ORIGINAL PAPER

SOME NEW APPLICATIONS OF LAPLACE-WEIERSTRASS TRANSFORM

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Abstract. *This paper is devoted to some new applications of Laplace-Weierstrass transform (i.e., for solving two dimensional diffusion equations). Solution of Cauchy's linear differential equation is also given. Some results are also given which are required for solving Cauchy's linear differential equation.*

Keywords: *Laplace transform; Weierstrass transform; Laplace-Weierstrass transform.*

1. INTRODUCTION

Integral transform is one of the most useful techniques in function transformation. It arises quite commonly not only in mathematics but also in optic, signals processing and many other areas of science and engineering. It provides new aspects to many mathematical disciplines such as transform theory, functional analysis, differential equation etc.

The theory of integral transform is flourished since the work of great mathematician Laplace on probability theory and yet it continuous to do so since having lot of applications. Like Fourier transform, Laplace transform is used for solving the differential and integral equations. The technique of solving differential equation was developed by the physicists, Oliver Heaviside in 1893. His approach was purely operational.

Bhosale [1] studied application of generalized fractional Fourier transform for solving particular types of partial differential equations. Khairnar et al. [2] explained bilateral Laplace-Mellin integral transform and its application. Applications of Laplace-Weierstrass transform are discussed in [3]. Integral transformation of generalized functions and its applications are given by Pathak [4]. Diffusion of solid particles confined in a viscous fluid are prepared and written by Ursell [5].

This paper is planned as follows: in section 2, the solution of Cauchy's linear differential equation is given; two dimensional diffusion equations are explained in section 3; lastly, conclusion is given. Notations and terminology used as in Zemanian [6].

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2. APPLICATION OF LAPLACE-WEIERTRASS TRANSFORM TO SOLVE CAUCHY'S LINEAR DIFFERENTIAL EQUATION

First some results which is required for solving Cauchy's linear differential equation were proved.

2.1. LAPLACE-WEIERSTRASS TRANSFORM OF $y\, f_y(t,y)$

For $f(t,y) \in LW^{*}_{a,b}$,

$$LW\{y\, f_y(t,y)\} = 2LW\{ f_{yy}(t,y)\} + x\, LW\{ f_y(t,y)\} + k$$

where $k = \frac{e^{\frac{-x^2}{4}}}{\sqrt{\pi}} \int_0^{\infty} e^{-st} f_y(t,0)\,dt$

Proof: We have,

$$LW\{y\, f_y(t,y)\} = \frac{1}{\sqrt{4\pi}} \int_0^{\infty}\int_0^{\infty} e^{-st-\frac{(x-y)^2}{4}}\, y\, f_y(t,y)\,dt\,dy$$

$$= \frac{1}{\sqrt{4\pi}} \left[2\int_0^{\infty} e^{-st} f_y(t,0) e^{\frac{-x^2}{4}} dt + 2\int_0^{\infty}\int_0^{\infty} f_{yy}(t,y) e^{-st-\frac{(x-y)^2}{4}} dy\,dt + x\int_0^{\infty}\int_0^{\infty} e^{-st-\frac{(x-y)^2}{4}} f_y(t,y)\,dy\,dt \right]$$

$$= 2\,LW\{ f_{yy}(t,y)\} + x\,LW\{ f_y(t,y)\} + k$$

where

$$k = \frac{e^{\frac{-x^2}{4}}}{\sqrt{\pi}} \int_0^{\infty} e^{-st} f_y(t,0)\,dt$$

2.2. LAPLACE-WEIERSTRASS TRANSFORM OF $y^2 f_{yy}(t,y)$

For $f(t,y) \in LW^{*}_{a,b}$

$$LW\{y^2 f_{yy}(t,y)\} = 4\,LW\{ f_{yyyy}(t,y)\} + 4x\,LW\{ f_{yyy}(t,y)\} + (2+x^2)LW\{ f_{yy}(t,y)\} + k_1$$

where

$$k_1 = \frac{x e^{\frac{-x^2}{4}}}{\sqrt{\pi}} \int_0^{\infty} e^{-st} f_{yy}(t,0)\,dt + \frac{2e^{\frac{-x^2}{4}}}{\sqrt{\pi}} \int_0^{\infty} e^{-st} f_{yyy}(t,0)\,dt$$

