ORIGINAL PAPER

ANALYTICAL TECHNIQUE WITH LAGRANGE MULTIPLIER FOR SOLVING SPECIFIC NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract. *This paper will use Lagrange parameter in Adomain decomposition method to suggest new method for solving nonlinear differential equation. This method will be highly order convergent. Also, this method will be compared with old existence method. At last, some numerical examples will be given to illustrate the efficiency of newly developed method.*

Keywords: *variational iteration method; Lagrange multiplier; differential equation; Maple 18.*

1. INTRODUCTION

In recent years, applications of the nonlinear differential problems [1-12] have very important role in different sciences. Analytic solutions of the problem play significant role in understanding colorful features of the phenomenon involved. There are many methods had been developed to nonlinear differential equation, but the most important methods are Variational iterative method and Adomain decomposition method [2, 5].

The variational iteration method is the most comprehensive, simple and user friendly technique to solve the differential equations. For the first time introduced by JI-Huan He [8-10]. It has been extensively used by many authors to solve problems with high non-linearity. JI-Huan He used this technique for approximate solutions for non-linear differential equations. The involvement of Lagrange Multipliers in VIM reduces successive integration and the cumbersome of huge computational work while still maintaining a very high level of accuracy.

2. ANALYSIS OF THE METHOD

Consider the general differential equation as:

$$LH + NH = g(x), \tag{1}$$

In above equation H is unknown function which is to be determined, L is linear operator and N are linear and operator, and $g(x)$ is the inhomogeneous term. The correction functional for above equation [8-11] is given by

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$$H_{n+1} = v_n(x) + \int_0^x \lambda(t)\left[LH_n(t) + NH_n(t) - g(t)\right]dt, \tag{2}$$

where λ is Lagrange's multiplier and it can be a constant or functions.

In this method, first was determined the value of Lagrange multiplier which can be determined optimally via integration by parts and by using restricted variation. By using the value of Lagrange multiplier [9] determine the successive approximations $H_{n+1}(x)$ of the solution $H(x)$. The zero*th* ordered approximation $H_0(x)$ can be any selective function. Finally the solution is given by

$$H(x) = \lim_{n\to\infty} H_n(x). \tag{3}$$

3. SOLUTION PROCEDURE

In this section, some numerical examples were presented to show the efficiency and high accuracy of the proposed method. Maple is use to give the numerical values in easy way.

Problem 3.1

Solve the following linear homogeneous ordinary differential equation

$$H'''(x) - H(x) = 0, \tag{4}$$

with conditions

$$H(0)=1, \;\; H'(0)=1, \; H''(0)=1. \tag{5}$$

This problem has exact solution as

$$H(x)=e^x.$$

Substitute the given eq. (4) in the iterative formula

$$H_{n+1}(x) = H_n(x) + \int_0^x \lambda(s)\left(-\frac{d^3H_n(s)}{ds^3} - \tilde{H}_n(s)\right)ds, \; n \geq 0. \tag{6}$$

The approximate Lagrange multiplier is

$$\lambda(s) = \frac{(s-x)^2}{2}\,. \tag{7}$$

Using the eq. (7), the eq. (6) becomes

$$H_{n+1}(x)=H_n(x)+\int_0^x\left(\frac{(s-x)^2}{2}\right)\left(\frac{d^3H_n(s)}{ds^3}-H_n(s)\right)ds,\ n\geq 0. \tag{8}$$

Using initial conditions in eq. (8) and solving, the following approximations were obtained:

$$\begin{aligned}
H_0 &= 1+x+\frac{1}{2}x^2,\\
H_1 &= 1+x+\frac{1}{2}x^2-\frac{1}{6}x^3-\frac{1}{24}x^4-\frac{1}{120}x^5,\\
H_2 &= 1+x+\frac{1}{2}x^2-\frac{1}{2}x^3-\frac{1}{8}x^4-\frac{1}{40}x^5,\\
&\vdots\\
H_n(x) &= 1+x+\frac{1}{2}x^2-\frac{1}{2}x^3-\frac{1}{8}x^4-\frac{1}{40}x^5\cdots
\end{aligned} \tag{9}$$

