

EXAMINATION OF SETS OF POINTS AND NON-NULL LINES UNDER THE TWO-PARAMETER LORENTZIAN MOTION

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Abstract. *In this paper, firstly, we interested in finding the relationships among the densities of sets of collinear points, among the densities of non-collinear points and among the densities of sets of intersecting non-null lines in Lorentzian plane. Furthermore, we concerned with the density formulas of sets of points and the sets of non-null lines under the two-parameter planar Lorentzian motion.*

Keywords: *point density, line density, two-parameter motion, Lorentzian plane.*

1. INTRODUCTION

Between the years 1935 and 1939, W. Blaschke and his colleagues presented a series of 'Integral Geometry' presentations at the Mathematical Seminars at the University of Hamburg. Thereafter, W. Blaschke published this work in book entitled "Vorlesungen über Integralgeometrie" [1]. In this book, he gave the density for sets of points/lines and the geometric meanings of the integrals of these differential forms. Undoubtedly, his studies have been accepted as the main classical references for integral geometry. Integral geometry has improved by the researchers of W. Blaschke, A. Weil, S. S. Chern, L. A. Santalo. After their contributions, numerous studies have been examined on integral geometry in Euclidean and non-Euclidean planes/spaces. L. A. Santalo published the book "Integral geometry and geometric probability" which is one of the comprehensive books on this area [2]. He, who is one of the student of W. Blaschke, referred both classical and modern Euclidean and non-Euclidean integral geometries. The products of the densities of sets of points/lines are given in [2]. Using the studies [2, 3], G. S. Birman introduced the equivalent expressions of some of the formulas of integral geometry in the Lorentzian plane [4]. The densities for a set of points and non-null lines and the products of the densities of sets of points and non-null lines are defined in the Lorentzian plane by Birman [4].

Kinematically, the two-parameter planar Lorentzian motions are introduced with different but equivalent definitions and examined with different viewpoint in the studies [5-7]. The geometry of such a two-parameter motion of points, objects/bodies and systems of objects/bodies has a number of applications in physics, bio mechanics, geometric design and trajectory of robotics. In [8], the authors give the density formulas for the sets of points and lines under the two-parameter planar Euclidean motion and obtain the connection among the densities of the sets of points and among the densities of the sets of intersecting lines under the two-parameter planar Euclidean motion.

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We wish to investigate the relations between the densities of sets of collinear/non-collinear points and intersecting non-null lines under the two-parameter planar Lorentzian motion. Due to these, firstly, we give the the relations between the densities of sets of collinear/non-collinear points and intersecting non-null lines in the Lorentzian plane. After that, we define the density formulas for sets of points and non-null lines under the two-parameter planar Lorentzian motion. Using all these results, we finally get the relations between the densities of sets of collinear/non-collinear points, the relations between densities of sets of intersecting non-null lines, the product of the densities of sets of points and non-null lines under the two-parameter planar Lorentzian. Under the motion, the density of points, except for the pole points, and the density of non-null lines, except for the polar axes, are not invariant. This leads to differences in our study. Since Lorentzian plane geometry has similarities and differences from the Euclidean plane geometry, this situation is much more complicated than the Euclidean case.

2. PRELIMINARIES

This section presents some basic notions on density of sets of points and non-null lines and two-parameter planar motions in the Lorentzian plane.

2.1. BASIC NOTATIONS

The Lorentzian plane \mathbb{L} is the vector space \mathbb{R}^2 provided with Lorentzian inner product $\langle \cdot, \cdot \rangle_L$ given by

$$\langle X, Y \rangle_L = x_1x_2 - y_1y_2,$$

where $X(x_1, y_1)$ and $Y(x_2, y_2)$. Since $\langle \cdot, \cdot \rangle_L$ is an indefinite metric, a vector X in the Lorentzian plane \mathbb{L} can have one of three casual characters: it can be spacelike if $\langle X, X \rangle_L > 0$ or $V = 0$; timelike if $\langle X, X \rangle_L < 0$; null(lightlike) if $\langle X, X \rangle_L = 0$ [3].

Definition 1: The measure of a set A of points $X(x, y)$ in the Lorentzian plane is defined by the Lebesque integral of the differential form

$$dX = dx \wedge dy, \tag{1}$$

over the set A , i.e., $m(A) = \int_A dX$. The differential form in the equation (1) is called the density for points X [4].

Definition 2: A timelike line (or a spacelike line) with respect to the coordinate system $\{O; \mathbf{l}_1, \mathbf{l}_2\}$ in Lorentzian plane \mathbb{L} is denoted by L_t (or L_s) and it is defined by the equation

$$x \cosh \Psi - y \sinh \Psi = p \text{ (or } x \sinh \Psi - y \cosh \Psi = p)$$

with the Hesse-coordinates (p, Ψ) , where p is the distance from the line L_i (or L_s) to the origin and $\Psi \in \mathbb{R}$ [4].

Definition 3: The measure of a set B of non-null lines $L(p, \Psi)$ in the Lorentzian plane \mathbb{L} is defined by the Lebesgue integral of the differential form

$$dL = dp \wedge d\Psi,$$

over the set B , i.e., $m(B) = \int_L dL$. The differential form dL is called the density for non-null lines L [4].

Let $X(x_1, y_1), Y(x_2, y_2)$ be sets of points and $L(p, \Psi)$ be a set of non-null lines passing through these points in the Lorentzian plane \mathbb{L} . The pairs of points X, Y can be determined by the four coordinates x_1, y_1, x_2, y_2 , as well as by the Hesse-coordinates p, Ψ of the non-null line L and the distances λ_1, λ_2 from the points X, Y to the foot H of the perpendicular from the origin O to L . The density for a pair of points $dX \wedge dY$

$$dX \wedge dY = |\lambda_1 - \lambda_2| dL \wedge d\lambda_1 \wedge d\lambda_2 \tag{2}$$

can be calculated in two ways depending on whether the line L is timelike or spacelike, respectively. We write the absolute value of $\lambda_1 - \lambda_2$, because all densities are assumed to be positive.

Note that the equation (2) is the same expressions as the density of a pair of points in Euclidean plane [4].

