ORIGINAL PAPER

ON THE UNITARILY INVARIANT NORMS OF THE MATRICES CONNECTED TO COMPLEX NUMBER SEQUENCES

MUSTAFA BAHSI[1]

Manuscript received: 04.12.2019; Accepted paper: 29.03.2020;
Published online: 30.06.2020.

Abstract. *In this study, we compute the unitarily invariant norms of the matrices* $A_z=(z_iz_j)_{i,j=1}^n$, $B_z=(z_i-z_j)_{i,j=1}^n$ *and* $C_z=(\frac{z_i}{z_j})_{i,j=1}^n$, *where* $z_i s$ *are ith components of any complex sequence* (z_n). *Moreover, we give some corollaries and numerical examples related to norms of these matrices.*

Keywords: *Unitarily invariant norms, Singular values, Complex sequence.*

1. INTRODUCTION

Let $\|.\|$ be a unitarily invariant norm on M_n the space of $n\times n$ complex matrices. Then, $\|UAV\|=\|A\|$ for all $A\in M_n$ and for all unitary matrices $U,V\in M_n$. The well-known two classes of unitarily invariant norms are Ky Fan *k*-norm and Schatten *p*-norm. The Ky Fan *k*-norm and Schatten *p*-norm of the matix *A* are defined as [1]:

$$\|A\|_{(k)}=\sum_{j=1}^{k}s_j(A),\qquad k=1,2,...,n$$

and

$$\|A\|_p=\left(\sum_{j=1}^{n}s_j^p(A)\right)^{1/p},\qquad 1\le p<\infty,$$

respectively, where $s_i(i=1,2,...,n)$ are the singular values of *A* with $s_1\ge s_2\ge\cdots\ge s_n$, which are the eigenvalues of the matrix $(AA^H)^{\frac{1}{2}}$. When we take $k=1$ and $p=2,$ we have the well-known spectral norm $\|.\|_s$ and Euclidean norm $\|.\|_E$, respectively. That is,

$$\|A\|_s=s_1(A)=\|A\|_{(1)}$$

and

$$\|A\|_E=\sqrt{\sum_{j=1}^{n}s_j^2(A)}=\|A\|_2.$$

[1] Aksaray University, Education Faculty, 68100, Aksaray, Turkey. E-mail:mhvbahsi@yahoo.com

The equation $\det(\lambda \mathrm{I}-A)=0$, the polynomial $p(\lambda)=\det(\lambda \mathrm{I}-A)$ and the solutions of the equation $\det(\lambda \mathrm{I}-A)=0$ are known as characteristic equation, characteristic polynomial and eigenvalues of matrix A, respectively. The characteristic polynomial of the matrix A is a monic polynomial and has the following form:

$$p(\lambda)=\lambda^n+a_1\lambda^{n-1}+a_2\lambda^{n-2}+\cdots+a_{n-1}\lambda+a_n.$$

The coefficients of $p(\lambda)$ are calculated by using principal minors of the matrix A. That is

$$a_r=\sum_{1\le i_1<i_2<\ldots<i_r\le n}(-1)^r A\begin{pmatrix} i_1 i_2\ldots i_r \\ i_1 i_2\ldots i_r\end{pmatrix}, \quad (1\le r\le n)$$

where $A\begin{pmatrix} i_1 i_2\ldots i_r \\ i_1 i_2\ldots i_r\end{pmatrix}$ is $r-$principal minor of the matrix A and it is denoted by

$$A\begin{pmatrix} i_1 i_2\ldots i_r \\ i_1 i_2\ldots i_r\end{pmatrix}=\begin{vmatrix} a_{i_1,i_1} & a_{i_1,i_2} & \cdots & a_{i_1,i_r} \\ a_{i_2,i_1} & a_{i_2,i_2} & \cdots & a_{i_2,i_r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_r,i_1} & a_{i_r,i_2} & \cdots & a_{i_r,i_r}\end{vmatrix},$$

where $1\le i_1<i_2<\ldots<i_r\le n,\ (1\le r\le n)$ [2].

To compute the norms of some special matrices and matrices related to integer sequences has been attractive for some researchers [3-7]. Solak [3] has found out some bounds for the spectral and Euclidean norms of the circulant matrices $A=(F_{\mathrm{mod}(j-i,n)})$ and $B=(L_{\mathrm{mod}(j-i,n)})$. Shen and Cen [4] have obtained bounds for the norms of r-circulant matrices of the forms $A=C_r(F_0,F_1,...,F_{n-1})$ and $B=C_r(L_0,L_1,...,L_{n-1})$. Solak and Bahşi [5] have given the equalities for Toeplitz matrices, such that $A=(F_{i-j})$ and $B=(L_{i-j})$. Recently, Solak [6] have computed the spectral norm of the matrix $A_x=(x_i x_j)_{i,j=1}^n$.