If the exact Lagrange multiplier for eq. (6) is as

$$\lambda(s)=-\frac{1}{3}e^{\frac{-x}{2}}\left(e^{\frac{3x}{2}}e^{-s}-e^{\frac{s}{2}}\cos\left(\frac{2\sqrt{3}}{2}s-\frac{2\sqrt{3}}{2}x\right)+\sqrt{3}e^{\frac{s}{2}}\sin\left(\frac{2\sqrt{3}}{2}s-\frac{2\sqrt{3}}{2}x\right)\right) \tag{10}$$

Now the iteration formula for given eq. (4) and for the exact Lagrange multiplier is

$$\begin{aligned}
H_{n+1}(x)=H_n(x)+\int_0^x&\left(-\frac{1}{3}e^{\frac{-x}{2}}\left(e^{\frac{3x}{2}}e^{-s}-e^{\frac{s}{2}}\cos\left(\frac{2\sqrt{3}}{2}s-\frac{2\sqrt{3}}{2}x\right)+\sqrt{3}e^{\frac{s}{2}}\sin\left(\frac{2\sqrt{3}}{2}s-\frac{2\sqrt{3}}{2}x\right)\right)\right)\\
&\left(\frac{d^3H_n(s)}{ds^3}-H_n(s)\right)ds,\ n\geq 0.
\end{aligned} \tag{11}$$

Table 1. Error Analysis of problem (3.1)

X	**Exact Solution *U(X)***	**Solution by VIM using ELM *V(X)***	**Solution by VIM using ALM *W(X)***	**Error |*U(X)- V(X)*|**	**Error |*U(X)- W(X)*|**
0.00	1.00000	1.00000	1.00000	0.00000E+00	0.00000E+00
0.10	1.10517	1.10517	1.104487	0.00000E+00	6.83668E-04
0.20	1.22140	1.22140	1.21579	0.00000E+00	5.61076E-03
0.30	1.34986	1.34986	1.33043	0.00000E+00	1.94320E-02
0.40	1.49182	1.49182	1.44454	0.00000E+00	4.72806E-02
0.50	1.64872	1.64872	1.55391	0.00000E+00	9.48150E-02
0.60	1.82212	1.82212	1.65385	0.00000E+00	1.68262E-01
0.70	2.01375	2.01375	1.73928	0.00000E+00	2.74467E-01
0.80	2.22554	2.22554	1.80461	0.00000E+00	4.20933E-01
0.90	2.45960	2.45960	1.84372	0.00000E+00	6.15877E-01
1.00	2.71828	2.71828	1.85000	0.00000E+00	8.68282E-01

Again, applying the initial guess, the approximations were obtained, as given below

$$\begin{aligned} &H_0 = 1 + x + \frac{1}{2}x^2, \\ &H_1 = e^x, \\ &H_2 = e^x, \\ &\vdots \\ &H_n(x) = e^x. \end{aligned} \tag{12}$$

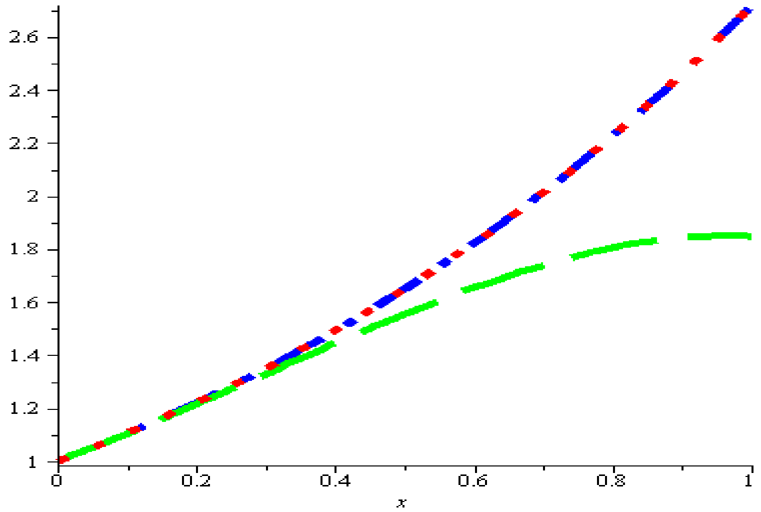

Figure 1. The 2D plot of problem (3.1) for exact solution (blue), solution with approximate Lagrange multiplier (green) and solution with exact Lagrange Multiplier (red).