Let $L_i(p_i, \Psi_i), i = 1, 2$, be two sets of different non-null lines that intersect at the points $X(x, y)$ and make angles Φ_1, Φ_2 with the x axis in Fig. 1, respectively. The density for a pair of non-null lines L_1 and L_2 can be investigated in three ways depending on whether the lines are timelike or spacelike.

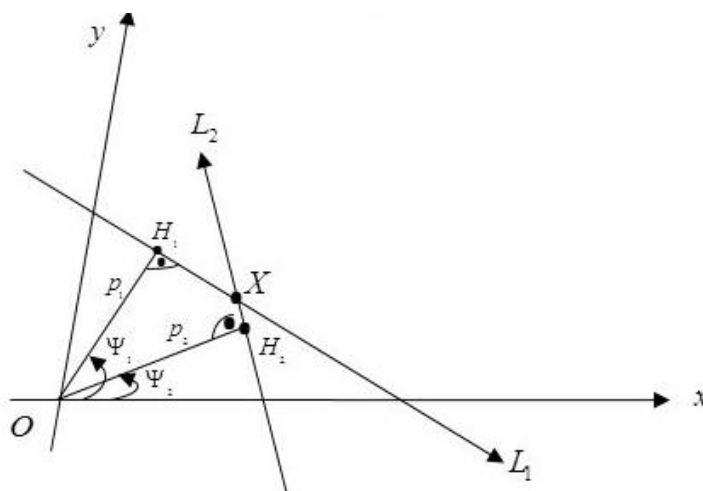


Figure 1. A pair of non-null lines in the Lorentzian plane \mathbb{L}

i. A pair of timelike lines

Let $L_i(p_i, \Psi_i), i=1,2$, be two sets of timelike lines that intersect at the points $X(x, y)$. Hence, we can write

$$\begin{aligned} L_{t_1} \dots x \cosh(\Phi_1 - a) - y \sinh(\Phi_1 - a) &= p_1, \\ L_{t_2} \dots x \cosh(\Phi_2 - a) - y \sinh(\Phi_2 - a) &= p_2, \end{aligned}$$

for the timelike lines, where $a \in \mathbb{R}$. Since, $d\Phi_1 = d\Psi_1, d\Phi_2 = d\Psi_2$ can be written, we get

$$dL_{t_1} \wedge dL_{t_2} = |\sinh(\Phi_2 - \Phi_1)| dX \wedge d\Phi_1 \wedge d\Phi_2, \quad (3)$$

for the density of a pair of timelike lines [4].

ii. A pair of spacelike lines

If $L_s(p_i, \Psi_i), i=1,2$, are two sets of spacelike lines that intersect at the points $X(x, y)$, then we have the following equation for the density of a pair of spacelike lines [4]

$$dL_{s_1} \wedge dL_{s_2} = |\sinh(\Phi_1 - \Phi_2)| dX \wedge d\Phi_1 \wedge d\Phi_2. \quad (4)$$

iii. A pair of timelike and spacelike lines

Note that, unlike the article [4], if $L_t(p_1, \Psi_1)$ and $L_s(p_2, \Psi_2)$ are sets of timelike lines and spacelike lines that intersect at the points $X(x, y)$, we can write for the density of a pair of timelike and spacelike lines,

$$dL_t \wedge dL_s = \cosh(\Phi_1 - \Phi_2) dX \wedge d\Phi_1 \wedge d\Phi_2. \quad (5)$$

2.2. TWO-PARAMETER PLANAR LORENTZIAN MOTIONS

Definition 4: Let \mathbb{L} and \mathbb{L}' be moving and fixed Lorentzian planes; and $\{O; \mathbf{i}_1, \mathbf{i}_2\}$ and $\{O'; \mathbf{i}'_1, \mathbf{i}'_2\}$ be their orthonormal coordinate systems, respectively. Let us take the point $X(x, y) \in \mathbb{L}$ and $X'(x', y') \in \mathbb{L}'$ with respect to the moving and the fixed coordinate systems. A general planar Lorentzian motion is given by the equations

$$\begin{aligned} x' &= x \cosh \Theta + y \sinh \Theta - u_1 \cosh \Theta - u_2 \sinh \Theta, \\ y' &= x \sinh \Theta + y \cosh \Theta - u_1 \sinh \Theta - u_2 \cosh \Theta. \end{aligned} \quad (6)$$

If Θ, u_1, u_2 are given by continuously differentiable functions of parameters t_1 and t_2 , then the motion is called as two-parameter Lorentzian motion \mathbb{L} / \mathbb{L}' . We denote this motion

briefly by H_{II} . Here, if t_1 and t_2 are functions of the parameter t , then one-parameter planar Lorentzian motion H_I is obtained [5].

Two-parameter planar Lorentzian motion given by the equation $\{x'\}$ can be written in the form

$$X' = AX + C, X' = [x' \ y']^T, X = [x \ y]^T, C = [a \ b]^T \tag{7}$$

where, $A \in SO(2,1)$, $a = -u_1 \cosh \Theta - u_2 \sinh \Theta$, $b = -u_1 \sinh \Theta - u_2 \cosh \Theta$ and X' and X are the position vectors of the same point $X(x, y)$ with respect to the moving and fixed coordinate systems, respectively, and C is the translation vector of the motion. The general planar motion is defined by the combination of a rotation and translation.

By taking derivatives the equation (7), we get

$$dX' = dAX + AdX + dC$$

where the velocities $V_a = dX'$, $V_f = dAX + dC$, $V_r = dX$ are called as absolute, sliding and relative velocities of the point X [5].

Let $X'(x', y')$ be the image of the point X and

$$\begin{aligned} L_t' \dots x' \cosh \Psi' - y' \sinh \Psi' &= p', \\ L_s' \dots x' \sinh \Psi' - y' \cosh \Psi' &= p', \end{aligned}$$

are the images of timelike line $L_t(p, \Psi)$ and spacelike line $L_s(p, \Psi)$ under the motion H_{II} , respectively, where, $L_t'(p', \Psi')$ is defined with helps of the direction angle $\Theta + \Psi = \Psi'$ and the distance $p' = p - u_1 \cosh \Psi + u_2 \sinh \Psi$ from the origin O' . Similarly, $L_s'(p', \Psi')$ is defined with helps of the direction angle $\Theta + \Psi = \Psi'$ and the distance $p' = p - u_1 \sinh \Psi + u_2 \cosh \Psi$ from the origin O' [9].

