ORIGINAL PAPER

A NEW APPROXIMATION BASED ON RESIDUAL ERROR ESTIMATION FOR THE SOLUTION OF A CLASS OF UNSTEADY CONVECTION-DIFFUSION PROBLEM

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Abstract. *In this study, the unsteady convection-diffusion equation in one-dimension has been solved by using a hybrid matrix-collocation method which is based on Lerch and Taylor polynomials along with collocation points. The method reduces the solution of the given convection-diffusion equation with the initial and boundary conditions to the solution of a matrix equation corresponding to linear algebraic equations system with unknown Lerch coefficients. Also, the error estimation on technique related with residual functions is developed and some illustrative examples to show the effectiveness and convenience of the method are fulfilled. Moreover, the proposed algorithm can be used to solve other linear or nonlinear physical problems.*

Keywords: *Lech and Taylor series, convection-diffusion problems, matrix-collocation method, residual error analysis.*

1. INTRODUCTION

Convection-diffusion equation, which has been widely used to describe heat transfer, heat flow, astrophysics, hydraulics, air and river pollution, meteorology and oceanography, plays an important role in the modeling of several physical phenomena. Many physical problems have been simplified with convection-diffusion equation as one-dimensional issues. For instance, this equation is regarded as linearized version of one-dimensional Navier–Stokes equation [1-9].

The governing parabolic partial differential equation for unsteady one-dimensional time dependent convection-diffusion equation with constant coefficients can be denoted as

$$\kappa \frac{\partial^2 u(x,t)}{\partial x^2} - \nu \frac{\partial u(x,t)}{\partial x} = \frac{\partial u(x,t)}{\partial t} \tag{1}$$

where ν is the convective velocity and the constant κ is the diffusion coefficient. Depending upon a certain actual problem, $u(x,t)$ can be represented a concentration or temperature for mass or heat transfer [1-5]. The initial condition of the convection-diffusion problem is

$$u(x,0) = \varphi_1(x), \qquad t \in [0,T], \tag{2}$$

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with the following boundary conditions

$$u(a,t)=\varphi_2(t), \qquad u(b,t)=\varphi_1(t), \qquad x\in[a,b]. \tag{3}$$

Several numerical methods have been developed to solve convection-diffusion equation, such as Legendre wavelet method [8], nodal integral method [9], finite element method [10-12], finite difference method [13-15], finite volume method [16-18], restrictive Taylor's series method [19-20], Haar wavelet method [20-21], local series expansion method [22], boundary element method [23-25], Gaussian radial basis function method [26], half boundary method [27], cubic B-splines collocation method [28], Bessel collocation method [29], Laguerre collocation method [30-31], etc.

In this study, Lerch matrix collocation method is developed to solve one-dimensional time dependent convection-diffusion equation with constant coefficients. For this aim, the approximate solution of the problem (1)-(3) is assumed to be in the truncated Lerch series form as

$$u(x,t)\cong u_N(x,t,\lambda)=\sum_{m=0}^{N}\sum_{n=0}^{N}a_{m,n}L_{m,n}(x,t,\lambda), \qquad (N\geq 2) \tag{4}$$

where

$$L_{m,n}(x,t,\lambda)=L_m(x,\lambda)L_n(t,\lambda).$$

Here $u_N(x,t,\lambda)$ is the approximate solution of Eq. (1), $a_{m,n}$ are the unknown Lerch coefficients for $m,n=0,1,2,\ldots,N$, $L_m(x,\lambda)$ and $L_n(t,\lambda)$ are Lerch polynomials.

2. LERCH POLYNOMIALS

The Lerch polynomials are defined by the generating functions as follows [32-33]:

$$\left(1-x\log(1+t)\right)^{-\lambda}=\sum_{n\geq 0}L_n(x,\lambda)t^n.$$

By using standard methods, the explicit representations of the Lerch polynomials are obtained as

$$L_n(x,\lambda)=\sum_{k=1}^{n}\frac{k!}{n!}s(n,k)\binom{k+\lambda-1}{k}x^k. \tag{5}$$

Here $s(n,k)$ is Stirling numbers of the first kind. Some properties about the Stirling numbers of the first kind can be written as [34-35]

$$s(n+1,0)=0, \quad s(n,n)=1, \quad s(n,1)=(-1)^{n-1}(n-1)!, \quad s(n,n-1)=-\binom{n}{2}, \quad (n\geq 0).$$

Lerch polynomials for $\lambda = 1$, $\lambda = 10$ and $\lambda = 100$ are illustrated in Figs. 1-3.

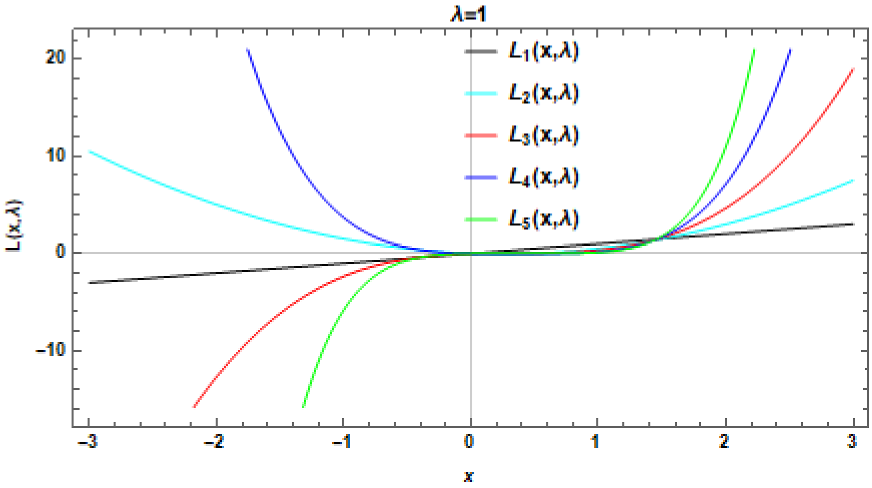

Figure 1. Lerch polynomials for $\lambda = 1$.

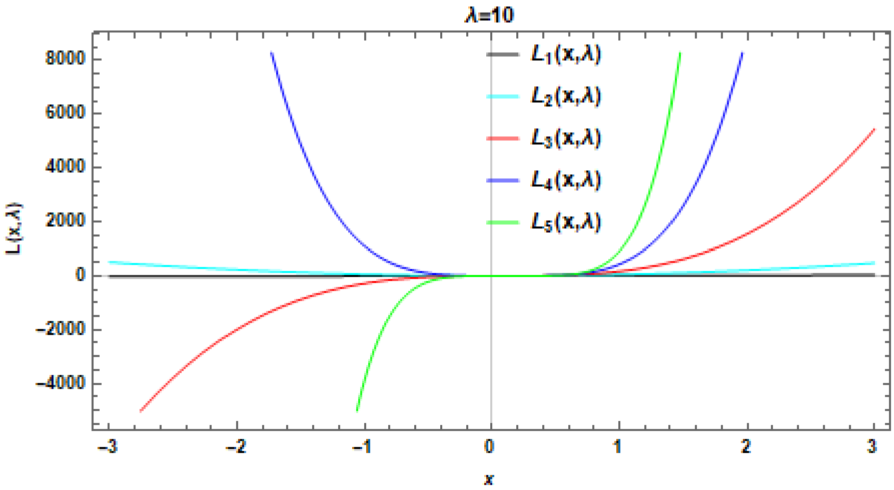

Figure 2. Lerch polynomials for $\lambda = 10$.