The main purpose of this paper is to compute Ky Fan k-norms and Schatten p-norms of the matrices

$$A_z=(z_i z_j)_{i,j=1}^n, \tag{1}$$

$$B_z=(z_i-z_j)_{i,j=1}^n \tag{2}$$

$$C_z=\left(\frac{z_i}{z_j}\right)_{i,j=1}^n \tag{3}$$

where z_i s $(z_i\neq 0)$ are ith components of any complex sequence (z_n).

2. MAIN RESULTS

Theorem 2.1. Let the matrix A_z be as in (1). Then

$$\|A_z\|_{(k)} = \sum_{i=1}^{n} |z_i|^2 , \qquad k = 1,2,...,n$$

and

$$\|A_z\|_p = \sum_{i=1}^{n} |z_i|^2 , \qquad 1 \le p < \infty,$$

where $n \ge 2$.

Proof: Since

$$A_z = \begin{pmatrix} z_1^2 & z_1 z_2 & z_1 z_3 & \cdots & z_1 z_n \\ z_2 z_1 & z_2^2 & z_2 z_3 & \cdots & z_2 z_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_n z_1 & z_n z_2 & z_n z_3 & \cdots & z_n^2 \end{pmatrix},$$

we have

$$A_z A_z^H = \begin{pmatrix} z_1^2 & z_1 z_2 & z_1 z_3 & \cdots & z_1 z_n \\ z_2 z_1 & z_2^2 & z_2 z_3 & \cdots & z_2 z_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_n z_1 & z_n z_2 & z_n z_3 & \cdots & z_n^2 \end{pmatrix} \begin{pmatrix} \overline{z_1^2} & \overline{z_2 z_1} & \overline{z_3 z_1} & \cdots & \overline{z_n z_1} \\ \overline{z_1 z_2} & \overline{z_2^2} & \overline{z_3 z_2} & \cdots & \overline{z_n z_2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{z_1 z_n} & \overline{z_2 z_n} & \overline{z_3 z_n} & \cdots & \overline{z_n^2} \end{pmatrix}$$

$$= \sum_{i=1}^{n} |z_i|^2 \begin{pmatrix} |z_1|^2 & z_1 \overline{z_2} & z_1 \overline{z_3} & \cdots & z_1 \overline{z_n} \\ z_2 \overline{z_1} & |z_2|^2 & z_2 \overline{z_3} & \cdots & z_2 \overline{z_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_n \overline{z_1} & z_n \overline{z_2} & z_n \overline{z_3} & \cdots & |z_n|^2 \end{pmatrix}.$$

If we add $\dfrac{-z_i \overline{z_1}}{|z_1|^2}$ multiple of first row to ith ($i = 2,3,...,n$) rows of $A_z A_z^H$, then we obtain

$$(A_z A_z^H)' = \sum_{i=1}^{n} |z_i|^2 \begin{pmatrix} |z_1|^2 & z_1 \overline{z_2} & z_1 \overline{z_3} & \cdots & z_1 \overline{z_n} \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Since $\operatorname{rank}(A_z A_z^H) = \operatorname{rank}(A_z A_z^H)' = 1$, for $r \ge 2$ all r-principal minors are zero. Hence the characteristic polynomial of the matrix $A_z A_z^H$ is

$$p(\lambda) = \lambda^n + a_1 \lambda^{n-1}$$

where $a_1 = -\sum_{1\le i\le n}(A_z A_z^H)\begin{pmatrix} i \\ i \end{pmatrix} = -trace(A_z A_z^H) = -\left(\sum_{i=1}^{n}|z_i|^2\right)^2$. Then the eigenvalues of the matrix $A_z A_z^H$ are $\lambda_1 = \lambda_2 = \ldots = \lambda_{n-1} = 0$ and $\lambda_n = \left(\sum_{i=1}^{n}|z_i|^2\right)^2$. Hence,

$$s_1(A_z) = \sum_{i=1}^{n}|z_i|^2 \text{ and } s_2(A_z) = s_3(A_z) = \ldots = s_n(A_z) = 0.$$

Thus,

$$\|A_z\|_{(k)} = \sum_{j=1}^{k} s_j(A_z) = \sum_{i=1}^{n}|z_i|^2, \quad k = 1,2,\ldots,n$$

and

$$\|A_z\|_p = \left(\sum_{j=1}^{n} s_j^p(A_z)\right)^{1/p} = \sum_{i=1}^{n}|z_i|^2, \quad 1 \le p < \infty.$$

Thus the proof is completed. □

Corollary 2.2. Let the matrix A_z be as in (1). Then, the spectral norm and Euclidean norm of A_z hold

$$\|A_z\|_s = \|A_z\|_E = \sum_{i=1}^{n}|z_i|^2$$

where $n \ge 2$.