3. ON THE DENSITY OF A SET OF POINTS IN THE LORENTZIAN PLANE

In this original section, our purpose is to give the relations between the densities of sets of points in the Lorentzian plane \mathbb{L} .

3.1 THE RELATION BETWEEN THE DENSITIES OF THE SETS OF POINTS

Theorem 5: Let $X(x_1, y_1)$ and $Y(x_2, y_2)$ be sets of different points, $L(p, \Psi)$ be a support function the points X and Y , namely, $L(p, \Psi)$ be a set of non-null lines that passes through these points. Let take λ as a distance function from X to Y in the Lorentzian plane \mathbb{L} . In this case, we can write

$$dY = dX + \lambda d\Psi \wedge d\lambda + \langle \mathbf{n}, dX \rangle_L \wedge d\lambda + \lambda \langle \mathbf{u}, dX \rangle_L \wedge d\Psi, \tag{8}$$

for density of a set of points Y , where $\mathbf{dX} = (dx, dy)$ is the relative velocity vector of the points X , \mathbf{u} is the directrix vector and \mathbf{n} is the normal vector of the line $L(p, \Psi)$.

Proof: Our aim to find the density of a set of points Y with the density of the set of points X . The theorem is another way of finding the density of the points Y . This proof will be divided into two parts depending on whether the line L is timelike or spacelike, respectively.

- i.* Let $L_t(p, \Psi)$ be a timelike line that passes through these points. In this case, the coordinates of the points Y can be written as

$$(x_2, y_2) = (x_1 + \lambda \sinh \Psi, y_1 + \lambda \cosh \Psi),$$

where $\mathbf{u} = (\sinh \Psi, \cosh \Psi)$ is the directrix vector of the timelike line $L_t(p, \Psi)$. Thereby, we have the following equation for the density of the set of points Y ,

$$dY = dX + \lambda d\Psi \wedge d\lambda + \langle \mathbf{n}, \mathbf{dX} \rangle_L \wedge d\lambda + \lambda \langle \mathbf{u}, \mathbf{dX} \rangle_L \wedge d\Psi, \quad (9)$$

where $\mathbf{dX} = (dx, dy)$ is the relative velocity vector of the points X , $\mathbf{n} = (\cosh \Psi, \sinh \Psi)$ is the normal vector of the timelike line $L_t(p, \Psi)$.

- ii.* Let $L_s(p, \Psi)$ be a spacelike line that passes through these points. So, we have

$$dY = dX + \lambda d\Psi \wedge d\lambda + \langle \mathbf{n}, \mathbf{dX} \rangle_L \wedge d\lambda + \lambda \langle \mathbf{u}, \mathbf{dX} \rangle_L \wedge d\Psi,$$

for the density of the points Y , where $\mathbf{n} = (\sinh \Psi, \cosh \Psi)$ is the normal vector and $\mathbf{u} = (\cosh \Psi, \sinh \Psi)$ is the directrix vector of the spacelike line $L_s(p, \Psi)$.

Corollary 6: Let $X(x_1, y_1)$ and $Y(x_2, y_2)$ be sets of different points, $L(p, \Psi)$ be a set of non-null lines that passes through the points X, Y . Let take λ as a constant distance function from X to Y . In this case, we get

$$dY = dX + \lambda \langle \mathbf{u}, \mathbf{dX} \rangle_L \wedge d\Psi,$$

for the density of a set of points Y , where $\mathbf{dX} = (dx, dy)$ is the relative velocity vector of the points X , \mathbf{u} is the directrix vector of the line $L(p, \Psi)$.

3.2 THE RELATION BETWEEN THE DENSITIES OF THE SETS OF COLLINEAR POINTS

Theorem 7: Let $X(x_1, y_1), Y(x_2, y_2)$ and $Q(x_3, y_3)$ be sets of collinear points in the Lorentzian plane \mathbb{L} . Then, the equivalent formulations of the density of a set of points Q can be given as follows:

$$i. \quad dQ = \lambda(2\lambda - 1)dX + (\lambda - 1)(2\lambda - 1)dY + 4\lambda(1 - \lambda)dM,$$

$$ii. \quad dQ = \lambda dX + (1 - \lambda)dY - \lambda(1 - \lambda)dXY,$$

where $(\lambda, 1 - \lambda)$ is the barycentric coordinate of the points Q with respect to the system $\{\mathbf{O}, \mathbf{X}, \mathbf{Y}\}$, M is a set of middle points of \overline{XY} and dXY a mixed point density of the points X and Y .

Proof: Using the barycentric coordinate system with respect to the system $\{\mathbf{O}, \mathbf{X}, \mathbf{Y}\}$, the following equation can be written

$$\mathbf{Q} = \lambda\mathbf{X} + (1 - \lambda)\mathbf{Y}.$$

Hence, we can write the following equation for the density of the set of points Q

$$dQ = \lambda^2 dX + (1 - \lambda)^2 dY + \lambda(1 - \lambda)(dx_1 \wedge dy_2 + dx_2 \wedge dy_1). \quad (10)$$

Here are another two ways of stating the equation (10):

i. Let M be a set of middle points of \overline{XY} . Then, we get the following equation

$$dM = d\left(\frac{x_1 + x_2}{2}\right) \wedge d\left(\frac{y_1 + y_2}{2}\right).$$

for density of the points M . Using this equality, we have

$$4dM - dX - dY = dx_1 \wedge dy_2 + dx_2 \wedge dy_1. \quad (11)$$

Substituting the equation (11) into the equation (10), we obtain

$$dQ = \lambda(2\lambda - 1)dX + (\lambda - 1)(2\lambda - 1)dY + 4\lambda(1 - \lambda)dM,$$

ii. Let dXY be a mixed point density of points X and Y which defined by the equation

$$-dx_1 \wedge dy_2 - dx_2 \wedge dy_1 + dX + dY = (dx_2 - dx_1) \wedge (dy_2 - dy_1) = dXY. \quad (12)$$

Substituting the equation (12) into the equation (10), we have

$$dQ = \lambda dX + (1 - \lambda)dY - \lambda(1 - \lambda)dXY,$$

for the density of the set of points Q .

