ Q\mathrm{B}_1 & Q\mathrm{B}_\mathrm{n} \end{bmatrix} &= \begin{bmatrix} 6 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 6 & -1 \\ 1 & 0 \end{bmatrix}^{n-1} \begin{bmatrix} Q\mathrm{B}_2 & Q\mathrm{B}_1 \\ Q\mathrm{B}_1 & Q\mathrm{B}_\mathrm{n} \end{bmatrix} \\ &= \begin{bmatrix} 6 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} Q\mathrm{B}_{n+1} & Q\mathrm{B}_\mathrm{n} \\ Q\mathrm{B}_n & Q\mathrm{B}_{\mathrm{n}-1} \end{bmatrix} \\ &= \begin{bmatrix} 6Q\mathrm{B}_{n+1} - Q\mathrm{B}_\mathrm{n} & 6Q\mathrm{B}_n - Q\mathrm{B}_{\mathrm{n}-1} \\ Q\mathrm{B}_{n+1} & Q\mathrm{B}_\mathrm{n} \end{bmatrix} \\ &= \begin{bmatrix} Q\mathrm{B}_{n+2} & Q\mathrm{B}_{\mathrm{n}+1} \\ Q\mathrm{B}_{n+1} & Q\mathrm{B}_\mathrm{n} \end{bmatrix} \end{aligned}$$

which ends the proof.

3. TRIDIAGONAL MATRIX WITH BICOMPLEX BALANCING QUATERNIONS

In this section, we give another way to obtain the n th term of the bicomplex balancing quaternion sequence as the computation of a tridiagonal matrix.

Theorem 10. [4] Let $\{x_n\}_{n=0}^{\infty}$ be any second order linear sequence defined recursively by the following ;

$$x_{n+1} = Ax_n + Bx_{n-1} \quad n \geq 1,$$

with $x_0 = C$, $x_1 = D$. Then for all $n \geq 0$

$$x_n = \begin{vmatrix} C & D & 0 & 0 & & 0 & 0 \\ -1 & 0 & B & 0 & \cdots & 0 & 0 \\ 0 & -1 & A & B & & 0 & 0 \\ & \vdots & & & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & A & B \\ 0 & 0 & 0 & 0 & & -1 & A \end{vmatrix}_{(n+1)\times(n+1)}$$

Proposition 11. For all $n \geq 0$

$$Q\mathrm{B}_\mathrm{n} = \begin{vmatrix} Q\mathrm{B}_0 & Q\mathrm{B}_1 & 0 & 0 & & 0 & 0 \\ -1 & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 6 & -1 & & 0 & 0 \\ & \vdots & & & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & 6 & -1 \\ 0 & 0 & 0 & 0 & & -1 & 6 \end{vmatrix}_{(n+1)\times(n+1)}$$

Proof: In Theorem 10, consider A = 6, B = -1, C = QB_0 and D = QB_1, then the proof is immediately seen.

4. BICOMPLEX LUCAS-BALANCING QUATERNIONS

Definition 12. The bicomplex Lucas-balancing quaternions are defined by

$$QC_n = C_n + iC_{n+1} + jC_{n+2} + ijC_{n+3} \tag{7}$$

where C_n is the n th Lucas-balancing number.

Theorem 13. For $n \geq 1$, $n \in \mathbb{Z}^+$

$$QC_{n+1} + QC_{n-1} = 6QC_n.$$

Proof: Considering the equality (7) and

$$C_{n+1} = 6C_n - C_{n-1}, (\mathrm{n} \geq 1)$$

then the proof is immediately seen.

We note that by the Theorem 13, we obtain the following recurrence relation:

$$QC_{n+1} = 6QC_n - QC_{n-1}, (n \geq 1), n \geq 1$$

with the initial values $QC_0 = 1 + 3i + 17j + 99ij$ and $QC_1 = 3 + 17i + 99j + 577ij$.

Theorem 14. The Binet- like formula for bicomplex Lucas-balancing quaternions is

$$QC_n = \frac{a\alpha^n + b\beta^n}{2}$$

where $a = 1 + i\alpha + j\alpha^2 + ij\alpha^3$,$b = 1 + i\beta + j\beta^2 + ij\beta^3$ and $\alpha = 3 + \sqrt{8}$, $\beta = 3 - \sqrt{8}$.