Proof: When we take $k = 1$ and $p = 2$ in Theorem 2.1, we have

$$\|A_z\|_s = \|A_z\|_E = \sum_{i=1}^{n}|z_i|^2. \quad \square$$

In Corollary 2.2, if we assume that imaginary part of $z_i (i = 1,2,\ldots,n)$ is zero $(\operatorname{Im}\{z_i\} = 0)$, then we have the result of Solak in [6] for the spectral norm.

Theorem 2.3. Let the matrix C_z be as in (3). Then

$$\|C_z\|_{(k)} = \left[\sum_{i=1}^{n}\sum_{j=1}^{n}|z_j|^2 |z_i|^{-2}\right]^{1/2}, \quad k = 1,2,\ldots,n$$

and

$$\|C_z\|_p = \left[\sum_{i=1}^{n}\sum_{j=1}^{n}|z_j|^2 |z_i|^{-2}\right]^{1/2}, \quad 1 \le p < \infty,$$

where $n \ge 2$.

Proof: This theorem can be proved by using a similar method to method of the proof of Theorem 2.1. □

Corollary 2.4. Let the matrix C_z be as in (3). Then, the spectral norm and Euclidean norm of C_z hold

$$\|C_z\|_s = \|C_z\|_E = \left[\sum_{i=1}^{n}\sum_{j=1}^{n}|z_j|^2 |z_i|^{-2}\right]^{1/2}$$

where $n \geq 2$.

Proof: When we take $k=1$ and $p=2$ in Theorem 2.3, we have

$$\|C_z\|_s = \|C_z\|_E = \left[\sum_{i=1}^{n}\sum_{j=1}^{n}|z_j|^2 |z_i|^{-2}\right]^{1/2}. \quad \square$$

Theorem 2.5. Let the matrix B_z be as in (2). Then

$$\|B_z\|_{(k)} = \begin{cases} \left[\sum_{1\le r<s\le n} |z_r - z_s|^2\right]^{1/2}, & \text{if } k=1, \\ 2\left[\sum_{1\le r<s\le n} |z_r - z_s|^2\right]^{1/2}, & \text{if } k=2,3,\dots,n, \end{cases}$$

and

$$\|B_z\|_p = \sqrt[p]{2}\left[\sum_{1\le r<s\le n} |z_r - z_s|^2\right]^{1/2}, \qquad 1 \le p < \infty,$$

where $n \geq 4$.

Proof: If we substract $(i-1)$th row from ith row of the matrix B_z for $i = n, n-1, \dots, 2$, then we obtain

$$B_z' = \begin{pmatrix} 0 & z_1 - z_2 & z_1 - z_3 & \cdots & z_1 - z_n \\ z_2 - z_1 & z_2 - z_1 & z_2 - z_1 & \cdots & z_2 - z_1 \\ z_3 - z_2 & z_3 - z_2 & z_3 - z_2 & \cdots & z_3 - z_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_{n-1} - z_{n-2} & z_{n-1} - z_{n-2} & z_{n-1} - z_{n-2} & \cdots & z_{n-1} - z_{n-2} \\ z_n - z_{n-1} & z_n - z_{n-1} & z_n - z_{n-1} & \cdots & z_n - z_{n-1} \end{pmatrix}$$

Evidently, $\operatorname{rank}(B_z') = \operatorname{rank}(B_z) = \operatorname{rank}(B_z B_z^H) = 2$, where

$$B_z B_z^H = \left(\sum_{k=1}^{n}(z_i - z_k)\overline{(z_j - z_k)}\right)_{i,j=1}^{n}.$$