Note that the Theorem 7 is the same expressions as the density of a pair of points in Euclidean plane [8].

3.3 THE RELATION BETWEEN THE DENSITIES OF THE SETS OF NON-COLLINEAR POINTS

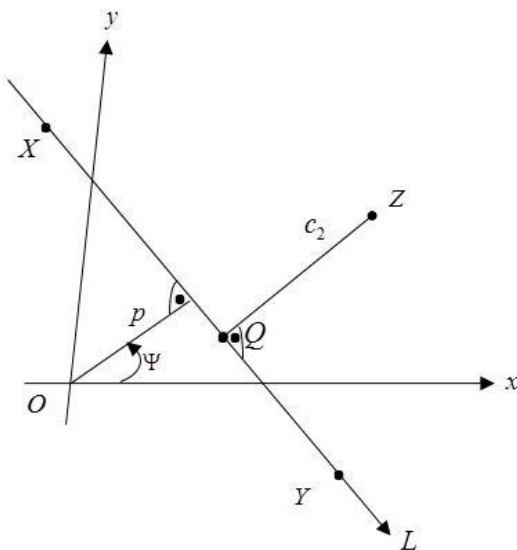


Figure 2. Non-collinear points in the Lorentzian plane \mathbb{L}

Theorem 8: Let $X(x_1, y_1), Y(x_2, y_2)$ and $Q(x_3, y_3)$ be sets of collinear points and $X(x_1, y_1), Y(x_2, y_2)$ and $Z(z_1, z_2)$ be sets of non-collinear points, c_2 be a constant distance from Q to Z and $L(p, \Psi)$ be a support function of the points X and Y in the Lorentzian plane \mathbb{L} . Then, we have

$$dZ = \lambda dX + (1 - \lambda) dY - \lambda(1 - \lambda) dXY + c_2 dL, \tag{13}$$

for the density of a set of points Z , where $(\lambda, 1 - \lambda)$ is the barycentric coordinate of the point Q with respect to the system $\{O, X, Y\}$ and dXY is a mixed density of points X and Y .

Proof: The aim of this theorem is to find the density of the set of points Z by using the densities of sets of points X and Y , in Fig.2. The proof will be divided in two ways depending on whether the line is timelike or spacelike, respectively.

i. Let $L_t(p, \Psi)$ be a set of timelike lines that passes through the points X, Y . Since using the barycentric coordinate system, the vector equation

$$\begin{aligned} \mathbf{Z} &= \lambda \mathbf{X} + (1 - \lambda) \mathbf{Y} + c_2 \mathbf{QZ} \\ &= (\lambda x_1 + (1 - \lambda) x_2 + c_2 \cosh \Psi, \lambda y_1 + (1 - \lambda) y_2 + c_2 \sinh \Psi) \end{aligned}$$

can be written, we can write

$$dZ = \lambda dX + (1 - \lambda) dY - \lambda(1 - \lambda) dXY + c_2 dL_t,$$

ii. Let $L_s(p, \Psi)$ be a set of spacelike lines that passes through the points X, Y . Similarly, we obtain

$$dZ = \lambda dX + (1 - \lambda) dY - \lambda(1 - \lambda) dXY + c_2 dL_s,$$

for the density of the point Z .

4. ON THE RELATION BETWEEN THE DENSITIES OF THE SETS OF INTERSECTING NON-NULL LINES IN LORENTZIAN PLANE

Let $L_i(p_i, \Psi_i), i = 1, 2$ be two sets of different non-null lines that intersect at the points $X(x, y)$ and make angles Φ_1, Φ_2 with the x axis, respectively, in the Lorentzian plane \mathbb{L} . The main idea of this section is to give the density of the set of lines L_1 with the density of the set of lines L_2 . This section will be divided in three cases depending on whether a pair of timelike lines, a pair of spacelike lines and a pair of timelike line and spacelike line.

i. A pair of timelike lines

Let $L_i(p_i, \Psi_i), i = 1, 2$, be two sets of timelike lines that intersect at the points $X(x, y)$. The density of the set of lines L_i is

$$dL_{t_1} = (dx \cosh \Psi_1 - dy \sinh \Psi_1) \wedge d\Psi_1,$$

or

$$\begin{aligned} dL_{t_1} &= (dx \cosh(\Psi_2 + \omega) - dy \sinh(\Psi_2 + \omega)) \wedge d(\Psi_2 + \omega) \\ &= \cosh \omega [(dx \cosh \Psi_2 - dy \sinh \Psi_2) \wedge d(\Psi_2 + \omega)] \\ &\quad + \sinh \omega [(dx \sinh \Psi_2 - dy \cosh \Psi_2) \wedge d(\Psi_2 + \omega)], \end{aligned} \tag{14}$$

where $\Phi_1 - \Phi_2 = \Psi_1 - \Psi_2 = \omega$. Let $L_{t_2}^\perp$ be set of spacelike lines which is orthogonal to the set of lines L_{t_2} and $X(x, y)$ be a intersection points of them. It can be given by the Hesse form $x \sinh \Phi_2 - y \cosh \Phi_2 = p_3$. Here is the another form of the equation (14):

$$dL_{t_1} = \cosh \omega [\langle \mathbf{n}_2, \mathbf{dX} \rangle_L \wedge d\omega + dL_{t_2}] + \sinh \omega [\langle \mathbf{u}_2, \mathbf{dX} \rangle_L \wedge d\omega + dL_{t_2}^\perp],$$

where $\mathbf{dX} = (dx, dy)$ is the relative velocity vector of the points X , $\mathbf{u}_2 = (\sinh \Psi_2, \cosh \Psi_2)$ is the directrix vector and $\mathbf{n}_2 = (\cosh \Psi_2, \sinh \Psi_2)$ is the normal vector of the lines L_{t_2} .