Problem 3.2

Solve the following linear non-homogeneous ordinary differential equation

$$H''(x) + H(x) = 1 \tag{13}$$

with conditions

$$H(0) = 0, \quad H'(0) = 0. \tag{14}$$

This problem has exact solution as

$$H(x) = -\cos x. \tag{15}$$

Substitute the given eq. (13) in the iterative formula

$$H_{n+1}(x) = H_n(x) + \int_0^x \lambda(s)\left(\frac{d^2 H_n(s)}{ds^2} + \tilde{H}_n(s) - \tilde{1}\right) ds, n \geq 0. \tag{16}$$

The approximate Lagrange multiplier is

$$\lambda(s) = s - x. \tag{17}$$

Using the eq. (17), the eq. (16) becomes

$$H_{n+1}(x)=H_n(x)+\int_0^x (s-x)\left(\frac{d^2H_n(s)}{ds^2}+y\tilde{H}_n(s)-1\right)ds, n\geq 0. \tag{18}$$

Using initial conditions in eq. (18) and solving, the following approximations were obtained

$$\begin{aligned} H_0 &= 0, \\ H_1 &= \frac{1}{2}x^2, \\ H_2 &= \frac{1}{2}x^2-\frac{1}{24}x^4, \\ &\vdots \\ H_n(x) &= \frac{1}{2}x^2-\frac{1}{24}x^4\cdots \end{aligned} \tag{19}$$

The exact Lagrange multiplier for eq. (18) is as

$$\lambda(s)=\sin(s-x). \tag{20}$$

Now the iteration formula for given eq. (16) and for the exact Lagrange multiplier is

$$H_{n+1}(x)=H_n(x)+\int_0^x (\sin(s-x))\left(\frac{d^2H_n(s)}{ds^2}+H_n(s)-1\right)ds, \ n\geq 0. \tag{21}$$

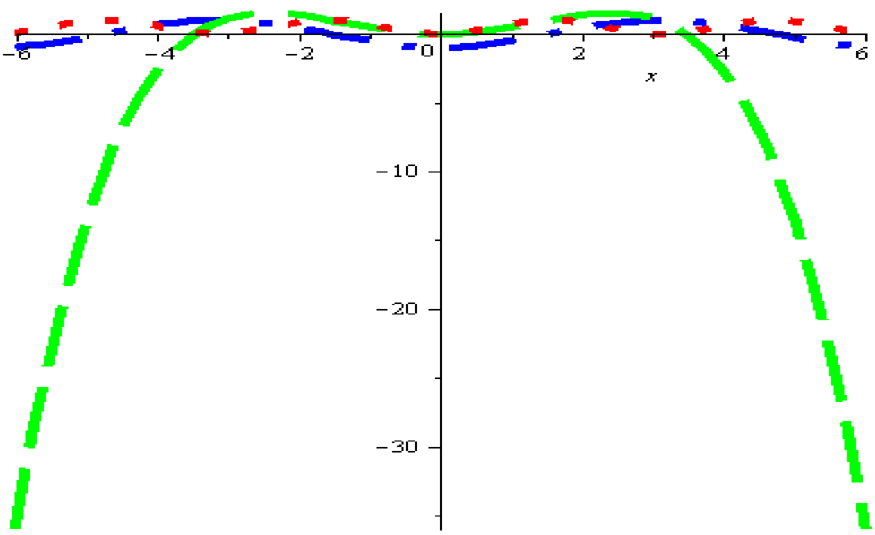

Figure 2.The 2D plot of problem (3.2) for exact solution (blue), solution with approximate Lagrange multiplier (green) and solution with exact Lagrange multiplier (red).