Proof: Using the following Binet-like formula for the Lucas- balancing numbers

$$C_n = \frac{\alpha^n + \beta^n}{2}$$

and considering the equality (7), the proof is easily seen.

Theorem 15. The generating function for bicomplex Lucas-balancing quaternions is

$$\sum_{n=0}^{\infty} QC_n x^n = \frac{QC_0 + (QC_1 - 6QC_0)x}{1 - 6x + x^2}$$

Proof: The proof of this theorem is similar to the proof of Theorem 4.

Theorem 16. (The Catalan's formula for bicomplex Lucas-balancing quaternions) For n > r and n, r $\in \mathbb{Z}^+$, we have

$$QC_{n-r}QC_{n+r} - (QC_n)^2 = 2abB_r^2 \tag{8}$$

where $a = 1 + i\alpha + j\alpha^2 + ij\alpha^3$and $b = 1 + i\beta + j\beta^2 + ij\beta^3$ and B_r is the *r* th balancing number.

Proof: Using the Binet-like formula for the bicomplex Lucas-balancing quaternions, the proof is immediately seen.

Corollary 17. The Cassini's-like formula for bicomplex Lucas-balancing quaternions is

$$QC_{n-1}QC_{n+1} - (QC_n)^2 = 2ab \tag{9}$$

Proof: We note that for r = 1, the equality (8) gives the formula (9). Further we remark that $B_1^2=1$.

Theorem 18. (D'ocagne's-like identity for bicomplex Lucas-balancing quaternions) If m > n and m, n$\in \mathbb{Z}^+$, then

$$QC_{\mathrm{m}}QC_{n+1} - QC_{\mathrm{m}+1}QC_n = -8(\mathrm{ab})\mathrm{B}_{m-n}$$

where $\mathrm{B}_{m-n}\}$ is (m-n)*th* balancing number.

Proof: Considering the Binet-like formula for bicomplex Lucas-balancing quaternions, the proof is easily seen.

Theorem 19. The following equalities are valid:

i) $2(Q\mathrm{B}_n)\mathrm{C}_\mathrm{n} - \mathrm{QB}_{2\mathrm{n}} = \frac{\mathrm{a}-\mathrm{b}}{2\sqrt{8}}$

ii) $2(Q\mathrm{C}_n)\mathrm{B}_\mathrm{n} - \mathrm{QB}_{2\mathrm{n}} = \frac{\mathrm{b}-\mathrm{a}}{2\sqrt{8}}$

where $\mathrm{a} = 1 + \mathrm{i}\alpha + \mathrm{j}\alpha^2 + \mathrm{ij}\alpha^3$ and$\mathrm{b} = 1 + \mathrm{i}\beta + \mathrm{j}\beta^2 + \mathrm{ij}\beta^3$.

Proof: Using the following formulas

$$QB_n = \frac{\mathrm{a}\alpha^\mathrm{n} - \mathrm{b}\beta^\mathrm{n}}{\alpha-\beta},\ B_n = \frac{\alpha^\mathrm{n} - \beta^\mathrm{n}}{\alpha-\beta}, QC_n = \frac{\mathrm{a}\alpha^\mathrm{n} + \mathrm{b}\beta^\mathrm{n}}{2}, C_n = \frac{\alpha^\mathrm{n} + \beta^\mathrm{n}}{2}$$

and αβ = 1, the proof is immediately seen.

Proposition 20. For n ≥ 0, we have

$$Q\mathrm{C}_\mathrm{n} = \begin{vmatrix} Q\mathrm{C}_0 & Q\mathrm{C}_1 & 0 & 0 & & 0 & 0 \\ -1 & 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 6 & -1 & & 0 & 0 \\ & \vdots & & & \ddots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & 6 & -1 \\ 0 & 0 & 0 & 0 & & -1 & 6 \end{vmatrix}_{(n+1)\times(n+1)}$$

Proof: In Theorem 3.1 considering A = 1, B = -6, C = QC_0 and D = QC_1 , then the proof is easily seen.

5. CONCLUSION

In this paper, we presented new two sequences, these are bicomplex balancing quaternions and bicomplex Lucas- balancing quaternions. Moreover we obtained some formulas for example; Binet formula, Catalan formula, Cassini formula etc. Furthermore we presented *n-th* term of bicomplex balancing quaternion sequences as computation of a tridiagonal matrix.

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