ii. A pair of spacelike lines

Let $L_s(p_i, \Psi_i), i = 1, 2$, be two sets of spacelike lines that intersect at the points $X(x, y)$ and $L_{s_2}^\perp$ be a set of timelike lines which is orthogonal to the set of spacelike lines to L_{s_2} that intersects at the same points. Then, we have

$$dL_{s_1} = \cosh \omega [\langle \mathbf{n}_2, \mathbf{dX} \rangle_L \wedge d\omega + dL_{s_2}] + \sinh \omega [\langle \mathbf{u}_2, \mathbf{dX} \rangle_L \wedge d\omega + dL_{s_2}^\perp]$$

for density of the lines dL_{s_1} , where $\mathbf{dX} = (dx, dy)$ is the relative velocity vector of the points X , $\mathbf{n}_2 = (\sinh \Psi_2, \cosh \Psi_2)$ is the normal vector and $\mathbf{u}_2 = (\cosh \Psi_2, \sinh \Psi_2)$ is the directrix vector of the lines L_{s_2} .

iii. A pair of timelike and spacelike lines

Let $L_t(p_1, \Psi_1)$ be set of timelike lines and $L_s(p_2, \Psi_2)$ be set of spacelike lines that intersect at the points $X(x, y)$ and L_s^\perp be orthogonal to set of spacelike lines L_s that intersects at the same points X . Then, we have the following equation for density of the lines L_t

$$dL_t = \cosh \omega [\langle \mathbf{u}_2, \mathbf{dX} \rangle_L \wedge d\omega + dL_s^\perp] + \sinh \omega [\langle \mathbf{n}_2, \mathbf{dX} \rangle_L \wedge d\omega + dL_s],$$

where $\mathbf{dX} = (dx, dy)$ is the relative velocity vector of the points X , \mathbf{n}_2 is the normal and \mathbf{u}_2 is the directrix vectors of the lines L_s .

5. ON THE DENSITIES OF THE SETS OF POINTS UNDER TWO-PARAMETER PLANAR LORENTZIAN MOTION

In this original section, as a beginning we give the density for a set of points $X'(x', y')$ in the fixed Lorentzian plane \mathbb{L}' . Following, Section 3 has been examined under the motion H_{II} . Namely, we investigate the relationships between densities of the sets of collinear and non-collinear points under the motion $H_{II} = \mathbb{L} / \mathbb{L}'$.

5.1 THE DENSITY FOR A SET OF POINTS

The purpose of this section is to find the density of a set of points X' with the density of the points X and the components of the motion H_{II} .

Theorem 9: Let $X(x, y) \in \mathbb{L}$ be set of points and $X'(x', y') \in \mathbb{L}'$ be an image of the set of points X under the motion $H_{II} = \mathbb{L} / \mathbb{L}'$. In this case, the density of a set of points X' in fixed plane \mathbb{L}' is

$$dX' = dX + dO' + [\langle X', \mathbf{dX} \rangle_L - \langle X, \mathbf{dO}' \rangle_L] \wedge d\Theta - \det(\mathbf{dX}, \mathbf{dO}'), \quad (15)$$

where $\mathbf{dX} = (dx, dy)$ is the relative velocity vector of the points X , $\mathbf{dO}' = (du_1, du_2)$ is the relative velocity vector of the points O' and Θ is rotation angle of the motion.

Proof: The density of a set of points $X'(x', y')$ in plane L' is calculated by

$$dX' = dx' \wedge dy'. \tag{16}$$

By taking differential of the equation (6) and substituting this into the equation (16), we get

$$dX' = dX + (x - u_1)[dx \wedge d\Theta + d\Theta \wedge du_1] - (y - u_2)[dy \wedge d\Theta + d\Theta \wedge du_2] + (du_1 \wedge du_2) - (dx \wedge du_2 + du_1 \wedge dy). \tag{17}$$

If we calculate the density of points O' , we can see that

$$dO' = (du_1 + u_2 d\Theta) \wedge (du_2 + u_1 d\Theta) = du_1 \wedge du_2 + u_1 du_1 \wedge d\Theta - u_2 du_2 \wedge d\Theta. \tag{18}$$

Using the equation (18), we get the following equation

$$dX' = dX + dO' + [\langle X', \mathbf{dX} \rangle_L - \langle X, \mathbf{dO}' \rangle_L] \wedge d\Theta - \det(\mathbf{dX}, \mathbf{dO}').$$

for the density of the points X' , where $\det(\mathbf{dX}, \mathbf{dO}') = dx \wedge du_2 - dy \wedge du_1$. This finishes the proof.

Corollary 10: The density of a set of pole points P in Lorentzian plane \mathbb{L} is equal to the density of a set of pole points P' in Lorentzian plane \mathbb{L}' . P' represent the point $P \in \mathbb{L}$ with respect to the fixed coordinate system under the motion H_{II} . Moreover, this density is zero,

$$dP = dP' = 0.$$

5.2 THE RELATION BETWEEN THE DENSITIES OF THE SETS OF POINTS

We can now give the analogue of the equation (8) in the Lorentzian plane \mathbb{L}' .

Theorem 11: Let $X(x_1, y_1)$ and $Y(x_2, y_2)$ be sets of different points, $L(p, \Psi)$ be a non-null support function of the points X, Y . Let take λ a distance function from X to Y in the moving plane \mathbb{L} . Under the motion H_{II} , let $L'(p', \Psi')$ be an image of the lines L and $X'(x_1', y_1')$, $Y'(x_2', y_2')$ be images of the points X, Y . In this case, the density of a set of points Y' is

$$dY' = dX' + \lambda d\lambda \wedge d\Psi' + \langle \mathbf{n}', \mathbf{dX}' \rangle_L \wedge d\lambda + \lambda \langle \mathbf{u}', \mathbf{dX}' \rangle_L \wedge d\Psi',$$

where $\mathbf{dX}' = (dx_1', dy_1')$ is the absolute velocity vector of the points X and \mathbf{n}' is the normal and \mathbf{u}' is the directrix vectors of the lines L .