Table 2. Error Analysis of problem (3.2)

X	Exact Solution $U(x)$	Solution by VIM using ELM $V(x)$	Solution by VIM using ALM $W(x)$	Error $\lvert U(x)- V(x)\rvert$	Error $\lvert U(x)- W(x)\rvert$
-6.00	-0.96017	0.07807	-36.00000	1.03824E+00	3.50398E+01
-4.00	-0.08750	0.99234	-10.59840	1.07984E+00	1.05109E+01
-3.00	0.896758	0.19582	-0.51840	7.00934E-01	1.41516E+00
-2.00	0.73739	0.45625	1.49760	2.81143E-01	7.60206E-01
-1.00	-0.36236	0.86870	0.63360	1.23105E+00	9.95958E-01
0.00	-1.00000	0.00000	0.00000	1.00000E+00	1.00000E+00
1.00	-0.36236	0.86870	0.63360	1.23105E+00	9.95958E-01
2.00	0.737394	0.45625	1.49760	2.81143E-01	7.60206E-01
3.00	0.89676	0.19582	-0.51840	7.00934E-01	1.41516E+00
4.00	-0.08750	0.99234	-10.59840	1.07984E+00	1.05109E+01
6.00	-0.96017	0.07807	-36.00000	1.03824E+00	3.50398E+01

Again, applying the initial guess, the following approximations were obtained:

$$\begin{aligned} &H_0 = 0, \\ &H_1 = \frac{1}{2}x^2 - \frac{1}{24}x^4, \\ &H_2 = x^2 - \frac{1}{3}x^4, \\ &\vdots \\ &H_n(x) = x^2 - \frac{1}{3}x^4. \end{aligned} \tag{22}$$

Problem 3.3

Solve the following linear non-homogeneous ordinary differential equation

$$H''(x) + H(x) = x, \tag{23}$$

with conditions

$$H(0) = 1,\ H'(0) = 1. \tag{24}$$

This problem has exact solution as

$$H(x) = x + \cos x.$$

Substitute the given eq. (24) in the iterative formula

$$H_{n+1}(x) = H_n(x) + \int_0^x \lambda(s)\left(\frac{d^2H_n(s)}{ds^2} + \tilde{H}_n(s) - \tilde{s}\right)ds, n \ge 0. \tag{25}$$

The approximate Lagrange multiplier is

$$\lambda(s) = s - x. \tag{26}$$

Using the eq. (25), the eq. (26) becomes

$$H_{n+1}(x) = H_n(x) + \int_0^x (s-x)\left(\frac{d^2 H_n(s)}{ds^2} + H_n(s) - s\right)ds, n \geq 0. \tag{27}$$

Using initial conditions in eq. (27) and solving, the following approximations were obtained

$$\begin{aligned} H_0 &= 1 + x, \\ H_1 &= 1 + x - \frac{1}{2}x^2, \\ H_2 &= 1 + x - \frac{1}{2}x^2 + \frac{1}{24}x^4, \\ &\vdots \\ H_n(x) &= 1 + x - \frac{1}{2}x^2 + \frac{1}{24}x^4 \cdots \end{aligned} \tag{28}$$

The exact Lagrange multiplier for eq. (27) is as

$$\lambda(s) = \sin(s-x). \tag{29}$$

Now the iteration formula for given eq. (23) and for the exact Lagrange multiplier is

$$H_{n+1}(x) = H_n(x) + \int_0^x \sin(s-x)\left(\frac{d^2 H_n(s)}{ds^2} + H_n(s) - s\right)ds, n \geq 0. \tag{30}$$

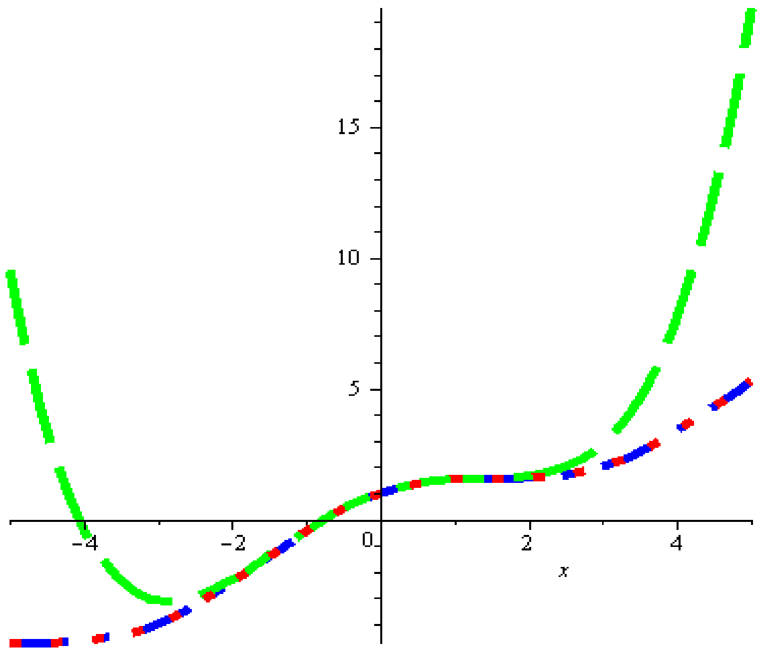

Figure 3.The 2D plot of problem (3.3) for exact solution (blue), solution with approximate Lagrange multiplier (green) and solution with exact Lagrange multiplier (red).