Since $\operatorname{rank}(B_z B_z^H) = 2$, for $r \geq 3$ all r-principal minors are zero. Hence the characteristic polynomial of the matrix $B_z B_z^H$ is

$$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2}$$

where

$$a_1 = -\sum_{1\le i\le n}(B_z B_z^H)\begin{pmatrix} i \\ i \end{pmatrix} = -trace(B_z B_z^H) = -\sum_{i=1}^{n}\sum_{k=1}^{n}\left|z_i - z_k\right|^2 = -2\sum_{1\le r<s\le n}\left|z_r - z_s\right|^2,$$

$$a_2 = \sum_{1\le r<s\le n}(B_z B_z^H)\begin{pmatrix} r & s \\ r & s \end{pmatrix} = \sum_{1\le r<s\le n}\begin{vmatrix} \sum_{k=1}^{n}(z_r - z_k)\overline{(z_r - z_k)} & \sum_{k=1}^{n}(z_r - z_k)\overline{(z_s - z_k)} \\ \sum_{k=1}^{n}(z_s - z_k)\overline{(z_r - z_k)} & \sum_{k=1}^{n}(z_s - z_k)\overline{(z_s - z_k)} \end{vmatrix}$$

$$= \left(\sum_{1\le r<s\le n}\left|z_r - z_s\right|^2\right)^2.$$

Then the eigenvalues of the matrix $B_z B_z^H$ are $\lambda_1 = \lambda_2 = \ldots = \lambda_{n-2} = 0$ and $\lambda_{n-1} = \lambda_n = \sum_{1\le r<s\le n}\left|z_r - z_s\right|^2$. Hence,

$$s_1(B_z) = s_2(B_z) = \left[\sum_{1\le r<s\le n}\left|z_r - z_s\right|^2\right]^{1/2} \text{ and } s_3(B_z) = s_4(B_z) = \ldots = s_n(B_z) = 0.$$

Thus,

$$\left\|B_z\right\|_{(k)} = \sum_{j=1}^{k} s_j\left(B_z\right) = \begin{cases} \left[\sum_{1\le r<s\le n}\left|z_r - z_s\right|^2\right]^{1/2}, & \text{if } k = 1, \\ 2\left[\sum_{1\le r<s\le n}\left|z_r - z_s\right|^2\right]^{1/2}, & \text{if } k = 2,3,\ldots,n \end{cases}$$

and

$$\left\|B_z\right\|_p = \left(\sum_{j=1}^{n} s_j^p\left(B_z\right)\right)^{1/p} = \sqrt[p]{2}\left[\sum_{1\le r<s\le n}\left|z_r - z_s\right|^2\right]^{1/2}, \quad 1 \le p < \infty.$$

Thus the proof is completed. □

Corollary 2.6. Let the matrix B_z be as in (2). Then, the spectral norm and Euclidean norm of B_z hold

$$\left\|B_z\right\|_s = \left[\sum_{1\le r<s\le n}\left|z_r - z_s\right|^2\right]^{1/2}$$

and

$$\left\|B_z\right\|_E = \sqrt{2}\left[\sum_{1\le r<s\le n}\left|z_r - z_s\right|^2\right]^{1/2},$$

where $n \ge 4$.

Proof: When we take $k = 1$ and $p = 2$ in Theorem 2.5, the proof is trivial. □

3. NUMERICAL EXAMPLES

In this section we give some numerical examples to illustrate our results. We will use some number sequences such as Fibonacci, Lucas and harmonic numbers in our examples. The well known F_n, L_n and H_n are the nth Fibonacci, Lucas and harmonic numbers defined as

$$F_n = \begin{cases} 0 & \text{if } n=0, \\ 1 & \text{if } n=1, \\ F_{n-1}+F_{n-2} & \text{if } n>1, \end{cases} \quad L_n = \begin{cases} 2 & \text{if } n=0, \\ 1 & \text{if } n=1, \\ L_{n-1}+L_{n-2} & \text{if } n>1, \end{cases} \quad H_n = \begin{cases} 0 & \text{if } n=0, \\ \sum_{k=1}^{n}\frac{1}{k} & \text{if } n\geq 1, \end{cases}$$

respectively.

Example 3.1. Let the general terms of complex sequences (f_n) and (l_n) be as $f_n = F_n + \mathrm{i}F_{n-1}$ and $l_n = L_n + \mathrm{i}L_{n-1}$, where i is complex unity. Then, the norms of the matrices $A_f = (f_i \overline{f_j})_{i,j=1}^{n}$ and $A_l = (l_i \overline{l_j})_{i,j=1}^{n}$ are

$$\|A_f\|_{(k)} = \|A_f\|_p = \sum_{i=1}^{n}|f_i|^2 = \sum_{i=1}^{n}(F_i^2+F_{i-1}^2) = \sum_{i=1}^{n}F_{2i-1} = F_{2n}$$

and

$$\|A_l\|_{(k)} = \|A_l\|_p = \sum_{i=1}^{n}|l_i|^2 = \sum_{i=1}^{n}(L_i^2+L_{i-1}^2) = \sum_{i=1}^{n}5F_{2i-1} = 5F_{2n},$$

where $k=1,2,...,n$, $1\leq p<\infty$, $F_k^2+F_{k-1}^2 = F_{2k-1}$, $\sum_{k=1}^{n}F_{2k-1} = F_{2n}$ and $L_k^2+L_{k-1}^2 = 5F_{2k-1}$.