Proof: Considering,

$$LW\{y^2 f_{yy}(t,y)\} = \frac{1}{\sqrt{4\pi}} \int_0^\infty \int_0^\infty e^{-st-\frac{(x-y)^2}{4}} y^2 f_{yy}(t,y)\,dt\,dy$$

$$= \frac{1}{\sqrt{4\pi}} \left[2\int_0^\infty e^{-st} \cdot e^{\frac{-x^2}{4}} x\, f_{yy}(t,0)\,dt + 4\int_0^\infty e^{-st} \cdot e^{\frac{-x^2}{4}} f_{yyy}(t,0)\,dt + 4\int_0^\infty \int_0^\infty e^{-st-\frac{(x-y)^2}{4}} f_{yyyy}(t,y)\,dy\,dt + (2+x^2)\int_0^\infty \int_0^\infty e^{-st-\frac{(x-y)^2}{4}} f_{yy}(t,y)\,dy\,dt + 4x\int_0^\infty \int_0^\infty f_{yyy}(t,y) e^{-st-\frac{(x-y)^2}{4}}\,dy\,dt \right]$$

$$= 4\,LW\{ f_{yyyy}(t,y) \} + 4x\; LW\{ f_{yyy}(t,y) \} + (2+x^2) LW\{ f_{yy}(t,y) \} + k_1$$

where

$$k_1 = \frac{x\,e^{\frac{-x^2}{4}}}{\sqrt{\pi}} \int_0^\infty e^{-st} f_{yy}(t,0)\,dt + \frac{2e^{\frac{-x^2}{4}}}{\sqrt{\pi}} \int_0^\infty e^{-st} f_{yyy}(t,0)\,dt$$

2.3. CAUCHY'S LINEAR DIFFERENTIAL EQUATION BY USING LAPLACE-WEIERSTRASS TRANSFORM

Here, the solution of Laplace-Weierstrass transform by using above two results was given. The Cauchy's linear differential equation is $y^2 f_{yy}(t,y) + y\, f_y(t,y) + f(t,y) = 0$, where t is constant variable. The Laplace-Weierstrass transform of

$$y^2 f_{yy}(t,y) + y\, f_y(t,y) + f(t,y) = 0$$

is

$$LW\{y^2\; f_{yy}(t,y)\} + LW\{y\, f_y(t,y)\} + LW\{f(t,y)\} = 0 \qquad (2.1)$$

Using result 2.1 and 2.2, equation (2.1) becomes,

$$4\,LW\{ f_{yyyy}(t,y) \} + (2+x^2) LW\{ f_{yy}(t,y) \} + 4x\,LW\{ f_{yyy}(t,y) \} +$$
$$k_1 + 2LW\{ f_{yy}(t,y) \} + x\,LW\{ f_y(t,y) \} + k + LW\{ f(t,y) \} = 0$$
$$4\,LW\{ f_{yyyy}(t,y) \} + 4x\,LW\{ f_{yyy}(t,y) \} + (4+x^2) LW\{ f_{yy}(t,y) \} +$$
$$x\,LW\{ f_y(t,y) \} + LW\{ f(t,y) \} = -(k+k_1)$$

where

$$k = \frac{e^{\frac{-x^2}{4}}}{\sqrt{\pi}} \int_0^\infty e^{-st} f_y(t,0)dt \text{ and } k_1 = \frac{x e^{\frac{-x^2}{4}}}{\sqrt{\pi}} \int_0^\infty e^{-st} f_{yy}(t,0)dt + \frac{2e^{\frac{-x^2}{4}}}{\sqrt{\pi}} \int_0^\infty e^{-st} f_{yyy}(t,0)dt \ .$$

3. SOLUTION OF TWO DIMENSIONAL DIFFUSION EQUATION BY USING LAPLACE-WEIERSTASS TRANSFORM

Consider the two dimensional diffusion equations in Cartesian coordinates

$$\nabla^2 P - \frac{1}{D}\frac{\partial P}{\partial t} = 0$$

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} - \frac{1}{D}\frac{\partial P}{\partial t} = 0$$

where the function $P(x,y,t)$ gives the probability of finding a perfectly average particle in the small vicinity of the point (x,y) at time $'t'$. The constant D is the diffusion coefficient whose nature will be explored in a moment.

Consider the following integral relations that define the two dimensional Laplace-Weierstrass transform in Cartesian coordinates. The function $\hat{P}$ will be called the Laplace-Weierstrass transform of the original function P :

$$\hat{P}(k_x, k_y, t) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-s k_x x - \frac{k_y (z-y)^2}{4}} P(x,y,t)\, dx\, dy \tag{3.1}$$

$$P(x,y,t) = \sqrt{4\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{s k_x x + \frac{k_y (z-y)^2}{4}} \hat{P}(k_x, k_y, t)\, dk_x\, dk_y \tag{3.2}$$

Notice that, the symmetry is going forward and backward in the transform. Integration by parts will be required. Using this, let’s examine the spatial derivatives of the diffusion equation, where the second derivative is considered to be the function of interest. It can be integrated these second derivatives by parts.