Table 3. Error Analysis of problem (3.3)

X	**Exact Solution** *U(X)*	**Solution by VIM using ELM** *V(X)*	**Solution by VIM using ALM** *W(X)*	**Error** *|U(X)- V(X)|*	**Error** *|U(X)- W(X)|*
-5.00	-4.71634	-4.71634	9.54167	0.00000E+00	1.42580E+01
-4.00	-4.65364	-4.65364	-0.33333	0.00000E+00	4.32031E+00
-3.00	-3.99000	-3.99000	-3.12500	0.00000E+00	8.64992E-01
-2.00	-2.41615	-2.41615	-2.33333	0.00000E+00	8.28135E-02
-1.00	-0.45970	-0.45970	-0.45833	0.00000E+00	1.36436E-03
0.00	1.00000	1.00000	1.00000	0.00000E+00	0.00000E+00
1.00	1.54030	1.54030	1.54167	0.00000E+00	1.36436E-03
2.00	1.58385	1.58385	1.66667	0.00000E+00	8.28135E-02
3.00	2.01001	2.01001	2.87500	0.00000E+00	8.64992E-01
4.00	3.34636	3.34636	7.66667	0.00000E+00	4.32031E+00
5.00	5.28366	5.28366	19.54166	0.00000E+00	1.42580E+01

Again, applying the initial guess, the following approximations were obtained

$$\begin{aligned} H_0 &= 1+x, \\ H_1 &= x-\sin x+\cos x+\cos x, \\ H_2 &= x+\cos x, \\ &\vdots \\ H_n &= x+\cos(x). \end{aligned} \tag{31}$$

Problem 3.4

The following non-homogeneous ordinary differential equation is given as

$$H''(x)\ -3H'(x)\ +2H(x)=2x-3, \tag{32}$$

with conditions

$$H(0)=1,\ \ H'(0)=2. \tag{33}$$

This problem has exact solution as

$$H(x)=x+e^{x}. \tag{34}$$

Solution: Substitute the given eq. (32) in the iterative formula

$$H_{n+1}(x)=H_n(x)+\int_0^x \lambda(s)\left(\frac{d^2H_n(s)}{ds^2}-3\frac{d\ \tilde{H}_n(s)}{ds}+2\tilde{H}_n(s)-2\tilde{s}+\tilde{3}\right)ds,\ n\geq 0. \tag{35}$$

The approximate Lagrange multiplier is

$$\lambda(s)=s-x. \tag{36}$$

Using the eq. (36), the eq. (35) becomes

$$H_{n+1}(x)=H_n(x)+\int_0^x (s-x)\left(\frac{d^2H_n(s)}{ds^2}-3\frac{d\ H_n(s)}{ds}+2y_n(s)-2s+3\right)ds, n\geq 0. \tag{37}$$

Using initial conditions in eq. (37) and solving, the following approximations were obtained

$$\begin{aligned}
H_0 &= 1+2x,\\
H_1 &= 1+2x+\frac{1}{2}x^2+\frac{1}{3}x^3,\\
H_2 &= 1+2x+\frac{1}{2}x^2+\frac{5}{6}x^3+\frac{1}{6}x^4-\frac{1}{30}x^5,\\
&\vdots\\
H_n &= 1+2x+\frac{1}{2}x^2+\frac{5}{6}x^3+\frac{1}{6}x^4-\frac{1}{30}x^5\ldots
\end{aligned} \tag{38}$$

