ORIGINAL PAPER

CERTAIN DEFINITE INTEGRALS OF *k* BESSEL FUNCTION OF FIRST KIND

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Abstract. *This paper deals with some new definite integrals of newly defined k Bessel function of first kind due to Gehlot [1]. The Laplace transform of the k Bessel function is also obtained. Certain new integral representation of k Bessel function are also investigated. The known results of classical Bessel functions are seen to follow as special cases.*
Keywords: *k Bessel function, Laplace Transform*

1. INTRODUCTION

Bessel functions need no introduction as they are very useful in mathematical literature especially in finding the solutions of certain differential equations. Like many other special function, Bessel function also get generalised in different ways. One useful generalization of Bessel function, called *k* Bessel function of first kind is recently done by [1] as

$$J_{\pm v}^{k}(x)=\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{x}{2}\right)^{2n\pm\frac{v}{k}}}{n!\Gamma_k(nk\pm v+k)},\ k\in\mathbb{R}^+,v>-1 \tag{1.1}$$

is the solution of *k* Bessel equation

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\frac{1}{k^2}\left(k-\frac{v^2}{x^2}\right)y=0. \tag{1.2}$$

It is noted that for $k\to1$, the *k* Bessel function (1.1) reduces to classical Bessel function. The symbol $\Gamma_k(x)$ is *k* Gamma function which is a generalization of classical gamma function introduced by Diaz and Pariguan [2], whose integral form defined as

$$\Gamma_k(z)=\int_0^\infty e^{-\frac{t^k}{k}}t^{z-1}dt. \tag{1.3}$$

The improper integral is convergent for $\operatorname{Re}(z)>0$. The *k* Gamma function reduced to classical Gamma function [3] i.e. $\Gamma_k\to\Gamma$ as $k\to1$. A simple change of variable reveals the relationship between *k* Gamma function and classical Gamma function:

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$$\Gamma_k(z) = k^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right). \tag{1.4}$$

In particular the k Bessel function of order zero is

$$J_0^k(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n}}{n!\Gamma_k(n+1)} \tag{1.5}$$

Making use of (1.4), we get

$$J_0^k(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2 k^n} \tag{1.6}$$

Similarly,

$$J_1^k(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n\pm k^{-1}}}{n!k^{n+k^{-1}}\Gamma\left(n+k^{-1}+1\right)} \tag{1.7}$$

Remarks: It can be readily seen that

$$\begin{aligned} J_0^k(0) &= 1 \\ J_1^k(0) &= 0 \end{aligned} \tag{1.8}$$

The following properties are well known and of particular interest [4]

$$\frac{d}{dx} J_v^k(x) = \frac{v}{kx} J_v^k(x) - J_{v+k}^k(x) \tag{1.9}$$

2. MAIN RESULTS

In this section, we will establish some definite integrals involving k Bessel function of first kind.

Theorem 2.1: For $k > 0$ and $a, b \in \mathbb{R},\ a > 0$ the following integral holds

$$\int_0^\infty e^{-ax} J_0^k(bx)\,dx = \frac{\sqrt{k}}{\sqrt{ka^2 + b^2}} \tag{2.1}$$

Proof: Consider

$$e^{\frac{ix\sin(\theta)}{\sqrt{k}}} = \sum_{n=0}^{\infty} \frac{i^n x^n \sin^n(\theta)}{n!k^{n/2}}.$$

Integrating both sides the above equation with respect to θ from 0 to 2π

$$\int_0^{2\pi} e^{\frac{ix\sin(\theta)}{\sqrt{k}}} d\theta = \int_0^{2\pi} \sum_{n=0}^{\infty} \frac{i^n x^n \sin^n(\theta)}{n!k^{n/2}} d\theta = \sum_{n=0}^{\infty} \frac{i^n x^n}{n!k^{n/2}} \sum_{n=0}^{\infty} \int_0^{2\pi} \sin^n(\theta) d\theta \tag{2.2}$$

using the formula

$$\int_0^{2\pi} \sin^n \theta d\theta = \begin{cases} \dfrac{(n-1)(n-3)\cdots 3.1}{n(n-2)\cdots 4.2}.2\pi & n \text{ is even} \\ 0 & n \text{ is odd} \end{cases} \tag{2.3}$$