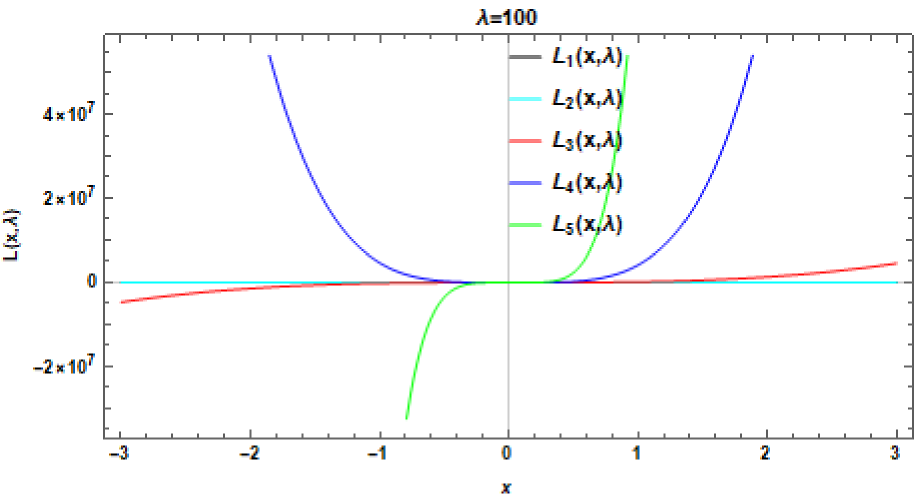

Figure 3. Lerch polynomials for $\lambda = 100$.

Also, from the relation (5), the matrix form of the Lerch polynomials can be written as following:

$$\underbrace{\begin{bmatrix} L_0(x,\lambda) \\ L_1(x,\lambda) \\ L_2(x,\lambda) \\ L_3(x,\lambda) \\ \vdots \\ L_N(x,\lambda) \end{bmatrix}}_{\mathbf{L}(x,\lambda)^T} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1!}{1!}s(1,1)\binom{\lambda}{1} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1!}{2!}s(2,1)\binom{\lambda}{1} & \frac{2!}{2!}s(2,2)\binom{\lambda+1}{2} & 0 & \cdots & 0 \\ 0 & \frac{1!}{3!}s(3,1)\binom{\lambda}{1} & \frac{2!}{3!}s(3,2)\binom{\lambda+1}{2} & \frac{3!}{3!}s(3,3)\binom{\lambda+2}{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{1!}{N!}s(N,1)\binom{\lambda}{1} & \frac{2!}{N!}s(N,2)\binom{\lambda+1}{2} & \frac{3!}{N!}s(N,3)\binom{\lambda+2}{3} & \cdots & \frac{N!}{N!}s(N,N)\binom{\lambda+N-1}{N} \end{bmatrix}}_{\mathbf{C}(\lambda)^T} \underbrace{\begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \\ \vdots \\ x^N \end{bmatrix}}_{\mathbf{X}(x)^T}. \quad (6)$$

Then, the expression (6) can be organized as

$$\mathbf{L}(x,\lambda)^T = \mathbf{C}(\lambda)^T \mathbf{X}(x)^T \Rightarrow \mathbf{L}(x,\lambda) = \mathbf{X}(x)\mathbf{C}(\lambda). \quad (7)$$

3. MATERIALS AND METHODS

3.1. FUNDAMENTAL MATRIX RELATIONS

By substituting the matrix relation (7) into the truncated Lerch series form (4), the matrix form of the approximate solution of the problem (1)-(3) is obtained as

$$u_N(x,t,\lambda) = \mathbf{L}(x,\lambda)\bar{\mathbf{L}}(t,\lambda)\mathbf{A} \Rightarrow u_N(x,t,\lambda) = \mathbf{X}(x)\mathbf{C}(\lambda)\bar{\mathbf{X}}(x)\bar{\mathbf{C}}(\lambda)\mathbf{A} \quad (8)$$

where

$$\mathbf{L}(x,\lambda) = \left[L_0(x,\lambda) \quad L_1(x,\lambda) \quad \cdots \quad L_N(x,\lambda)\right]^T,$$

$$\bar{\mathbf{L}}(t,\lambda) = diag\left[\mathbf{L}(t,\lambda), \mathbf{L}(t,\lambda), \cdots, \mathbf{L}(t,\lambda)\right],$$

$$\mathbf{A} = \left[a_{0,0} \quad a_{0,1} \quad \cdots \quad a_{0,N} \quad a_{1,0} \quad a_{1,1} \quad \cdots \quad a_{1,N} \quad \cdots \quad a_{N,0} \quad a_{N,1} \quad \cdots \quad a_{N,N}\right]^T,$$

$$\mathbf{A}_i = \begin{bmatrix} a_{i,0} & a_{i,1} & \cdots & a_{i,N} \end{bmatrix}^T \qquad (i=0,1,\ldots,N)\,,$$
$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_0 & \mathbf{A}_1 & \cdots & \mathbf{A}_N \end{bmatrix}^T,$$

$$\mathbf{X}(x) = \begin{bmatrix} 1 & x & \cdots & x^N \end{bmatrix}, \qquad \bar{\mathbf{X}}(t) = diag\left[\mathbf{X}(t),\mathbf{X}(t),\cdots,\mathbf{X}(t)\right],$$

$$\mathbf{C}(\lambda)^0 = \mathbf{I} = \mathbf{diag}[1,1,\cdots,1], \quad \bar{\mathbf{C}}(\lambda) = diag\left[\mathbf{C}(\lambda),\mathbf{C}(\lambda),\cdots,\mathbf{C}(\lambda)\right].$$

The relation between the matrices $\mathbf{X}(x)$ and its derivatives $\mathbf{X}'(x)$, $\mathbf{X}''(x)$,…, $\mathbf{X}^{(k)}(x)$ can be written as

$$\mathbf{X}'(x) = \mathbf{X}(x)\mathbf{B}\,,\ \mathbf{X}''(x) = \mathbf{X}(x)\mathbf{B}^2\,,\ldots,\ \mathbf{X}^{(k)}(x) = \mathbf{X}(x)\mathbf{B}^k\,. \tag{9}$$

Also, the relation between the matrices $\bar{\mathbf{X}}(t)$ and its derivatives $\overline{\mathbf{X}'}(t)$, $\overline{\mathbf{X}''}(t)$,…, $\overline{\mathbf{X}^{(k)}}(t)$ can be written as

$$\overline{\mathbf{X}'}(t) = \bar{\mathbf{X}}(t)\bar{\mathbf{B}}\,,\ \overline{\mathbf{X}''}(t) = \bar{\mathbf{X}}(t)\overline{\mathbf{B}^2}\,,\ldots,\ \overline{\mathbf{X}^{(k)}}(t) = \bar{\mathbf{X}}(t)\overline{\mathbf{B}^{(k)}}\,. \tag{10}$$