Therefore from equation (3.1), is obtained:

$$\frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-s k_x x - \frac{k_y (z-y)^2}{4}} \frac{\partial^2 P}{\partial x^2}\, dx\, dy$$

$$= \frac{1}{\sqrt{4\pi}} \left[e^{-s k_x x - \frac{k_y (z-y)^2}{4}} \frac{\partial P}{\partial x} \Big|_{-\infty(x,y)}^{\infty} - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial}{\partial x} e^{-s k_x x - \frac{k_y (z-y)^2}{4}} \frac{\partial P}{\partial x}\, dx\, dy \right]$$

$$= \frac{1}{\sqrt{4\pi}} e^{-s k_x x - \frac{k_y (z-y)^2}{4}} \frac{\partial P}{\partial x} \Big|_{-\infty(x,y)}^{\infty} + \frac{s k_x}{\sqrt{4\pi}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-s k_x x - \frac{k_y (z-y)^2}{4}} \frac{\partial P}{\partial x}\, dx\, dy \tag{3.3}$$

It was observed that, in the first term of (3.3) is effectively transferred one derivative off P and put it on the exponential of the Laplace-Weierstrass transform, but since the exponential is an explicit function, it can be performed just the derivative, giving us the constant on the right most integral of the second term. It should be physically intuitive that at an infinite distance from where the authors originally introduced the statistically perfect particle there should be zero chance for all time of it appearing. This leads to the boundary condition that the function P and all of its derivatives are zero at $|x|=|y|=\infty$, eliminating the first term of (3.3) and allowing to write,

$$\frac{1}{\sqrt{4\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-s k_x x-\frac{k_y(z-y)^2}{4}}\frac{\partial^2 P}{\partial x^2}dx\,dy=\frac{s\,k_x}{\sqrt{4\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-s k_x x-\frac{k_y(z-y)^2}{4}}\frac{\partial P}{\partial x}dx\,dy \qquad (3.4)$$

The right hand side of (3.4) looks like the Laplace-Weierstrass transform of first derivative of P. We pursue this integral in the same manner.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-s k_x x-\frac{k_y(z-y)^2}{4}}\frac{\partial P}{\partial x}dx\,dy=e^{-s k_x x-\frac{k_y(z-y)^2}{4}}P\,|_{-\infty(x,y)}^{\infty}+sk_x\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-s k_x x-\frac{k_y(z-y)^2}{4}}P\,dx\,dy$$

According to boundary condition, $P=0$ at $|x|=|y|=\infty$,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-s k_x x-\frac{k_y(z-y)^2}{4}}\frac{\partial P}{\partial x}dx\,dy=sk_x\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-s k_x x-\frac{k_y(z-y)^2}{4}}P\,dx\,dy$$

Finally, from (3.4) results:

$$\frac{1}{\sqrt{4\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-s k_x x-\frac{k_y(z-y)^2}{4}}\frac{\partial^2 P}{\partial x^2}dx\,dy=(sk_x)^2\,\hat{P}$$

In general, if the above procedure continue again and again results that:

$$\frac{\partial^n}{\partial x^n}P=(s\,k_x)^n\,\hat{P}$$

Similarly,

$$\frac{\partial^n}{\partial y^n}P=\left[\frac{k_y(z-y)}{2}\right]^n p_n k_y(z-y)\hat{P} \quad ,$$

where p_n is polynomial.

Now, the two-dimension diffusion equation can be written after Laplace-Weierstrass transform as:

$$(s\,k_x)^2\,\hat{P}+\left[\frac{k_y(z-y)}{2}\right]^2 p_2 k_y(z-y)\hat{P}-\frac{1}{D}\frac{\partial\hat{P}}{\partial t}=0$$

$$\frac{\partial \hat{P}}{\partial t} - \xi \hat{P} = 0$$

where

$$\xi = D\left\{ (s\,k_x)^2 + \left[\frac{k_y(z-y)}{2} \right]^2 p_2 k_y (z-y) \right\}$$

This is first order ordinary differential equation in time with the solution by separation of variables

$$\hat{P} = Be^{\xi t}$$

The constant B is called the normalization. Equation (3.2) gives,

$$P(x,y,t) = B^{'} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{s\,k_x x + \frac{k_y (z-y)^2}{4}} e^{\xi t}\, dk_x\, dk_y$$

where

$$B^{'} = B\sqrt{4\pi}$$

and

$$\xi = D\left\{ (s\,k_x)^2 + \left[\frac{k_y(z-y)}{2} \right]^2 p_2 k_y (z-y) \right\}$$

4. CONCLUSION

The solution of Cauchy's linear differential equation and two dimensional diffusion equations were have obtained. The Laplace-Weierstrass transform is usually used to simplify a differential equation into a simple and solvable algebra problem. Even when the algebra becomes a little complex, it is still easier to solve than solving a differential equation.

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