The exact Lagrange multiplier for eq. (35) is as

$$\lambda(s)=\frac{\sin\left(\sqrt{2}s-\sqrt{2}x\right)}{\sqrt{2}}. \tag{39}$$

Now the iteration formula for given eq. (32) and for the exact Lagrange multiplier is

$$\begin{aligned}
H_{n+1}(x)=H_n(x)+\int_0^x &\left(\frac{\sin\left(\sqrt{2}s-\sqrt{2}x\right)}{\sqrt{2}}\right)\\
&\left(\frac{d^2H_n(s)}{ds^2}-3\frac{d\ H_n(s)}{ds}+2H_n(s)-2s+3\right)ds,\ \ n\geq 0.
\end{aligned} \tag{40}$$

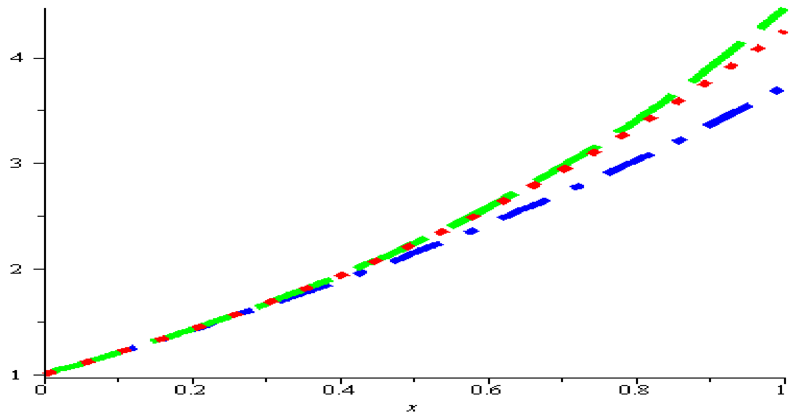

Figure 4. The 2D plot of Eq. (3.4) for exact solution(blue), solution with approximate Lagrange multiplier (green) and solution with exact Lagrange multiplier (red).

Table 4. Error Analysis of problem (3.4).

X	Exact Solution *U*(*X*)	Solution by VIM using ELM *V*(*X*)	Solution by VIM using ALM *W*(*X*)	Error |*U*(*X*)- *V*(*X*)|	Error |*U*(*X*)- *W*(*X*)|
0.00	1.00000	1.00000	1.00000	0.00000E+00	0.00000E+00
0.10	1.20517	1.20585	1.20585	6.77022E-04	6.78749E-04
0.20	1.42140	1.22140	1.42692	5.46286E-03	5.51991E-03
0.30	1.64986	1.66832	1.66877	1.84639E-02	1.89102E-02
0.40	1.89182	1.93533	1.93726	4.35017E-02	4.54340E-02
0.50	2.14872	2.23250	2.23854	8.37750E-02	8.98204E-02
0.60	2.42211	2.56362	2.57901	1.41500E-01	1.56889E-01
0.70	2.71375	2.93129	2.96525	2.17535E-01	2.51495E-01
0.80	3.02554	2.33654	3.40401	3.10999E-01	3.78470E-01
0.90	3.35960	3.77849	3.90217	4.18884E-01	5.42564E-01
1.00	3.71828	4.25397	4.46667	5.35686E-01	7.48385E-01

Again, applying the initial guess, the following approximations were obtained

$$\begin{aligned}
&H_0 = 1 + 2x,\\
&H_1 = 1 + 2x + \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{12}x^4 - \frac{1}{10}x^5 + \frac{1}{180}x^6 + \frac{1}{126}x^7,\\
&H_2 = 1 + 2x + \frac{1}{2}x^2 + \frac{5}{6}x^3 + \frac{1}{6}x^4 - \frac{1}{5}x^5 - \frac{11}{180}x^6 + \frac{19}{1260}x^7,\\
&\vdots\\
&H_n(x) = 1 + 2x + \frac{1}{2}x^2 + \frac{5}{6}x^3 + \frac{1}{6}x^4 - \frac{1}{5}x^5 - \frac{11}{180}x^6 + \frac{19}{1260}x^7.
\end{aligned} \tag{41}$$

4. CONCLUSION

It may be concluded that, the variational iteration method is very powerful and efficient analytical technique to finding the solution by using exact Lagrange multiplier, as well as, approximate Lagrange multipliers. The graphical comperision of all problems are given to elobrate the efficiency of verivational iteration method. This technique also motivates us to developed different highly convergent method by combining existence methods.

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