Here

$$\mathbf{B}^0 = \mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \bar{\mathbf{B}} = \begin{bmatrix} \mathbf{B} & 0 & \cdots & 0 \\ 0 & \mathbf{B} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{B} \end{bmatrix}.$$

By using the relations (8)-(10), the matrix relations of $\dfrac{\partial u(x,t)}{\partial x}$, $\dfrac{\partial u(x,t)}{\partial t}$ and $\dfrac{\partial^2 u(x,t)}{\partial x^2}$ are organized as follows:

$$\frac{\partial u(x,t)}{\partial x} = \mathbf{L}'(x,\lambda)\bar{\mathbf{L}}(t,\lambda)\mathbf{A} = \mathbf{X}'(x)\mathbf{C}(\lambda)\bar{\mathbf{X}}(t)\bar{\mathbf{C}}(\lambda)\mathbf{A} = \mathbf{X}(x)\mathbf{B}\mathbf{C}(\lambda)\bar{\mathbf{X}}(t)\bar{\mathbf{C}}(\lambda)\mathbf{A}$$

$$\frac{\partial u(x,t)}{\partial t} = \mathbf{L}(x,\lambda)\bar{\mathbf{L}}'(t,\lambda)\mathbf{A} = \mathbf{X}(x)\mathbf{C}(\lambda)\bar{\mathbf{X}}'(t)\bar{\mathbf{C}}(\lambda)\mathbf{A} = \mathbf{X}(x)\mathbf{C}(\lambda)\bar{\mathbf{X}}(t)\bar{\mathbf{B}}\bar{\mathbf{C}}(\lambda)\mathbf{A} \tag{11}$$

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \mathbf{L}''(x,\lambda)\bar{\mathbf{L}}(t,\lambda)\mathbf{A} = \mathbf{X}''(x)\mathbf{C}(\lambda)\bar{\mathbf{X}}(t)\bar{\mathbf{C}}(\lambda)\mathbf{A} = \mathbf{X}(x)\mathbf{B}^2\mathbf{C}(\lambda)\bar{\mathbf{X}}(t)\bar{\mathbf{C}}(\lambda)\mathbf{A}\,.$$

3.2. LERCH MATRIX COLLOCATION METHOD

By substituting the relations (11) into Eq.(1), we have the fundamental matrix form for Eq.(1):

$$\underbrace{\left\{\kappa\mathbf{X}(x)\mathbf{B}^2\mathbf{C}(\lambda)\overline{\mathbf{X}}(t)\overline{\mathbf{C}}(\lambda)-\nu\mathbf{X}(x)\mathbf{B}\mathbf{C}(\lambda)\overline{\mathbf{X}}(t)\overline{\mathbf{B}}\overline{\mathbf{C}}(\lambda)-\mathbf{X}(x)\mathbf{C}(\lambda)\overline{\mathbf{X}}(t)\overline{\mathbf{B}}\overline{\mathbf{C}}(\lambda)\right\}}_{\mathbf{W}(x,t)}\mathbf{A}=0;$$

$$\mathbf{W}(x,t)\mathbf{A}=0$$

and then, using the collocation points defined by

$$x_i=a+\frac{b-a}{N}i \quad , \quad t_j=c+\frac{d-c}{N}j \quad , \quad i,j=0,1,2,\ldots,N\,, \tag{12}$$

or shortly

$$\mathbf{W}\mathbf{A}=\mathbf{0} \text{ or } [\mathbf{W};\mathbf{0}]; \tag{13}$$

$$\mathbf{W}=\begin{bmatrix}\mathbf{W}_0\\ \mathbf{W}_1\\ \vdots\\ \mathbf{W}_N\end{bmatrix}; \quad \mathbf{W}_i=\begin{bmatrix}\mathbf{W}(x_i,t_0)\\ \mathbf{W}(x_i,t_1)\\ \vdots\\ \mathbf{W}(x_i,t_N)\end{bmatrix}, \quad \mathbf{0}=\begin{bmatrix}0\\0\\ \vdots\\0\end{bmatrix}, \qquad (i=0,1,2,\ldots,N).$$

Thus, the fundamental matrix equation of Eq.(1) is defined by (13). Similarly, by using the relations (8)-(10), the corresponding matrix forms for the initial condition (2) and the boundary conditions (3) can be obtained as follows:

$$\begin{aligned} u(x,0)&=\mathbf{L}(x,\lambda)\overline{\mathbf{L}}(0,\lambda)\mathbf{A}=\mathbf{X}(x)\mathbf{C}(\lambda)\overline{\mathbf{X}}(0)\overline{\mathbf{C}}(\lambda)\mathbf{A}\\ u(a,t)&=\mathbf{L}(a,\lambda)\overline{\mathbf{L}}(t,\lambda)\mathbf{A}=\mathbf{X}(a)\mathbf{C}(\lambda)\overline{\mathbf{X}}(t)\overline{\mathbf{C}}(\lambda)\mathbf{A}\\ u(b,t)&=\mathbf{L}(b,\lambda)\overline{\mathbf{L}}(t,\lambda)\mathbf{A}=\mathbf{X}(b)\mathbf{C}(\lambda)\overline{\mathbf{X}}(t)\overline{\mathbf{C}}(\lambda)\mathbf{A}\,. \end{aligned} \tag{14}$$

Eq. (14) can be organized as

$$\mathbf{U}_1(x,0)\mathbf{A}=\varphi_1(x), \quad \mathbf{U}_2(a,t)\mathbf{A}=\varphi_2(t) \quad \text{and} \quad \mathbf{U}_3(b,t)\mathbf{A}=\varphi_3(t)$$

where

$$\mathbf{U}_1(x,0)=\mathbf{X}(x)\mathbf{C}(\lambda)\overline{\mathbf{X}}(0)\overline{\mathbf{C}}(\lambda), \quad \mathbf{U}_2(a,t)=\mathbf{X}(a)\mathbf{C}(\lambda)\overline{\mathbf{X}}(t)\overline{\mathbf{C}}(\lambda),$$

$$\mathbf{U}_3(b,t)=\mathbf{X}(b)\mathbf{C}(\lambda)\bar{\mathbf{X}}(t)\bar{\mathbf{C}}(\lambda).$$