Into (2.2), we get

$$\int_0^{2\pi} e^{\frac{ix\sin(\theta)}{\sqrt{k}}} d\theta = \sum_{n=0}^{\infty} \frac{i^{2n} x^{2n}}{(2n)!k^n} \left[\frac{(2n-1)(2n-3)\cdots 3.1}{2n(2n-2)\cdots 4.2}.2\pi \right] \tag{2.4}$$

After simplification the series on the right of (2.4), we can write

$$\int_0^{2\pi} e^{\frac{ix\sin(\theta)}{\sqrt{k}}} d\theta = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!k^n} \left[\frac{(2n)!}{2^n n! 2^n n!}.2\pi \right] = 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2 k^n}$$

Using (1.5) we can write

$$J_0^k(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{\frac{ix\sin(\theta)}{\sqrt{k}}} d\theta$$

Equating the real part, we get

$$J_0^k(x) = \frac{1}{2\pi} \int_0^{2\pi} \cos\left(\frac{x\sin\theta}{\sqrt{k}} \right) d\theta$$

or we can write equivalently

$$J_0^k(x) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos\left(\frac{x\sin\theta}{\sqrt{k}} \right) d\theta = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos\left(\frac{x\sin\left(\frac{\pi}{2}-\theta\right)}{\sqrt{k}} \right) d\theta = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos\left(\frac{x\cos(\theta)}{\sqrt{k}} \right) d\theta \tag{2.5}$$

Since

$$\int_0^{\infty} e^{-ax} \cos(bx) dx = \frac{a}{a^2+b^2} \qquad (a>0) \tag{2.6}$$

we replace b by $\dfrac{b\cos\theta}{\sqrt{k}}$ into (2.6) to get

$$\int_0^\infty e^{-ax}\cos\left(\frac{bx\cos\theta}{\sqrt{k}}\right)dx = \frac{ka}{ka^2+b^2\cos^2\theta} \tag{2.7}$$

Integrating both sides of (2.7) with respect to θ from 0 to $\dfrac{\pi}{2}$

$$\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-ax}\cos\left(\frac{bx\cos\theta}{\sqrt{k}}\right)dx\,d\theta = \int_0^{\frac{\pi}{2}}\frac{ka}{ka^2+b^2\cos^2\theta}d\theta \tag{2.8}$$

Since the integrand on LHS of (2.8) is uniform continuous function, we can write

$$\int_0^\infty e^{-ax}\int_0^{\frac{\pi}{2}}\cos\left(\frac{bx\cos\theta}{\sqrt{k}}\right)d\theta\,dx = \int_0^{\frac{\pi}{2}}\frac{ka}{ka^2+b^2\cos^2\theta}d\theta \tag{2.9}$$

Using (2.5) we can write

$$\int_0^\infty e^{-ax}J_0^k(bx)\,dx = \int_0^{\frac{\pi}{2}}\frac{ka}{ka^2+b^2\cos^2\theta}d\theta \tag{2.10}$$

Evaluating the integral on the right of (2.10), we the required integral (2.1).

Remark 2.2: Letting $a\to 0$ in (2.1) we get for $b>0$

$$\int_0^\infty J_0^k(bx)\,dx = \frac{\sqrt{k}}{b} \tag{2.11}$$

In particular

$$b=1\quad \int_0^\infty J_0^k(x)\,dx = \sqrt{k} \tag{2.12}$$

Theorem 2.3: For $k>0$ and $a,c\in\mathbb{R}$ the following integrals hold

$$\int_0^\infty J_0^k(ax)\cos(cx)\,dx = \begin{cases}\dfrac{\sqrt{k}}{\sqrt{ka^2-c^2}} & ka>c\\ 0 & ka<c\end{cases} \tag{2.13}$$

$$\int_0^\infty J_0^k(ax)\sin(cx)\,dx = \begin{cases}0 & ka>c\\ \dfrac{\sqrt{k}}{\sqrt{c^2-ka^2}} & ka<c\end{cases} \tag{2.14}$$

Proof: Interchanging *a* and *b* in (2.1) and extending *b* to be a complex number we get

$$\int_0^\infty J_0^k(ax)e^{-(b+ic)x}dx = \frac{\sqrt{k}}{\sqrt{ka^2+(b+ic)^2}} \quad (b>0) \tag{2.15}$$