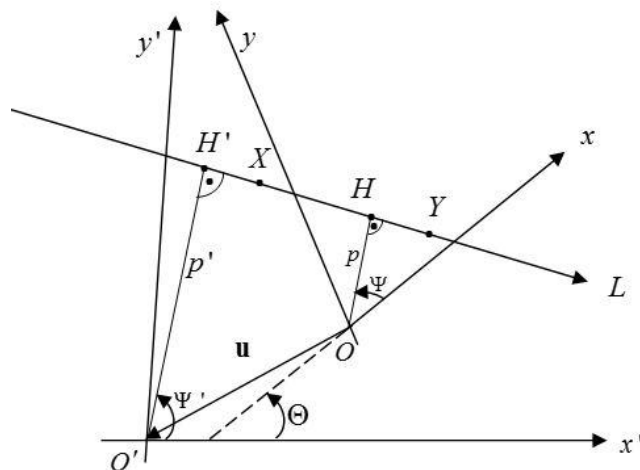


Figure 3. Under the motion H_{II} , X and Y points and non-null line L .

Proof: The main idea of this theorem is to give the density of a set of points Y' by using the density of a set of points in the Lorentzian plane \mathbb{L}' . We first say that the distance from X to Y is equal to the distance from X' to Y' , in Fig.3. The proof will be given in two ways depending on whether the line is timelike or spacelike respectively.

i. Let $L_t(p, \Psi)$ be a set of timelike lines that passes through the points X, Y and $L'_t(p', \Psi')$ be an image of timelike line L_t under the motion H_{II} . Similarly as in the equation (9), it is easy to check that

$$dY' = dX' + \lambda d\lambda \wedge d\Psi' + \langle \mathbf{n}', d\mathbf{X}' \rangle_L \wedge d\lambda + \lambda \langle \mathbf{u}', d\mathbf{X}' \rangle_L \wedge d\Psi'$$

is obtained for the density of the points Y' in the Lorentzian plane \mathbb{L}' .

ii. This case follows by the same method as before.

Corollary 12: Let $X(x_1, y_1)$ and $Y(x_2, y_2)$ be sets of different points, $L(p, \Psi)$ be a set of non-null lines that passes through the points X, Y . Let take λ as a constant distance function from X to Y in the moving plane \mathbb{L} . Under the motion H_{II} , let $L'(p', \Psi')$ be an image of the line L and $X'(x'_1, y'_1), Y'(x'_2, y'_2)$ be images of the points X, Y . The density of a set of points Y'

$$dY' = dX' + \lambda \langle \mathbf{u}', d\mathbf{X}' \rangle_L \wedge d\Psi',$$

where $d\mathbf{X}' = (dx'_1, dy'_1)$ is the absolute velocity vector of the points X and \mathbf{u}' is the directrix vector of the lines L .

5.3 THE RELATION BETWEEN THE DENSITIES OF THE SETS OF COLLINEAR POINTS

We can now state the analogue of the theorem 7.

Theorem 13: Let $X(x_1, y_1), Y(x_2, y_2)$ and $Q(x_3, y_3)$ be sets of collinear points in the Lorentzian plane \mathbb{L} . Under the motion H_{II} , let $X'(x_1', y_1'), Y'(x_2', y_2'), Q'(x_3', y_3')$ be images of the points X, Y and Q . Then, the equivalent formulations of the density of a set of points Q are

- i. $dQ' = \lambda(2\lambda - 1)dX' + (\lambda - 1)(2\lambda - 1)dY' + 4\lambda(1 - \lambda)dM'$,
- ii. $dQ' = \lambda dX' + (1 - \lambda)dY' - \lambda(1 - \lambda)dX'Y'$,

where $(\lambda, 1 - \lambda)$ is the barycentric coordinate of the points Q' with respect to the system $\{\mathbf{O}', \mathbf{X}', \mathbf{Y}'\}$, M' is a set of middle points of $\overline{X'Y'}$ and $dX'Y'$ a mixed point density of the points X' and Y' .

Proof: The proof runs as in Section 3.2.

Note that, the exterior product of the densities dX', dY' and dQ' .

5.4 THE RELATION BETWEEN THE DENSITIES OF THE SETS OF NON-COLLINEAR POINTS

We can now give the analogue of the equation (13) in the Lorentzian plane \mathbb{L}' .

Theorem 14: Let $X(x_1, y_1), Y(x_2, y_2)$ and $Q(x_3, y_3)$ be sets of collinear points, $X(x_1, y_1), Y(x_2, y_2)$ and $Z(z_1, z_2)$ be sets of non-collinear points and $L(p, \Psi)$ be a non-null support function of the points X, Y in the moving Lorentzian plane \mathbb{L} . Under the motion H_{II} , $X'(x_1', y_1'), Y'(x_2', y_2'), Q'(x_3', y_3')$ and $Z'(z_1', z_2')$ are image of the points X, Y, Q, Z and $L'(p', \Psi')$ is an image of the line L . Then, the density of a set of points Z' is

$$dZ' = \lambda dX' + (1 - \lambda)dY' - \lambda(1 - \lambda)dX'Y' + c_2 dL',$$

where $(\lambda, 1 - \lambda)$ is the barycentric coordinate of the points Q' with respect to the system $\{\mathbf{O}', \mathbf{X}', \mathbf{Y}'\}$, c_2 is a constant distance from Q to Z and Q' to Z' .

Proof: We want to find the density of a set of points Z' by using the densities of sets of points X' and Y' . The theorem can be proved in two ways depending on whether the line is timelike or spacelike, respectively.

i. Let $L_t(p, \Psi)$ be a set of timelike lines that passes through the points X, Y in the Lorentzian plane \mathbb{L} . Then, $L_t'(p', \Psi')$ is an image of the line L_t under the motion H_{II} . As in the Section 3.3, it is obvious that

$$(z_1, z_2) = (\lambda x_1' + (1 - \lambda)x_2' + c_2 \cosh \Psi', \lambda y_1' + (1 - \lambda)y_2' + c_2 \sinh \Psi').$$

We thus get

$$dZ' = \lambda dX' + (1 - \lambda) dY' - \lambda(1 - \lambda) dX'Y' + c_2 dL_t'$$

for the density of the points Z' .

- ii. Let $L_s(p, \Psi)$ be a set of spacelike lines that passes through the points X, Y . In this case, $L_s'(p', \Psi')$ is an image of the line L_s under the motion H_{II} , so we have

$$dZ' = \lambda dX' + (1 - \lambda) dY' - \lambda(1 - \lambda) dX'Y' + c_2 dL_s'$$

for the density of the point Z' .