Then, by using the collocation points (12) for $i,j=1,2,\ldots,N-1$, the matrix forms of the conditions can be written

$$\mathbf{U}_1(x_i,0)\mathbf{A}=\varphi_1(x_i),\quad \mathbf{U}_2(a,t_j)\mathbf{A}=\varphi_2(t_j)\quad \text{and}\quad \mathbf{U}_3(b,t_j)\mathbf{A}=\varphi_3(t_j)$$

or shortly

$$\mathbf{U}_\mu\mathbf{A}=\Phi_\mu \quad \text{or} \quad \left[\mathbf{U}_\mu;\Phi_\mu\right];\quad (\mu=1,2,3) \tag{15}$$

$$\mathbf{U}_1=\begin{bmatrix}\mathbf{U}_1(x_1,0)\\ \mathbf{U}_1(x_2,0)\\ \vdots\\ \mathbf{U}_1(x_N,0)\end{bmatrix},\quad \mathbf{U}_2=\begin{bmatrix}\mathbf{U}_2(a,t_1)\\ \mathbf{U}_2(a,t_2)\\ \vdots\\ \mathbf{U}_2(a,t_N)\end{bmatrix},\quad \mathbf{U}_3=\begin{bmatrix}\mathbf{U}_3(b,t_1)\\ \mathbf{U}_3(b,t_2)\\ \vdots\\ \mathbf{U}_3(b,t_N)\end{bmatrix},$$

$$\Phi_1=\begin{bmatrix}\varphi_1(x_1)\\ \varphi_1(x_2)\\ \vdots\\ \varphi_1(x_N)\end{bmatrix},\quad \Phi_2=\begin{bmatrix}\varphi_2(t_1)\\ \varphi_2(t_2)\\ \vdots\\ \varphi_2(t_N)\end{bmatrix},\quad \Phi_3=\begin{bmatrix}\varphi_3(t_1)\\ \varphi_3(t_2)\\ \vdots\\ \varphi_3(t_N)\end{bmatrix}.$$

To obtain the solution of Eq.(1) under the initial condition (2) and the boundary conditions (3), the following augmented matrix is constructed by replacing the row matrices (13) by the $3(N-1)$ rows of the matrix (14); so we have the new augmented matrix

$$\tilde{\mathbf{W}}\mathbf{A}=\tilde{\mathbf{0}} \quad \text{or} \quad \left[\tilde{\mathbf{W}};\tilde{\mathbf{0}}\right]. \tag{16}$$

If $rank(\tilde{\mathbf{W}})=rank(\tilde{\mathbf{W}};\tilde{\mathbf{G}})=(N+1)^2$, then $\mathbf{A}=(\tilde{\mathbf{W}})^{-1}\tilde{\mathbf{G}}$ is solved and $\mathbf{A}$ is uniquely determined. In this way, the unknown Lerch coefficients are obtained. Thus, the approximate solution $u_N(x,t,\lambda)$ is found in the form (4).

4. ERROR ANALYSIS

Accuracy of the presented method can be checked. Since the truncated Lerch series solution (4) is the approximate solution of Eq.(1), when the approximate solution $u_N(x,t,\lambda)$ and its derivatives are substituted in Eq.(1), the resulting equation must be satisfied approximately; that is,for $x=x_i$, $a\le x_i\le b$ and $t=t_j$, $c\le t_j\le d$ [36-40]:

$$R_N\left(x_i,t_j,\lambda\right)=\left|\kappa\frac{\partial^2 u_N\left(x_i,t_j,\lambda\right)}{\partial x^2}-\nu\frac{\partial u_N\left(x_i,t_j,\lambda\right)}{\partial x}-\frac{\partial u_N\left(x_i,t_j,\lambda\right)}{\partial t}-g\left(x,t\right)\right|\cong 0$$

where $R_N\left(x_i,t_j,\lambda\right)\le 10^{-k_{ij}}=10^{-k}$ (k is positive integer) . If $\max 10^{-k_{ij}}=10^{-k}$ is prescribed, then the truncation limit N is increased until the difference $R_N\left(x_i,t_j,\lambda\right)$ at each of the points becomes smaller than the prescribed 10^{-k}.

4.1. CORRECTED ERROR

The error can be estimated by means of the residual error function $R_N\left(x,t,\lambda\right)$. By using the linear operator L, the residual function $R_N\left(x,t,\lambda\right)$ is obtained as [38-40]

$$R_N\left(x,t,\lambda\right)=L\left[u_N\left(x,t,\lambda\right)\right]-g\left(x,t\right)$$

where $L\left[u\left(x,t\right)\right]=g\left(x,t\right)$. Afterwards, the error function $E_N\left(x,t,\lambda\right)$ and the error partial differential equation can be written as follows:

$$E_N\left(x,t,\lambda\right)=u\left(x,t\right)-u_N\left(x,t,\lambda\right)$$

and

$$L\left[E_N\left(x,t,\lambda\right)\right]=L\left[u\left(x,t\right)\right]-L\left[u_N\left(x,t,\lambda\right)\right]=-R_N\left(x,t,\lambda\right). \qquad (17)$$

Similarly, the initial condition (2) and the boundary conditions (3) related to the error function $E_N\left(x,t,\lambda\right)$ can be organized as

$$E_N\left(x,0,\lambda\right)=u\left(x,0\right)-u_N\left(x,0,\lambda\right)=\varphi_1\left(x\right)-\varphi_1\left(x\right)=0,$$

$$E_N\left(a,t,\lambda\right)=u\left(a,t\right)-u_N\left(a,t,\lambda\right)=\varphi_2\left(t\right)-\varphi_2\left(t\right)=0,$$

$$E_N\left(b,t,\lambda\right)=u\left(b,t\right)-u_N\left(b,t,\lambda\right)=\varphi_3\left(t\right)-\varphi_3\left(t\right)=0.$$

The error partial differential equation (17) subject to the initial condition (2) and the boundary conditions (3) is become an error problem, which is solved by following same procedure described in Section 3. So, an estimated error function $E_{N,M}\left(x,t,\lambda\right)$ is obtained as

$$E_N\left(x,t,\lambda\right)=\sum_{m=0}^{M}\sum_{n=0}^{M}a_{m,n}^{*}L_{m,n}\left(x,t,\lambda\right),\quad M>N$$

where $a_{m,n}^{*}$ are the unknown Lerch coefficients for $m,n=0,1,2\ldots,M$. Thus, the approximate solution can be developed as the following:

$$u_{N,M}\left(x,t,\lambda\right)=u_N\left(x,t,\lambda\right)+E_{N,M}\left(x,t,\lambda\right)$$

where $u_{N,M}\left(x,t,\lambda\right)$ is a corrected approximate solution.

4.2. ESTIMATION THE UPPER BOUND OF THE ERROR BASED ON RESIDUAL FUNCTION

By means of residual function $R_N\left(x,t,\lambda\right)$ and the mean value of the absolute function $\left|R_N\left(x,t,\lambda\right)\right|$, the accuracy of the solution can be controlled. Thus, the upper bound of the mean error can be estimated, that is, $\overline{R_N}$ as follows:

$$\left|\iint_D R_N\left(x,t,\lambda\right)dA\right|\le\iint_D\left|R_N\left(x,t,\lambda\right)\right|dA.$$

The mean value theorem for double integrals says that if R_N is a continuous function on a plane region D, then there is exists a point $\left(x_0,t_0\right)\in D$ such that

$$\iint_D R_N\left(x,t,\lambda\right)dA=R_N\left(x_0,t_0,\lambda\right)A(D),$$

$$\left|\iint_D R_N\left(x,t,\lambda\right)dA\right|=\left|R_N\left(x_0,t_0,\lambda\right)\right|\left|A(D)\right|\ \Rightarrow\ \left|R_N\left(x_0,t_0,\lambda\right)\right|\left|A(D)\right|\le\iint_D\left|R_N\left(x,t,\lambda\right)\right|dA$$

$$\Rightarrow\left|R_N\left(x_0,t_0,\lambda\right)\right|\le\frac{\iint_D\left|R_N\left(x,t,\lambda\right)\right|dA}{A(D)}=\overline{R_N}\ .$$

where $A(D)$ denotes the area of D.