Hence, for $b>0$

$$\int_0^\infty J_0^k(ax)e^{-(b+ic)x}dx = \frac{\sqrt{k}}{A+iB} = \frac{\sqrt{k}\,(A-iB)}{A^2+B^2} \tag{2.16}$$

where

$$A+iB = \sqrt{ka^2+(b+ic)^2} \tag{2.17}$$

Equating real and imaginary parts of (2.16), we get two definite integrals as below

$$\int_0^\infty J_0^k(ax)e^{-bx}\cos(cx)dx = \frac{A\sqrt{k}}{A^2+B^2} \tag{2.18}$$

$$\int_0^\infty J_0^k(ax)e^{-bx}\sin(cx)dx = \frac{B\sqrt{k}}{A^2+B^2} \tag{2.19}$$

From (2.17) we can write

$$\begin{cases} A^2-B^2 = ka^2+b^2-c^2 \\ AB = bc \end{cases} \tag{2.20}$$

Eliminating *A* and *B* from (2.20), we readily see that

$$\begin{cases} 2A^2 = ka^2+b^2-c^2+\sqrt{\left(ka^2+b^2-c^2\right)^2+4b^2c^2} \\ 2B^2 = -\left(ka^2+b^2-c^2\right)+\sqrt{\left(ka^2+b^2-c^2\right)^2+4b^2c^2} \end{cases} \tag{2.21}$$

Now suppose $a>0,\ c>0$ then from (2.17) both $A>0, B>0$. If we let $b\to 0$ then, if $ka>c$, then from (2.17) it is obvious that $A\to\sqrt{ka^2-c^2}, B\to 0$ but if $ka<c$, $A\to 0, B\to\sqrt{c^2-ka^2}$ Therefore from (2.17) and (2.18), we conclude that integral (2.13) and (2.14) hold.

Special Case 2.4: If we let $a=0$ in (2.17), we get $A=b$ and $B=c$ then from (2.18) and (2.19), we deduce, for $k=1$, the following well known integrals [5]

$$\int_0^\infty e^{-bx}\cos(cx)dx = \frac{b}{b^2+c^2}$$
$$\int_0^\infty e^{-bx}\sin(cx)dx = \frac{c}{b^2+c^2} \tag{2.22}$$

Theorem 2.5: Show that for *A* defined as in (2.21),

$$\int_0^\infty J_0^k(ax)e^{-bx}\frac{\sin(cx)}{x}dx = \sqrt{k}\tan^{-1}\left(\frac{c}{A}\right) \tag{2.23}$$

Proof: Integrating (2.18) with respect to c and noting that the integrand is uniformly convergent so we can integrate under the integral sign, we get

$$\int_0^\infty J_0^k(ax)e^{-bx}\frac{\sin(cx)}{x}dx = \sqrt{k}\int_0^c \frac{A}{A^2+B^2}dc \tag{2.24}$$

Now taking differentials of (2.20),

$$\begin{cases} AdA - BdB + cdc = 0 \\ BdA + AdB - bdc = 0 \end{cases} \tag{2.25}$$

Solving the system of linear equations in *dA* and *dB* in (2.25), and employing second equation of (2.20), we get

$$\frac{dA}{(bB-cA)dc} = \frac{1}{A^2+B^2} \Rightarrow \frac{Adc-cdA}{A\left(A^2+c^2\right)+(AB)B-bcB} = \frac{Adc-cdA}{A\left(A^2+c^2\right)} \tag{2.26}$$

So that

$$\int_0^c \frac{A}{A^2+B^2}dc = \int_0^c \frac{Adc-cdA}{A^2+c^2}dc = \int_0^c d\tan^{-1}\left(\frac{c}{A}\right)dc = \tan^{-1}\left(\frac{c}{A}\right) \tag{2.27}$$

Hence using (2.27) into (2.24) we get (2.23) as claimed.

