ORIGINAL PAPER

PROPERTIES OF LUCAS-SUM GRAPH

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Abstract. *For each positive integer n, the Lucas-sum graph H_n on vertices $1,2,\dots,n$ is defined by two vertices forming an edge if and only if they sum two Lucas number. In this paper, Lucas-sum graph was defined and some properties of this graph were examined.*
Keywords: *Lucas sequence, Lucas-sum graph, Hamiltonian path.*

1. INTRODUCTION

For each $n \geq 1$, $G(V,E)$ structure is defined as a graph with the vertex set $V = \{1,2,\dots,n\}$ and the edge set $E = \{ij: i,j \in V\}$. In any $G(V,E)$ graph, for $i,j \in V$ if $ij \in E$ then, i and j vertices are called as adjacent vertices and indicated by $i \sim j$. In graph theory, the number of edges that are incident to i-vertex is called the degree of i-vertex and denoted by $d(i)$ [1-3].

In a graph, the sequence $v_i e_{i+1} v_{i+1} \dots e_j v_j$ is called as a walk. If all the vertices and edges of a walk are distinct, then this walk called as a path. Moreover, a graph path between two vertices of a graph that visits each vertex exactly once is called as a Hamiltonian path [1-3].

In [4], Fibonacci-sum graph was defined and was examined its properties. The definition of Fibonacci-sum graph is as follows:

For each $n \geq 1$, the graph $G_n = (V,E)$ is defined as Fibonacci-sum graph with the vertex set $V = [n] = \{1,2,\dots,n\}$ and the edge set $E = \{\mathrm{ij}: \mathrm{i},\mathrm{j} \in \mathrm{V}, \mathrm{i} \neq \mathrm{j}, \mathrm{i}+\mathrm{j}$ is a Fibonacci number$\}$.

Inspired by this, we defined the Lucas-sum graph and investigated some properties.

Lucas sequence is defined as by the recurrence relation $\mathrm{L}_{n+1} = \mathrm{L}_n + \mathrm{L}_{n-1}$ with initial values $\mathrm{L}_0 = 2$ and $\mathrm{L}_1 = 1$ and has the following fundamental properties [5-7]:

- Binet Formula: Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$, so that α and β are both roots of the equation $x^2 = x + 1$. Then, $L_n = \alpha^n + \beta^n$, for all $n \geq 1$.
- $2F_{m+n} = F_m L_n + F_n L_m$
- $\mathrm{L}_n{}^2 - 5F_n{}^2 = 4(-1)^n$

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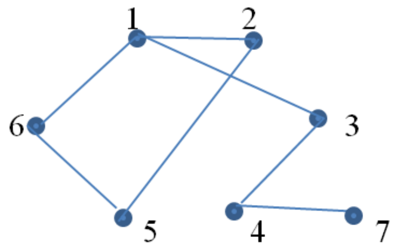
2. THE PROPERTIES OF LUCAS-SUM GRAPH

Definition 2.1. For each $n \geq 1$, the graph $H_n = (V, E)$ is defined as Lucas-sum graph with the vertex set $V = [n] = \{1,2, \dots, n\}$ and the edge set $E = \{\mathrm{ij} : \mathrm{i}, \mathrm{j} \in \mathrm{V}, \mathrm{i} \neq \mathrm{j}, \mathrm{i} + \mathrm{j} \text{ is a Lucas number}\}$.

Example 2.2. A Lucas-sum graph for $n = 7$ is as follows



Theorem 2.3. For any integer $k \geq 2$, the subgraph of H_{L_k}, formed by the vertices whose sum is in $\{L_{k-1}, L_k, L_{k+1}\}$ is a Hamiltonian path.

Proof: Let P_k be a subgraph of H_{L_k} whose sum is in $\{L_{k-1}, L_k, L_{k+1}\}$. We claim that P_k is a path. We will show that P_k is a Hamiltonian path. To see this, we will show that two vertices of H_{L_k} have degree 1 in P_k(endpoints) and the others have degree 2.

We will show that each vertex lies on at most two edges having sums in $\{L_{k-1}, L_k, L_{k+1}\}$ to see that each vertex of H_{L_k} have degree at most 2 in P_k. By the definition of Lucas-sum graph, no vertex in $\{1, \dots, L_{k-1} - 1\}$ lies on an edge having sum L_{k+1} and no vertex in $\{L_{k-1}, \dots, L_k\}$ lies on an edge having sum L_{k-1}. Contrary, each vertex smaller than L_{k-1} lies on an edge having sum L_{k-1}, except for the vertex $L_{k-1}/2$ when L_{k-1} is even. Similarly, each vertex smaller than L_k lies on an edge having sum L_k, except for the vertex $L_k/2$ when L_k is even. Lastly, each vertex in $\{L_{k-1}, \dots, L_k\}$ lies on an edge having sum L_{k+1}, except for the vertex $L_{k+1}/2$ when L_{k+1} is even. Hence, each vertex of H_{L_k} has degree at least 2 in P_k, except possibly for L_k, $L_{k-1}/2$, $L_k/2$ and $L_{k+1}/2$. Because of the definition of Lucas-sum graph there is no cycle in P_k. The vertex L_k must have degree 1, and the vertices $L_{k-1}/2$, $L_k/2$ and $L_{k+1}/2$ have degree 1 precisly when L_{k-1}, L_k and L_{k+1}, respectively, are even. Since exactly one of L_{k-1}, L_k and L_{k+1} is even, P_k has exactly two vertices of degree 1.