5. NUMERICAL EXAMPLES

In this section some examples are given to show the effectiveness and reliability of the Lerch matrix collocation method.

Example 1. [41] Consider the convection-diffusion equation

$$0.01\frac{\partial^2u\left(x,t\right)}{\partial x^2}-0.1\frac{\partial u\left(x,t\right)}{\partial x}=\frac{\partial u\left(x,t\right)}{\partial t}$$

under the initial condition

$$u\left(x,0\right)=e^{5x}\sin\left(\pi x\right)\ ,\qquad 0\le x\le1$$

and the boundary conditions

$$u(0,t)=0\ ,\quad u(1,t)=0\ ,\quad t\geq 0\ .$$

The exact solution of the problem is given as

$$u(x,t)=e^{5x-\left(0.25+0.01\pi^2\right)t}\sin(\pi x).$$

The approximate solution $u_N(x,t,\lambda)$ is investigated in the truncated Lerch series form (4). By applying the Lerch matrix collocation method described in the Section 3, the fundamental matrix representation of the problem and conditions are calculated. Afterward, the unknown Lerch coefficients are found. Numerical results can be seen in Tables 1-2 and Figs. 4-5.

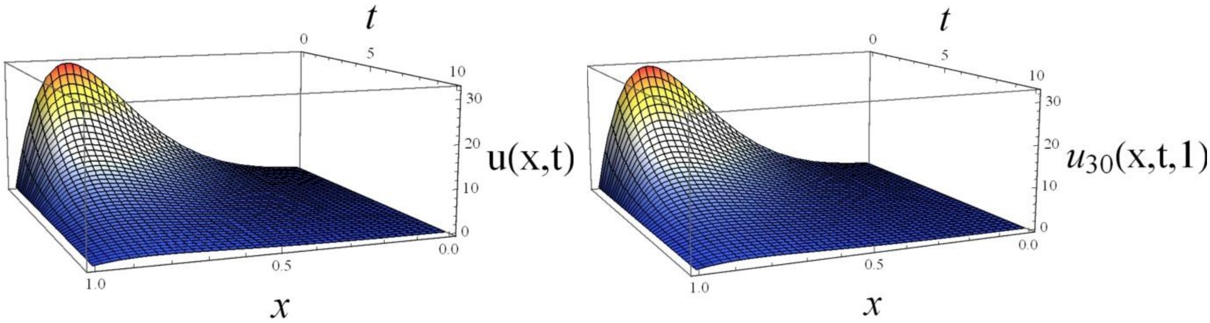

Figure 4. Comparison between exact and numerical solutions in Example 1.

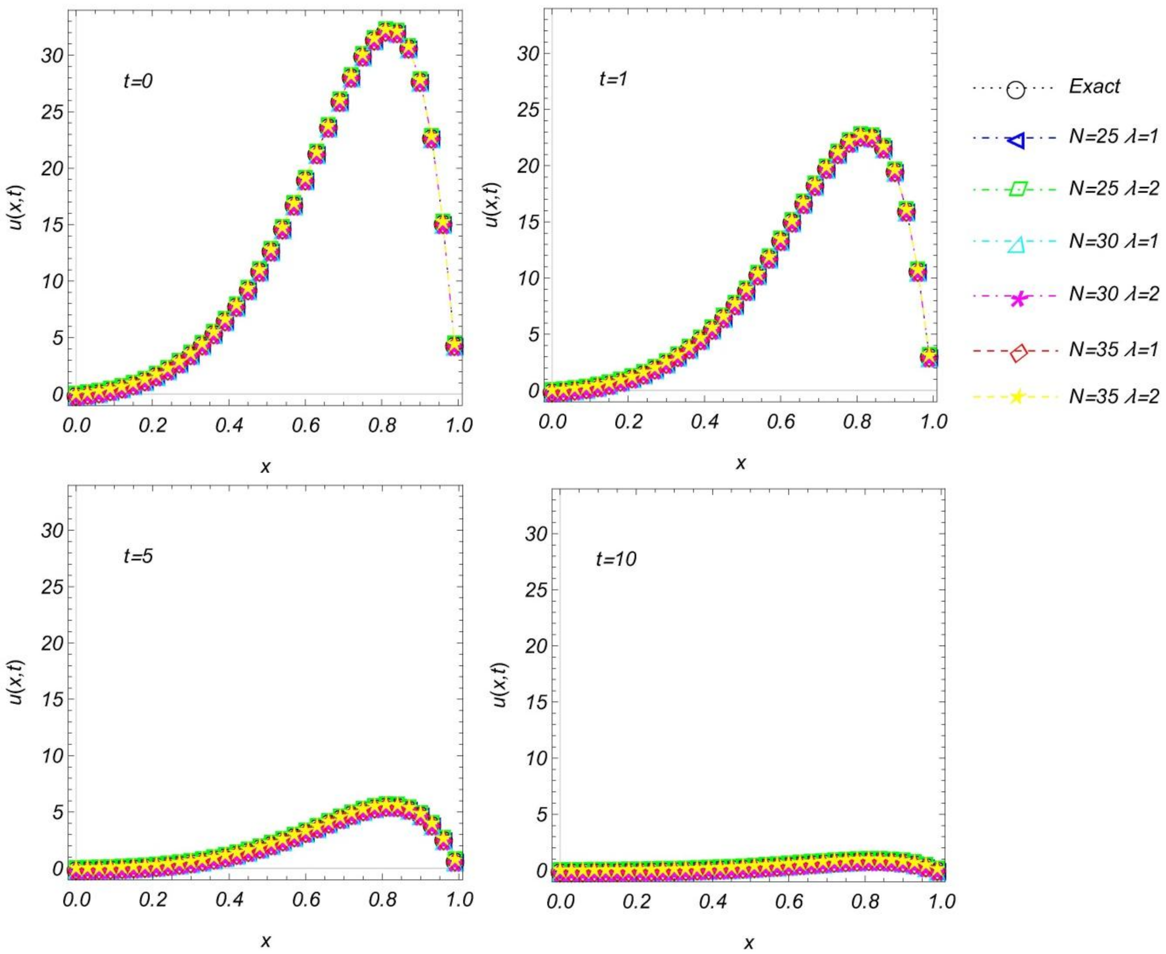

Figure 5. Comparison of the exact and numerical solutions for various values of t .

In Table 1, the absolute errors of our approximate solutions have compared with the absolute errors of the approximate solutions of Krylov subspace method and Crank Nicolson method. In Table 2, the relative errors of the corrected approximate solutions have given for $0 \le x \le 1$ and $0 \le t \le 1$.