Special case 2.6: If we let $a=0$ in (2.23), we deduce, for $k=1$, the following well known integrals [5]

$$\int_0^\infty e^{-bx}\frac{\sin(cx)}{x}dx = \tan^{-1}\left(\frac{c}{b}\right)$$

Corollary 2.7: Suppose $a>0,\ c>0$ in (2.23). If we let $b\to 0$ then, if $ka>c,$ then from (2.17) it is obvious that $A\to\sqrt{ka^2-c^2}, \tan^{-1}\left(\frac{c}{A}\right)\to\tan^{-1}\left(\frac{c}{\sqrt{ka^2-c^2}}\right)=\sin^{-1}\left(\frac{c}{\sqrt{k}a}\right)$. On the other hand if $ka<c,\ A\to 0, \tan^{-1}\left(\frac{c}{A}\right)\to\frac{\pi}{2}$. Therefore from (2.23), we conclude that

$$\int_0^\infty J_0^k(ax)\frac{\sin(cx)}{x}dx=\begin{cases}\dfrac{\pi}{2} & ka<c\\ \sin^{-1}\left(\dfrac{c}{\sqrt{k}a}\right) & ka>c\end{cases}\tag{2.28}$$

Remark 2.8: If $c=0$, then the integral (2.28) vanishes. If $c<0$, then the integral (2.28) is an odd function of c. In case $a=0$ then the integral (2.28) turns to be

$$\begin{aligned}&\int_0^\infty \frac{\sin(cx)}{x}dx=\frac{\pi}{2} \qquad (c>0)\\ &\int_0^\infty \frac{\sin(cx)}{x}dx=-\frac{\pi}{2} \qquad (c<0)\end{aligned}\tag{2.29}$$

3. LAPLACE TRANSFORM OF *k* BESSEL FUNCTION

In this section, we will derive the Laplace transform of *k* Bessel function. Recall that the Laplace transform of a piece wise continuous function is denoted and defined as

$$L(f(x))=\int_0^\infty e^{-sx}f(x)dx,\ \mathrm{Re}(s)>0\tag{3.1}$$

Before we derive the Laplace transform of *k* Bessel function, we need to establish integral representation of *k* Bessel function as below:

Theorem 3.1: For $k>0, n\in\mathbb{Z}_{\geq 0}$ the following integral holds

$$J_{nk}^k(x)=\frac{1}{2\pi k^{\frac{n+1}{2}}}\int_0^{2\pi} e^{i\left[\frac{x}{\sqrt{k}}\sin\left(\frac{\theta}{\sqrt{k}}\right)-\frac{\theta n}{\sqrt{k}}\right]}d\theta\tag{3.2}$$

Proof: From complex analysis, due to the residue theorem we know that if a function $f(z)$ defined on the complex plane has the Laurent Series representation

$$f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n\tag{3.3}$$

and if C is a contour with positive orientation in the complex plane containing the point z_0 then

$$\oint_C f(z)dz = 2\pi i a_{-1} \tag{3.4}$$

where a_{-1} is called the residue of $f(z)$ at z_0. Now dividing (2.1) by z^{n+1} and then taking integral over C

$$\int_C \frac{e^{\frac{x}{2\sqrt{k}}\left(\frac{z}{\sqrt{k}}-\frac{\sqrt{k}}{z}\right)}}{z^{n+1}}dz = \int_C \sum_{m=-\infty}^{\infty} J_{mk}^{k}(x) z^{m-n-1}dz$$

Using (3.4), we can write

$$\int_C \frac{e^{\frac{x}{2\sqrt{k}}\left(\frac{z}{\sqrt{k}}-\frac{\sqrt{k}}{z}\right)}}{z^{n+1}}dz = 2\pi i J_{nk}^{k}(x) \tag{3.5}$$

Letting $z=\sqrt{k}e^{\frac{\theta}{\sqrt{k}}i}$ in the LHS of (3.5), we get

$$J_{nk}^{k}(x) = \frac{1}{2\pi}\int_C e^{\frac{ix}{\sqrt{k}}\sin\left(\frac{\theta}{\sqrt{k}}\right)+\frac{i\theta}{\sqrt{k}}}\left(\sqrt{k}e^{\frac{i\theta}{\sqrt{k}}}\right)^{-n-1} d\theta \tag{3.6}$$

On simplifying we get the (3.2) and this completes the proof.