Example 3.2. Let the general terms of complex sequences (f_n) and (l_n) be as $f_n = F_n + \mathrm{i}F_n$ and $l_n = L_n + \mathrm{i}L_n$ where i is complex unity. Then, the norms of the matrices $B_f = (f_i - f_j)_{i,j=1}^{n}$ and $B_l = (l_i - l_j)_{i,j=1}^{n}$ are

$$\|B_f\|_{(1)}^2 = \sum_{1\leq r<s\leq n}|f_r - f_s|^2 = \sum_{1\leq r<s\leq n}|(F_r - F_s)(1+\mathrm{i})|^2$$

$$= 2\sum_{1\leq r<s\leq n}(F_r-F_s)^2 = 2(n-1)F_{n+1}F_n - 4\sum_{r=1}^{n-1}\sum_{s=r+1}^{n}F_rF_s,$$

$$\|B_f\|_{(k)}^2 = 4\sum_{1\leq r<s\leq n}|f_r - f_s|^2 = 8(n-1)F_{n+1}F_n - 16\sum_{r=1}^{n-1}\sum_{s=r+1}^{n}F_rF_s, \qquad (k=2,3,...,n),$$

$$\|B_f\|_p^2 = \sqrt[p/2]{2}\sum_{1\leq r<s\leq n}|f_r - f_s|^2 = 2\sqrt[p/2]{2}(n-1)F_{n+1}F_n - 4\sqrt[p/2]{2}\sum_{r=1}^{n-1}\sum_{s=r+1}^{n}F_rF_s, \qquad (1\leq p<\infty),$$

$$\|B_l\|_{(1)}^2 = \sum_{1\le r<s\le n} |l_r - l_s|^2 = \sum_{1\le r<s\le n} |(L_r - L_s)(1+\mathrm{i})|^2$$

$$= 2\sum_{1\le r<s\le n} (L_r - L_s)^2 = (n-1)(L_{n+1}L_n - 2) - 2\sum_{r=1}^{n-1}\sum_{s=r+1}^{n} L_r L_s,$$

$$\|B_l\|_{(k)}^2 = 4\sum_{1\le r<s\le n} |l_r - l_s|^2 = 4(n-1)(L_{n+1}L_n - 2) - 8\sum_{r=1}^{n-1}\sum_{s=r+1}^{n} L_r L_s, \qquad (k = 2,3,\dots,n)$$

and

$$\|B_l\|_p^2 = \sqrt[p/2]{2}\sum_{1\le r<s\le n} |l_r - l_s|^2 = \sqrt[p/2]{2}(n-1)(L_{n+1}L_n - 2) - 2\sqrt[p/2]{2}\sum_{r=1}^{n-1}\sum_{s=r+1}^{n} L_r L_s, \qquad (1 \le p < \infty).$$

Example 3.3. Let the general term of complex sequence (h_n) be as $h_n = 1 + \mathrm{i}\sqrt{n-1}$ where i is complex unity. Then, the norms of the matrix $C_h = \left(\frac{h_i}{h_j}\right)_{i,j=1}^{n}$ is

$$\|C_h\|_{(k)}^2 = \|C_h\|_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} |h_j|^2 |h_i|^{-2} = \sum_{i=1}^{n}\frac{1}{i}\sum_{j=1}^{n} j = \frac{n(n+1)}{2} H_n,$$

where $k = 1,2,\dots,n$, $1 \le p < \infty$.

CONCLUSION

In this study, we compute Ky Fan *k*-norms and Schatten *p*-norms of the some matrices connected to complex number sequences. Moreover, we give some corollaries and numerical examples. Our some results generalize results of Solak in [6] from integer sequence to complex sequence.

REFERENCES

[1] Zou, L., Hea, C., *Linear Algebra and Its Applications*, **436**, 3354, 2012.
[2] Barrett, W.W., *Linear Algebra and Its Applications,* **27**, 211, 1979.
[3] Solak, S., *Applied Mathematics and Computation*, **160**, 125, 2005.
[4] Shen, S., Cen, J., *Applied Mathematics and Computation*, **205**, 391, 2008.
[5] Solak, S., Bahşi, M., *Hacettepe Journal of Mathematics and Statistics* **42**(1), 15, 2013.
[6] Solak, S., *Applied Mathematics and Computation*, **232,** 919, 2014.
[7] Altınısık E., *Mathematical Inequalities and Applications,* **7**(4), 491, 2004.