6. ON THE DENSITIES FOR THE SETS OF NON-NULL LINES UNDER THE TWO-PARAMETER PLANAR LORENTZIAN MOTIONS

In this section, as a beginning we calculate the density for sets of non-null lines under the motion $H_{II} = \mathbb{L} / \mathbb{L}'$. Following, we investigate the relation between densities of the sets of non-null intersecting lines under the motion H_{II} . Namely, Section 4 has been examined under the motion H_{II} . Finally, we give the relations between the wedge product of the densities of sets of non-null lines and points under the motion H_{II} .

6.1 THE DENSITY FOR A SET OF NON-NULL LINES

The aim of this section is to calculate the density of set of lines $L'(p', \Psi') \in \mathbb{L}'$, which is an image of set of lines $L(p, \Psi) \in \mathbb{L}$, by using the density of lines $L(p, \Psi)$.

Theorem 15: Let $L(p, \Psi) \in \mathbb{L}$ be set of non-null lines and $L'(p', \Psi') \in \mathbb{L}'$ be an image of the lines $L(p, \Psi)$, under the motion H_{II} . Then, we have

$$dL' = dL + dp \wedge d\Theta - \langle \mathbf{n}, \mathbf{d}'O' \rangle_L \wedge d\Psi - \langle \mathbf{n}, \mathbf{d}O' \rangle_L \wedge d\Theta,$$

for the density of set of the lines L' , where $\mathbf{d}O' = (du_1, du_2)$ is the relative velocity vector of the points O' , $\mathbf{d}'O' = (du_1 + u_2 d\Theta, du_2 + u_1 d\Theta)$ is the absolute velocity the vector of points O' , Θ is the rotation angle of the motion and \mathbf{n} is the normal vector of the lines $L(p, \Psi)$.

Proof: The density of set of non-null lines $L'(p', \Psi') \in \mathbb{L}'$ is calculated by

$$dL' = dp' \wedge d\Psi'.$$

The proof is straightforward. It will be divided into two cases depending on whether lines timelike or spacelike.

6.2 THE RELATION BETWEEN THE DENSITIES OF THE SETS OF INTERSECTING NON-NULL LINES

Let $L_i(p_i, \Psi_i), i=1,2$, be sets of different non-null lines that intersect at the points $X(x, y)$ in the moving plane \mathbb{L} . Let $L_i'(p_i', \Psi_i')$ be an image of the lines L_i that intersect at the points $X'(x', y')$ in the fixed plane \mathbb{L}' under the motion H_{II} , respectively, in Fig. 4.

Our aim is to give the density of a set of non-null lines L_1' with the density of a set of non-null lines L_2' . This section falls naturally into three parts depending on whether pair of timelike lines, pair of spacelike lines and pair of timelike and spacelike lines, respectively.

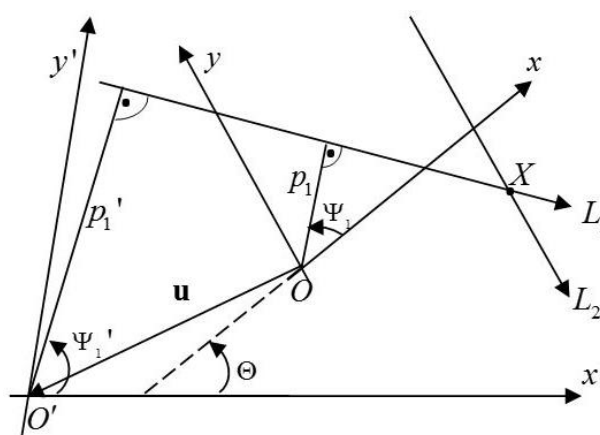


Figure 4. Under the motion H_{II} , intersecting non-null lines L_1, L_2 .

i. A pair of timelike lines

Let $L_i(p_i, \Psi_i), i=1,2$, be sets of timelike lines that intersect at the points $X(x, y)$. We thus get

$$dL_i' = (dx' \cosh \Psi_1' - dy' \sinh \Psi_1') \wedge d\Psi_1'$$

for the density of the lines L_i' . We have $\Psi_1' = \Psi_2' + \omega$, because

$$\Phi_1' - \Phi_2' = \Psi_1' - \Psi_2' = \Phi_1 - \Phi_2 = \Psi_1 - \Psi_2 = \omega.$$

Let us take $(L_{t_2}')^\perp$ that an orthogonal set of lines to the lines L_{t_2}' that intersects at the points $X(x, y)$. Using these, we can obtain the density of the lines L_{t_1}' as

$$dL_{t_1}' = \cosh \omega [\langle \mathbf{n}_2', d\mathbf{X}' \rangle_L \wedge d\omega + dL_{t_2}'] + \sinh \omega [\langle \mathbf{u}_2', d\mathbf{X}' \rangle_L \wedge d\omega + d(L_{t_2}')^\perp]$$

where $d\mathbf{X}' = (dx', dy')$ is the absolute velocity vector of the points X , \mathbf{u}_2' is the directrix vector and \mathbf{n}_2' is the normal vector of the lines $L_{t_2}'(p_2', \Psi_2')$.

ii. A pair of spacelike lines

As in the pair of timelike lines, we have

$$dL_{s_1}' = \cosh \omega \left[\langle \mathbf{n}_2', d\mathbf{X}' \rangle_L \wedge d\omega + dL_{s_2}' \right] + \sinh \omega \left[\langle \mathbf{u}_2', d\mathbf{X}' \rangle_L \wedge d\omega + d(L_{s_2}')^\perp \right]$$

where $L_{s_i}(p_i, \Psi_i), i=1,2$ are two sets of different spacelike lines that intersect at the points $X(x, y)$, $(L_{s_2}')^\perp$ is an orthogonal set of lines to the lines L_{s_2}' that intersects at the points $X(x, y)$, $d\mathbf{X}' = (dx', dy')$ is the absolute velocity vector of the points X , \mathbf{u}_2' is the directrix and \mathbf{n}_2' is the normal vectors of the lines $L_{s_2}'(p_2', \Psi_2')$.

iii. A pairs of timelike and spacelike lines

Let $L_t(p_1, \Psi_1)$ and $L_s(p_2, \Psi_2)$ be sets of timelike and spacelines, respectively, that intersect at the points $X(x, y)$. We thus write the density of the lines (L_t') as

$$dL_t' = \cosh \omega \left[\langle \mathbf{u}_2', d\mathbf{X}' \rangle_L \wedge d\omega + d(L_s')^\perp \right] + \sinh \omega \left[\langle \mathbf{n}_2', d\mathbf{X}' \rangle_L \wedge d\omega + dL_s' \right]$$

where $d\mathbf{X}' = (dx', dy')$ is the absolute velocity vector of the points X , $(L_s')^\perp$ is an orthogonal set of lines to the lines L_s' that intersects at the points $X(x, y)$, \mathbf{u}_2' is the directrix and \mathbf{n}_2' is the normal vectors of the lines $L_s'(p_2', \Psi_2')$.