Corollary 3.2: For $k>0, n\in\mathbb{Z}_{\geq 0}$

$$\left|J_{nk}^{k}(x)\right| \leq k^{-\frac{n+1}{2}} \; x\in\mathbb{R} \tag{3.7}$$

From (3.2), we see that

$$\left|J_{nk}^{k}(x)\right| = \frac{1}{2\pi k^{\frac{n+1}{2}}}\left|\int_0^{2\pi} e^{i\left[\frac{x}{\sqrt{k}}\sin\left(\frac{\theta}{\sqrt{k}}\right)-\frac{\theta n}{\sqrt{k}}\right]}d\theta\right| \leq \frac{1}{2\pi k^{\frac{n+1}{2}}}\int_0^{2\pi}\left|e^{i\left[\frac{x}{\sqrt{k}}\sin\left(\frac{\theta}{\sqrt{k}}\right)-\frac{\theta n}{\sqrt{k}}\right]}\right|d\theta = \frac{1}{2\pi k^{\frac{n+1}{2}}}.2\pi = k^{-\frac{n+1}{2}}$$

Theorem 3.3: For $k>0, n\in\mathbb{Z}_{\geq 0}$ the Laplace transform of *k* Bessel function is given by

$$L\left(J_{nk}^{k}(x)\right) = \frac{k\left[\sqrt{s^2k^2+k}-sk\right]^n}{\sqrt{s^2k^2+k}} \tag{3.8}$$

Proof: Taking Laplace of (3.2), we get

$$L\left(J_{nk}^{k}(x)\right)=\frac{1}{2\pi k^{\frac{n+1}{2}}}\int_{0}^{\infty}e^{-sx}\int_{0}^{2\pi}e^{i\left[\frac{x}{\sqrt{k}}\sin\left(\frac{\theta}{\sqrt{k}}\right)-\frac{\theta n}{\sqrt{k}}\right]}d\theta dx \tag{3.9}$$

Rearranging the integrals we get

$$L\left(J_{nk}^{k}(x)\right)=\frac{1}{2\pi k^{\frac{n+1}{2}}}\int_{0}^{2\pi}e^{-\frac{i\theta n}{\sqrt{k}}}\left(\int_{0}^{\infty}e^{-x\left[s-\frac{i}{\sqrt{k}}\sin\left(\frac{\theta}{\sqrt{k}}\right)\right]}dx\right)d\theta \tag{3.10}$$

Evaluating the inner integral

$$L\left(J_{nk}^{k}(x)\right)=\frac{1}{2\pi k^{\frac{n+1}{2}}}\int_{0}^{2\pi}\frac{e^{-\frac{i\theta n}{\sqrt{k}}}}{s-\frac{i}{\sqrt{k}}\sin\left(\frac{\theta}{\sqrt{k}}\right)}d\theta \tag{3.11}$$

Taking unit circle as our contour so that $z=\sqrt{k}e^{-\frac{\theta}{\sqrt{k}}i}$ then the RHS of (3.11) takes the form

$$L\left(J_{nk}^{k}(x)\right)=\frac{1}{2\pi\sqrt{k}}\int_{|z|=1}\frac{z^{n}}{s+\frac{1}{2\sqrt{k}}\left(\frac{z}{\sqrt{k}}-\frac{\sqrt{k}}{z}\right)}\left(-\frac{i\sqrt{k}}{z}dz\right) \tag{3.12}$$

On simplifying we get

$$L\left(J_{nk}^{k}(x)\right)=-\frac{ik}{\pi}\int_{|z|=1}\frac{z^{n}}{z^{2}+2ksz-k}dz \tag{3.13}$$

The integrand has simple poles at $z_1=-sk+\sqrt{s^2k^2+k}$ and $z_2=-sk-\sqrt{s^2k^2+k}$ and for $s,k>0$ only z_1 lies inside of the unit circle $|z|=1$, so by residue theorem we can write

$$\begin{aligned}L\left(J_{nk}^{k}(x)\right)&=\frac{-ik}{\pi}2\pi i\operatorname{Res}\left[\frac{z^{n}}{(z-z_1)(z-z_2)},z=z_1\right]\\&=2k\left[\lim_{z\to z_1}\frac{-ik}{\pi}\frac{(z-z_1)z^{n}}{(z-z_1)(z-z_2)}\right]=2k\left[\frac{z_1{}^{n}}{z_1-z_2}\right]\end{aligned} \tag{3.14}$$

Substituting the values we get (3.8).