Result 2.4. For $k \geq 2$ in H_{L_k} graph, L_k is adjacent to only L_{k-1}.

Proof: The result can be easily seen by the recurrence relation of Lucas sequence.

Theorem 2.5. Let $n \geq 2$ and $L_k \leq n < L_{k+1}$. In H_n, the vertex n is adjacent to only

$$\begin{cases} L_{k+1} - n; & if \quad n \leq \dfrac{L_{k+2}}{2} \\ L_{k+1} - n \;\; and \;\; L_{k+2} - n; & if \;\; n > \dfrac{L_{k+2}}{2} \end{cases}.$$

Proof: Let $x \in [1, n]$ be adjacent to n; in other words, let i be so that $x + n = L_i$. Since

$$L_k < 1 + L_k \leq x + n$$

and

$$x + n \leq 2n - 1 < 2L_{k+1} < L_{k+1} + L_{k+2} = L_{k+3}$$

it follows that $i \in \{k+1, k+2\}$.

When $n \leq \frac{L_{k+2}}{2}$, $x + n < 2n \leq L_{k+2}$, we get $L_i < L_{k+2}$. So $i \neq k+2$. Thus $x = L_{k+1} - n$ is the only possible solution. When $n > \frac{L_{k+2}}{2}$, the possible solution is also $L_{k+2} - n$.

Lemma 2.6. Let $n \geq 2$ and $L_k \leq n < L_{k+1}$. Then in H_n, the vertex L_k is adjacent to only L_{k-1}.

Proof: If $n = L_k$, then by the previous result it is true. Suppose that $L_k < n < L_{k+1}$. In H_n, let any vertex v satisfy $L_k < v \leq n$. Then, since

$$L_{k+1} < L_k + L_k < L_k + v \leq L_k + n < L_k + L_{k+1} = L_{k+2}$$

we can see that $L_k + v$ is not a Lucas number, and so v is not adjacent to L_k. Thus in H_n, L_k has only one neighbour which is smaller than L_k, namely L_{k-1}.

Theorem 2.7. Let $n \geq 1$ and let $x \in [1, n]$. Let $k \geq 1$ satisfy $L_k \leq x < L_{k+1}$ and $l \geq k$ satisfy $L_s \leq x + n < L_{s+1}$. Then the degree of x in H_n is

$$deg_{H_n}(x) = \begin{cases} s - k - 1\,; & if\ x = \frac{1}{2}L_{k+2} \\ s - k; & otherwise\,. \end{cases}$$

Proof: For each $m \in [1, n]$, since $L_k < x + m \leq x + n < L_{s+1}$, if $x + m$ is a Lucas number, then $k < s$ and $x + m \in \{L_{k+1}, \dots, L_s\}$. Hence,

$$deg_{H_n}(x) = |\{m \in [1, n]: m \neq x, x + m \in \{L_{k+1}, \dots, L_s\}\}| \qquad (1)$$

and so $deg_{H_n}(x) \leq s - k$.

Assume that $2x$ is a Lucas number. Then we get $2x < 2L_{k+1} < L_{k+3}$, so either $2x = L_{k+1}$ or $2x = L_{k+2}$. If $2x = L_{k+1}$ then $2L_k \leq 2x = L_{k+1} = L_{k-1} + L_k$. This implies that $L_k \leq L_{k-1}$, and this is a contradiction since there is no Lucas number which satisfy this equality. So, we hold that $2x = L_{k+2}$. Then $L_{k+2} = x + x \leq x + n < L_{s+1}$, and so $k + 2 \leq s$. Thus one value for m in (2) is lost and so $deg_{H_n}\left(\frac{1}{2}L_{k+2}\right) = s - k - 1$.

In other cases, $2x$ is not a Lucas number, by (2) $deg_{H_n}(x) = s - k$.

3. CONCLUSION

In this paper we have defined Lucas-sum graph using Lucas sequence. Also we have obtained some properties of these graphs. In future studies, new graphs can be defined and their properties can be examined using different recurrence sequences.

REFERENCES

[1] Brouwer, A.E., Haemers, W.H., *Spectra of graphs*, Springer, 2011.
[2] Bandy, J.A., Murty, U.S.R., *Graph theory with applications*, North-Holland, 1982.
[3] Diestel, R., *Graph theory, 5th Edition*, Springer-Verlag, 2016.
[4] Arman, A., Gunderson, D.S., Li, P.C., *Properties of the Fibonacci-sum graph*, arXiv:1710.10303v1[math.CO] 27 Oct 2017 https://arxiv.org/abs/1710.10303.
[5] Benjamin, A.T., Quinn, J.J., *The College Mathematics Journal*, **30**(5), 359, 1999.
[6] Robbins, N., *The Fibonacci Quaterly*, **29**(4), 362, 1991.
[7] Vajda, S., *Fibonacci and Lucas numbers, and the golden section: Theory and applications*, Ellis Horwood Limited, 1989.