Table 1. Comparison of absolute error and $\overline{R}_N$ in Example 1.

x	t	Lerch Matrix Collcation Method						Krylov Subspace Method [41]	Crank Nicolson Method [41]
		$N=25$		$N=30$		$N=35$			
		$\lambda=1$	$\lambda=2$	$\lambda=1$	$\lambda=2$	$\lambda=1$	$\lambda=2$		
0.1	0.2	2.44E-10	1.70E-10	2.83E-10	9.13E-11	1.04E-10	1.11E-11	6.54E-10	1.88E-05
0.1	0.6	7.21E-10	3.65E-10	5.10E-10	1.79E-10	1.93E-10	2.25E-11	1.47E-09	4.15E-05
0.1	1	5.29E-10	2.51E-10	2.46E-10	7.95E-11	1.29E-10	3.78E-13	1.89E-09	5.26E-05
0.1	10	6.83E-09	2.18E-09	2.18E-10	2.21E-09	1.73E-10	6.60E-11	3.37E-10	7.17E-06
0.5	0.2	1.71E-09	1.64E-10	1.09E-11	3.62E-11	7.68E-11	2.21E-11	2.09E-09	1.39E-05
0.5	0.6	5.24E-09	4.35E-10	8.49E-11	3.43E-11	2.03E-10	5.34E-11	5.49E-09	3.66E-05
0.5	1	5.27E-09	4.99E-10	1.65E-10	2.30E-12	2.19E-10	5.79E-11	7.96E-09	5.31E-05
0.5	10	7.46E-07	3.50E-07	3.09E-07	1.15E-07	4.99E-08	1.91E-08	3.46E-09	2.31E-05
0.9	0.2	3.82E-08	4.55E-08	4.51E-10	3.82E-08	2.59E-10	2.66E-09	2.61E-08	9.67E-04
0.9	0.6	5.62E-08	1.55E-07	2.37E-09	1.05E-07	4.69E-09	3.18E-08	5.53E-08	2.10E-03
0.9	1	6.12E-09	3.41E-07	3.24E-08	1.60E-07	1.17E-08	1.01E-07	6.71E-08	2.63E-03
0.9	10	4.59E-04	1.89E-04	5.10E-05	2.22E-05	6.57E-06	4.13E-07	2.60E-09	2.86E-04

Table 2. Relative errors of the corrected approximate solutions.

x	**t**	$N=15$ $\lambda=1$	$N=15$ $\lambda=1$		
			$M=21$	$M=22$	$M=23$
0.2	0.80	4.30E-4	2.08E-4	2.27E-5	1.89E-5
0.7	0.80	1.02E-3	6.86E-4	3.71E-5	4.38E-6
0.2	0.85	7.88E-4	3.51E-4	2.82E-5	1.49E-5
0.7	0.85	2.25E-3	1.67E-3	1.13E-4	1.28E-5
0.2	0.90	1.39E-3	8.46E-4	5.12E-5	2.01E-5
0.7	0.90	4.67E-3	3.77E-3	2.75E-4	5.21E-5

Example 2. [2] Consider the convection-diffusion equation

$$\kappa \frac{\partial^2 u(x,t)}{\partial x^2} - \nu \frac{\partial u(x,t)}{\partial x} = \frac{\partial u(x,t)}{\partial t}$$

with the initial condition and boundary conditions

$$u(x,0) = \sin(2\pi x) \ , \qquad 0 \le x \le 1$$

$$u(0,t) = e^{\left(-\kappa 4\pi^2 t\right)} \sin(-2\pi \upsilon t) \ , \qquad u(1,t) = e^{\left(-\kappa 4\pi^2 t\right)} \sin\left(2\pi(1-\upsilon t)\right) \ , \quad 0 \le t \le T \ .$$

The exact solution of the problem is given as $u(x,t)=e^{\left(-\kappa 4\pi^2 t\right)}\sin\left(2\pi\left(x-\upsilon t\right)\right)$. The approximate solution of the problem for $\nu=1$, $\kappa=0.05$ and $\kappa=0.01$ are found by using the Lerch matrix collocation. Numerical results can be seen in Tables 1-4 and Figs. 6-9.

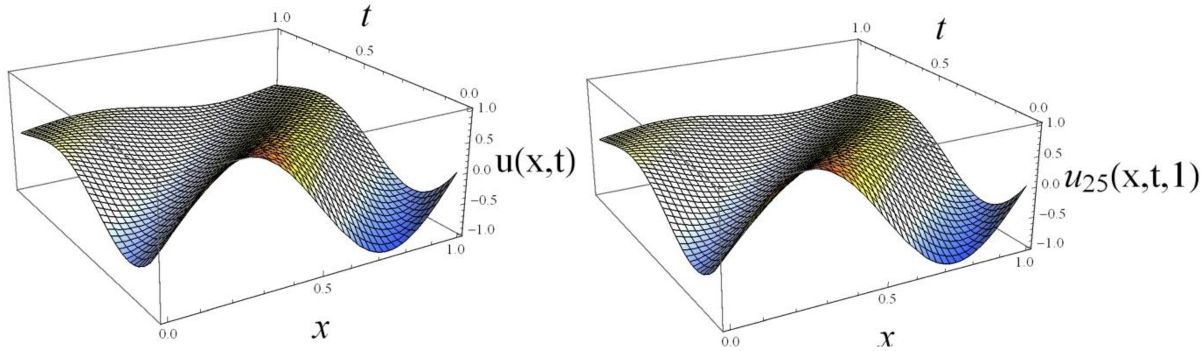

Figure 6. Comparison between exact and numerical solution in Example 2.