Remark 3.4: For $k=1$ we get the Laplace transform of classical Bessel functions [6]

$$L\left(J_n(x)\right)=\frac{\left[\sqrt{s^2+1}-s\right]^n}{\sqrt{s^2+1}} \tag{3.15}$$

Proposition 3.5: Show that

$$\int_0^\infty J_k^k(x)\,dx=k \tag{3.16}$$

Substituting $n=1$ in (3.8), and then

$$\int_0^\infty e^{-sx}J_k^k(x)\,dx=\frac{k\left[\sqrt{s^2k^2+k}-sk\right]}{\sqrt{s^2k^2+k}} \qquad \text{n(3.17)}$$

letting $s\to 0$ we get (3.16)

Corollary 3.6: Substituting $n=\pi$ in (3.17), we get nice representation of π

$$\int_0^\infty J_\pi^\pi(x)\,dx=\pi \tag{3.18}$$

Proposition 3.7: Show that

$$\int_0^\infty J_1^k(x)dx=k^{\frac{1}{2}\left(1+\frac{1}{k}\right)} \tag{3.19}$$

Substituting $n=\frac{1}{k}$ in (3.8), and then letting $s\to 0$ we get (3.19)

Proposition 3.8: Show that

$$\lim_{x\to\infty}J_0^k(x)=1-k \tag{3.20}$$

From (1.9) for $v=0$ we get $\frac{d}{dx}J_0^k(x)=-J_k^k(x)$. From (3.17) we have

$$\int_0^\infty J_k^k(x)\,dx=-\int_0^\infty\frac{d}{dx}J_0^k(x)\,dx=-\lim_{x\to\infty}J_0^k(x)+J_0^k(0)=k$$

Using (1.8) we get (3.20)

Proposition 3.9: Show that

$$\lim_{x\to\infty} \dot{J}_0^k(x) = 0 \tag{3.21}$$

where dot represents the differentiation with respect to *x*.

Considering $L\left(\dot{J}_k^k(x)\right) = sL\left(J_k^k(x)\right) - J_k^k(0)$ and using (3.8)

$$\int_0^\infty e^{-sx}\,\dot{J}_k^k(x)\,dt = s\left[\frac{k\left[\sqrt{s^2k^2+k}-sk\right]}{\sqrt{s^2k^2+k}}\right] = s\left[k - \frac{sk^2}{\sqrt{s^2k^2+k}}\right]$$

Letting $s \to 0 \int_0^\infty \dot{J}_k^k(x)\,dx = \lim_{x\to\infty} J_k^k(x) - J_k^k(0) \Rightarrow \lim_{x\to\infty} J_k^k(x) = 0 \Rightarrow \lim_{x\to\infty} \dot{J}_0^k(x) = 0$

Proposition 3.10: Show that

$$\int_0^\infty \frac{\dot{J}_k^k(x)}{x} = \frac{1}{\sqrt{k}} \tag{3.22}$$

Since $J_0^k(x)$ is the solution of zero order *k* Bessel differential equation $y'' + \frac{1}{x}y' + \frac{1}{k}y = 0$ then we can write

$$\ddot{J}_0^k(x) + \frac{1}{x}\dot{J}_0^k(x) + \frac{1}{k}J_0^k = 0 \tag{3.23}$$

Now integrating (3.22) with respect to *x* from 0 to ∞ and using (2.12) and (3.21)

$$\lim_{x\to\infty}\left(\dot{J}_0^k(x)\right) + \int_0^\infty \frac{1}{x}\dot{J}_0^k(x)\,dx + \frac{1}{k}\left(\sqrt{k}\right) = 0$$

$$0 + \int_0^\infty \frac{1}{x}\dot{J}_0^k(x)\,dx + \frac{1}{\sqrt{k}} = 0$$

By using the relation $\frac{d}{dx}J_0^k(x) = -J_k^k(x)$ we get the desired result.

Theorem 3.11: If u, v, w denote the integrals

$$u = \int_0^\infty \frac{xJ_0^k}{k\sqrt{a^2+x^2}}dx, \quad v = \int_0^\infty \frac{\dot{J}_0^k(x)}{\sqrt{a^2+x^2}}dx, \quad w = \int_0^\infty \frac{x\ddot{J}_0^k(x)}{\sqrt{a^2+x^2}}dx \tag{3.24}$$

then the following equations hold.

a) $u + v + w = 0$ (3.25a)

b) $\frac{du}{da} + \frac{av}{k} + \frac{1}{k} = 0$ (3.25b)

c) $a\frac{dv}{da} - w = 0$ (3.25c)

Proof: The equation (3.25a) is obvious from the combination of (3.23) and (3.24). For (3.25b), differentiate the integral of *u* from (3.24) with respect to *a*, and employing Leibniz rule of integration under integral sign, we get

$$\frac{du}{da} = \int_0^\infty \frac{\partial}{\partial a}\frac{xJ_0^k(x)}{k\sqrt{a^2+x^2}} = -\frac{a}{k}\int_0^\infty J_0^k(x)\left(a^2+x^2\right)^{-\frac{3}{2}} xdx$$