6.3 THE DENSITY FOR PAIRS OF NON-NULL LINES

Let $L_i(p_i, \Psi_i), i=1,2$, be sets of different non-null lines that intersect at the points $X(x, y)$ and make angles Φ_1, Φ_2 with the x axis in the moving plane \mathbb{L} . Let $L_i'(p_i', \Psi_i')$ be an image of the lines that intersect at the points $X'(x', y')$ in the fixed plane \mathbb{L}' under the motion $H_{II} = \mathbb{L} / \mathbb{L}'$, respectively.

The main idea of this subsection is to give the relation between the wedge products $dL_1' \wedge dL_2'$ to $dL_1 \wedge dL_2$. We have divided this section into three parts.

i. A pair of timelike lines

Let $L_i(p_i, \Psi_i), i=1,2$, be sets of timelike lines that intersect at the points $X(x, y)$. Similarly as in Section 2.1, it is easily seen that

$$dL_{t_1}' \wedge dL_{t_2}' = |\sinh(\Phi_1' - \Phi_2')| dX' \wedge d\Phi_1' \wedge d\Phi_2'.$$

Using the equality $\Phi_1' - \Phi_2' = \Psi_1' - \Psi_2' = \omega$ and the equation (17), we can write

$$dL_{t_1}' \wedge dL_{t_2}' = dL_{t_1} \wedge dL_{t_2} + |\sinh \omega| (dX + dO' - \det(\mathbf{dX}, \mathbf{dO}')) d\omega \wedge d\Theta$$

$$+ |\sinh \omega| (dO' + [\langle X', \mathbf{dX} \rangle_L - \langle X, \mathbf{dO}' \rangle_L]) \wedge d\Theta - \det(\mathbf{dX}, \mathbf{dO}') d\Phi_1 \wedge d\Phi_2,$$

ii. A pair of spacelike lines

Let $L_{s_i}(p_i, \Psi_i), i = 1, 2$, be two sets of spacelike lines that intersect at the points $X(x, y)$. We thus write

$$dL_{t_1}' \wedge dL_{t_2}' = dL_{t_1} \wedge dL_{t_2} + |\sinh \omega| (dX + dO' - \det(\mathbf{dX}, \mathbf{dO}')) d\omega \wedge d\Theta$$

$$+ |\sinh \omega| (dO' + [\langle X', \mathbf{dX} \rangle_L - \langle X, \mathbf{dO}' \rangle_L]) \wedge d\Theta - \det(\mathbf{dX}, \mathbf{dO}') d\Phi_1 \wedge d\Phi_2.$$

iii. A pair of timelike and spacelike lines

Let $L_t(p_1, \Psi_1)$ be set of timelike and $L_s(p_2, \Psi_2)$ be set of spacelike lines that intersect at the points $X(x, y)$. Then, we can write

$$dL_t' \wedge dL_s' = dL_t \wedge dL_s + \cosh \omega (dX + dO' - \det(\mathbf{dX}, \mathbf{dO}')) d\omega \wedge d\Theta$$

$$+ \cosh \omega (dO' + [\langle X', \mathbf{dX} \rangle_L - \langle X, \mathbf{dO}' \rangle_L]) \wedge d\Theta - \det(\mathbf{dX}, \mathbf{dO}') d\Phi_1 \wedge d\Phi_2,$$

where $\Psi_1 = \Psi + \omega$, $\mathbf{dX} = (dx, dy)$ is the relative velocity vector of the points X , Θ is rotation angle of the motion H_{II} and $\mathbf{dO}' = (du_1, du_2)$ is the relative velocity vector of the points O' .

Lemma 16: Let $X(x_1, y_1), Y(x_2, y_2)$ be sets of different points and $L(p, \Psi)$ be a set of non-null lines that passes through the points X and Y in the moving plane \mathbb{L} . $X'(x_1', y_1')$, $Y'(x_2', y_2')$ and $L'(p', \Phi')$ represent the points X, Y and the non-null line $L(p, \Psi)$ with respect to the fixed coordinate system under the motion $H_{II} = \mathbb{L} / \mathbb{L}'$, respectively. Then, the relation between the wedge products $dX' \wedge dY'$ to $dX \wedge dY$ is

$$dX' \wedge dY' = dX \wedge dY - |\lambda| dL' \wedge d\lambda \wedge dq +$$

$$+ |\lambda| [dp \wedge d\Theta - \langle \mathbf{n}, \mathbf{dO}' \rangle_L \wedge d\Psi - \langle \mathbf{n}, \mathbf{dO}' \rangle_L \wedge d\Theta] \wedge d\lambda_1 \wedge d\lambda_2,$$

where λ_1, λ_2 are distance functions from the points X, Y to the foot H of the perpendicular from the origin O to L , $\lambda = \lambda_2 - \lambda_1$, $\mathbf{dO}' = (du_1, du_2)$ is the relative velocity vector of the points O' , $\mathbf{d}'O' = (du_1 + u_2 d\Theta, du_2 + u_1 \Theta)$ is the absolute velocity vector of the points O' , Θ is rotation angle of the motion and \mathbf{n} is the normal vector of the lines L .

7. CONCLUSION

We developed the theories of the density of points and the density of non-null lines in the Lorentzian plane. It has been shown that under two-parameter planar Lorentzian motion, the density of points, except for the pole points, and the density of non-null lines, except for the polar axes, are not invariant.

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