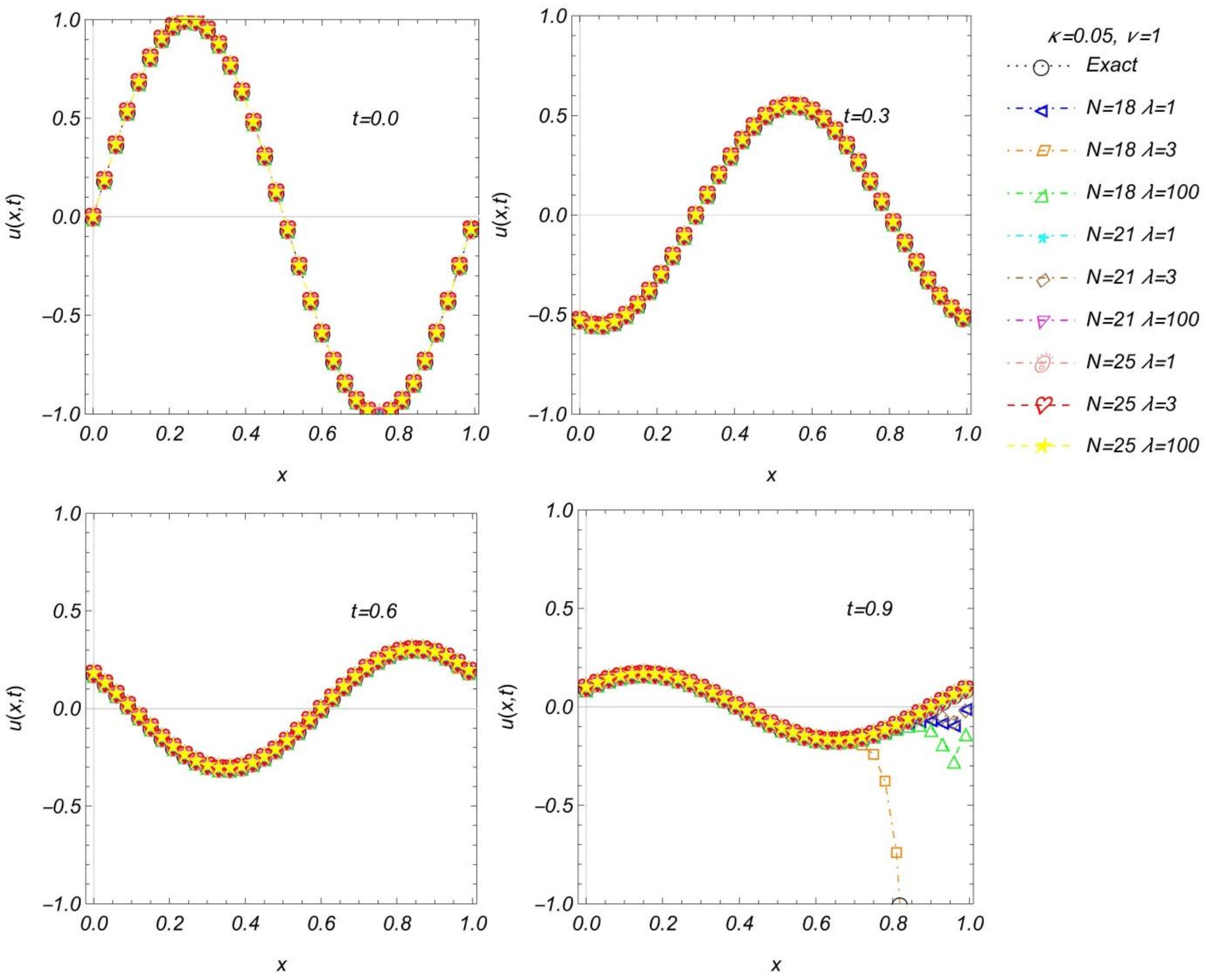

Figure 7. Comparison of the exact and numerical solutions for various values of t.

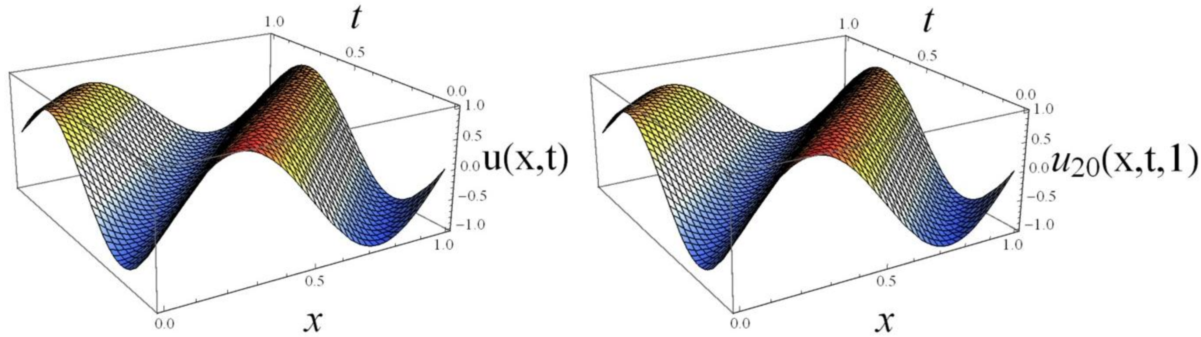

Figure 8. Comparison between exact and numerical solutions in Example 2.

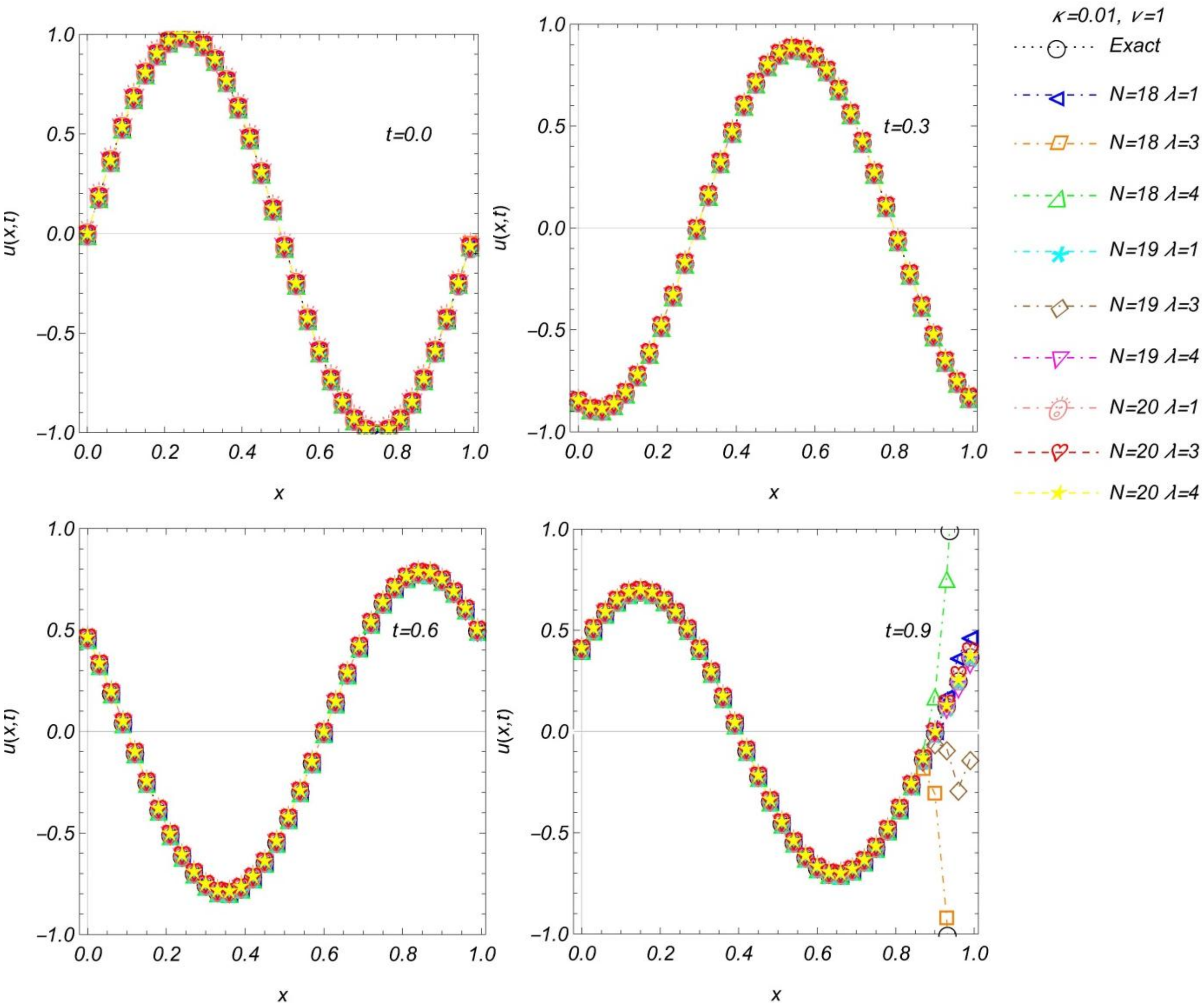

Figure 9. Comparison of the exact and numerical solutions for various values of t.