Integrating by parts, we get

$$\frac{du}{da} = -\frac{a}{2k}\left[-2\left.\frac{J_0^k(x)}{\left(a^2+x^2\right)^{\frac{1}{2}}}\right|_0^\infty + 2\int_0^\infty \frac{\dot{J}_0^k(x)}{\left(a^2+x^2\right)^{\frac{1}{2}}}dx\right] = -\frac{1}{k} - \frac{a}{k}\int_0^\infty \frac{\dot{J}_0^k(x)}{\sqrt{a^2+x^2}}dx$$

Then by using the definition (3.24), we get (3.25b). To establish (3.25c), we observe that

$$a\frac{dv}{da} = -a^2\int_0^\infty \dot{J}_0^k(x)\left(a^2+x^2\right)^{-\frac{3}{2}}dx \tag{3.26}$$

Now integrating by parts the integral of *w* in (3.24), we get

$$w = \left|\frac{x\dot{J}_0^k(x)}{\sqrt{a^2+x^2}}\right|_0^\infty - \int_0^\infty \dot{J}_0^k(x)\left(-x^2\left(a^2+x^2\right)^{-\frac{3}{2}} + \left(a^2+x^2\right)^{-\frac{1}{2}}\right)dx$$

Since $\frac{x\dot{J}_0^k(x)}{\sqrt{a^2+x^2}} \to 0$ as $x \to \infty$, we conclude that

$$w = -a^2 \int_0^\infty \dot{J}_0^k(x)\left(a^2+x^2\right)^{-\frac{3}{2}} dx \tag{3.27}$$

Now from (3.26) and (3.27), we get (3.25c).

Corollary 3.12: Establish the integrals for $a > 0$

$$\int_0^\infty \frac{xJ_0^k}{k\sqrt{a^2+x^2}} dx, = -\frac{(1-k)^2}{2\sqrt{k}} e^{\frac{1}{\sqrt{k}}a} + \frac{1+k^2}{2\sqrt{k}} e^{-\frac{1}{\sqrt{k}}a} \tag{3.28}$$

$$\int_0^\infty \frac{\dot{J}_0^k(x)}{\sqrt{a^2+x^2}} dx = \frac{1}{2a}\left(\left(1-k\right)^2 e^{\frac{1}{\sqrt{k}}a} + \left(1+k^2\right)e^{-\frac{1}{\sqrt{k}}a} - 2\right) \tag{3.29}$$

Differentiating (3.25b), with respect to *a* and using (3.25a), we get second order homogeneous linear differential equation

$$\frac{d^2u}{da^2} - \frac{1}{k}u = 0 \tag{3.30}$$

Letting $a \to 0$ into the *u* integral and in (3.25b), we get readily the initial conditions

$$u(0) = k^{\frac{3}{2}}, \ \frac{du(0)}{dx} = -\frac{1}{k} \tag{3.31}$$

Solving (3.30) together with initial conditions (3.31), we get the solution

$$u(a) = -\frac{(1-k)^2}{2\sqrt{k}} e^{\frac{1}{\sqrt{k}}a} + \frac{1+k^2}{2\sqrt{k}} e^{-\frac{1}{\sqrt{k}}a} \tag{3.32}$$

Differentiating (3.32) with respect to *a* and plugging the value into (3.25b), we get

$$\frac{(1-k)^2}{2} e^{\frac{1}{\sqrt{k}}a} + \frac{1+k^2}{2} e^{-\frac{1}{\sqrt{k}}a} = av + 1$$

or

$$v = \frac{1}{2a}\left(\left(1-k\right)^2 e^{\frac{1}{\sqrt{k}}a} + \left(1+k^2\right)e^{-\frac{1}{\sqrt{k}}a} - 2\right)$$

Now by the definitions of *u* and *v* from (3.24), we get the integrals (3.28) and (3.29).

4. CONCLUSION

We established certain new definite integrals of *k* Bessel function of first kind. We also derived the Laplace transform of *k* Bessel function and used it to inverstigate some new propertoes of *k* Bessel function of first kind. We found that all these definite integrals do dollow the classical Bessel function as special cases.

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