In Table 3, the absolute errors of our approximate solutions have compared with the absolute errors of the approximate solutions of FDTDQS and DQTFDS. Moreover, the upper bound of the error $\overline{R}_N$ have computed for $\kappa = 0.05$ and $\nu = 1$ in a domain $0 \le x \le 1$ and $0 \le t \le 1$. In Table 4, the absolute errors and the upper bound of the error $\overline{R}_N$ have evaluated for $\kappa = 0.01$ and $\nu = 1$ in a domain $0 \le x \le 1$ and $0 \le t \le 1$.

Table 3. Comparison of absolute error and $\overline{R}_N$ for $\kappa = 0.05$ and $\nu = 1$ in Example 2.

x	t	Lerch Matrix Collcation Method						FDTDQS [2]	DQTFDS [2]
		$N=18$		$N=21$		$N=25$			
		$\lambda=1$	$\lambda=100$	$\lambda=1$	$\lambda=100$	$\lambda=1$	$\lambda=100$		
0.25	0.4	2.65E-8	8.46E-6	1.23E-7	4.61E-8	2.24E-9	1.55E-9	1.1E-3	1.1E-2
0.25	0.6	9.87E-9	6.98E-6	8.78E-9	4.19E-8	1.26E-9	2.32E-9	3.4E-3	1.2E-2
0.25	0.8	6.81E-6	1.19E-5	1.29E-7	1.19E-8	3.54E-10	5.60E-9	2.1E-3	1.0E-2
0.25	1.0	1.09E-3	1.07E-3	1.58E-5	1.52E-5	8.85E-8	1.01E-7	6.0E-4	1.3E-3
0.50	0.4	2.83E-7	1.19E-5	4.12E-7	9.35E-8	2.50E-8	9.83E-9	1.0E-2	6.6E-2
0.50	0.6	8.80E-8	9.48E-6	1.96E-7	5.27E-8	1.03E-8	2.09E-10	3.9E-3	5.4E-2
0.50	0.8	1.64E-7	7.53E-6	5.10E-8	5.27E-8	3.80E-9	4.30E-9	3.4E-3	4.2E-3
0.50	1.0	3.69E-5	4.21E-5	4.99E-7	2.70E-7	1.90E-9	1.25E-9	3.4E-3	2.5E-2
0.75	0.4	1.18E-5	3.60E-5	1.81E-7	6.40E-7	4.92E-8	1.44E-7	2.9E-3	1.9E-2
0.75	0.6	4.19E-6	3.35E-5	6.48E-8	8.09E-7	4.25E-8	5.40E-7	1.1E-2	2.7E-2
0.75	0.8	3.45E-4	2.90E-4	8.78E-6	6.47E-7	3.14E-7	9.01E-7	6.4E-3	3.2E-2
0.75	1.0	1.87E-3	5.01E-3	8.58E-5	6.07E-6	1.83E-6	9.06E-7	1.9E-3	6.3E-3
$\overline{R}_N$		1.79E-1	4.49E-1	1.08E-2	8.88E-4	2.86E-4	2.90E-4	—	—

Table 4. Comparison of absolute error and $\overline{R}_N$ for $\kappa = 0.01$ and $\nu = 1$ in Example 2.

x	t	Lerch Matrix Collcation Method								
		$N=18$			$N=19$			$N=20$		
		$\lambda=1$	$\lambda=3$	$\lambda=4$	$\lambda=1$	$\lambda=3$	$\lambda=4$	$\lambda=1$	$\lambda=3$	$\lambda=4$
0.25	0.4	6.49E-8	1.01E-7	2.12E-7	3.86E-8	9.65E-8	5.71E-8	5.29E-9	2.12E-8	8.25E-10
0.25	0.6	1.09E-7	2.47E-7	7.94E-8	9.63E-10	1.35E-7	9.35E-8	3.02E-9	8.79E-8	2.02E-10
0.25	0.8	6.64E-7	2.75E-6	3.97E-6	1.21E-7	3.50E-6	1.89E-7	2.89E-9	1.42E-7	1.36E-7
0.25	1.0	7.31E-5	7.08E-4	5.79E-4	3.40E-5	1.99E-4	1.31E-5	6.37E-7	7.88E-6	3.00E-6
0.50	0.4	1.31E-6	3.62E-6	8.40E-8	6.33E-8	5.91E-7	9.89E-8	2.28E-8	2.48E-7	5.94E-8
0.50	0.6	4.98E-7	8.85E-7	4.00E-7	3.73E-8	9.87E-8	2.95E-8	2.04E-8	9.80E-8	1.58E-8
0.50	0.8	9.87E-8	2.84E-8	1.23E-7	1.59E-8	1.41E-8	9.73E-8	7.10E-10	6.71E-8	1.88E-9
0.50	1.0	7.98E-7	7.26E-6	6.20E-6	1.22E-7	2.47E-6	1.11E-7	6.87E-9	1.26E-7	1.02E-7
0.75	0.4	3.54E-7	9.17E-7	1.42E-7	1.43E-7	1.37E-7	6.68E-8	2.30E-8	1.63E-8	3.36E-8
0.75	0.6	8.94E-7	2.75E-6	3.38E-8	1.01E-7	7.45E-7	3.92E-8	2.51E-8	1.61E-7	3.26E-8
0.75	0.8	2.25E-6	1.82E-5	1.53E-5	3.04E-8	1.70E-6	8.04E-8	2.50E-8	1.71E-7	2.09E-8
0.75	1.0	1.36E-5	3.98E-4	2.03E-4	6.57E-8	1.71E-5	1.20E-6	1.32E-8	3.30E-7	3.30E-9
$\overline{R}_N$		6.42E-2	1.56E-0	8.62E-1	8.15E-4	3.23E-1	2.34E-2	8.30E-4	1.49E-2	1.34E-3

6. CONCLUSIONS

In this study Lerch matrix collocation method has been introduced in order to solve the unsteady one-dimensional time dependent convection-diffusion equation with constant coefficients on a rectangular domain. Numerical examples have applied to show usefulness this method. The numerical results show that the accuracy improves when N is increased. Also, the presented method has been compared with other numerical methods such as Krylov subspace method, Crank Nicolson method, FDTDQS and DQTFDS. Our method can be easily extended to solve model equations such as nonlinear convection, reaction, linear diffusion and dispersion; but some modifications are required.

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