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STABILITY AND HOPF BIFURCATION ANALYSIS OF DELAY PREY-PREDATOR MODEL

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Abstract. *In this study, an analysis of the dynamic behavior of the delay prey-predator population model is presented. The delay parameter is selected as the value of the bifurcation; and the bifurcation value of the coexistence equilibrium point of the system is calculated.*

Keywords: *Dynamic system, Equilibrium point, Hopf bifurcation, Stability*

1. INTRODUCTION

Lotka-Volterra model is one of the first continuous prey-predator population models, it is frequently used in the dynamic studies of population models [1, 2]. To have a deep knowledge of the dynamics of population models requires stability and bifurcation analysis of the system. To answer the question of “How does the system's dynamics change when a parameter of the system changes?” is important. Hopf bifurcation analysis is related to the investigation of the change in the topological structure of the system together with the changing parameter in continuous systems [3-5]. Most authors have tried the stability and Hopf-Bifurcations analysis of population models with delay [6-14]. Because one factor in creating more realistic models in population dynamics is that the population includes the past status. The time mentioned as the delay time; refers to the fact that the rate of change of the population depends not only on the current population but also on the past population.

2. METHODS

Let's consider a modified Lotka –Volterra model with delay effect as follows:

$$\begin{aligned}\frac{dx(t)}{dt} &= ax(t)(1-x(t)) - x(t-\tau)y(t-\tau) \\ \frac{dy(t)}{dt} &= \frac{1}{b}x(t)y(t) - ey(t)\end{aligned} \tag{1}$$

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where all parameters are positive, a; the maximum growth rate of the prey population, $\frac{1}{b}$; the maximum growth rate of the predator population, e is the death rate of the predator when there is no prey. $x(t)$ is the prey population in time t, $y(t)$ is the predator population at time t. τ is the delay parameter.

If the equilibrium point definition is considered [15], then we get the equilibrium point $(0,0)$, $(1,0)$ *ve* $(be, \mathrm{a}-\mathrm{abe})$ of system (1). We will do our analysis for the positive equilibrium point $\mathrm{E}_2 = (be, \mathrm{a}-\mathrm{abe})$ where the prey and predator population coexists.

With the linearization of the system at a point (x, y), we reach

$$\begin{aligned} \frac{du}{dt} &= (a-2ax)u - yu(t-\tau) - x\vartheta(t-\tau) \\ \frac{d\vartheta}{dt} &= (\frac{1}{b}y)u + (\frac{1}{b}x - e)\vartheta. \end{aligned} \tag{2}$$

If the system (2) is written in a different form, then we get as follows:

$$\begin{bmatrix} \frac{du}{dt} \\ \frac{d\vartheta}{dt} \end{bmatrix} = \begin{bmatrix} a-2ax & 0 \\ \frac{1}{b}y & \frac{1}{b}x-e \end{bmatrix} \begin{bmatrix} u \\ \vartheta \end{bmatrix} + \begin{bmatrix} -y & -x \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u(t-\tau) \\ \vartheta(t-\tau) \end{bmatrix}. \tag{3}$$

From

$$\det \begin{bmatrix} a-2ax-ye^{-\tau\lambda}-\lambda & -xe^{-\tau\lambda} \\ \frac{1}{b}y & \frac{1}{b}x-e-\lambda \end{bmatrix} = 0 \tag{4}$$

we have the following the characteristic equation of the system (3):

$$\lambda^2 + A_1\lambda + A_2 + e^{-\lambda\tau}(B_1\lambda + B_2) = 0 \tag{5}$$

where the coefficient

$$\begin{aligned} A_1 &= -a + 2ax + e - \frac{x}{b} \\ A_2 &= -ae + 2axe + \frac{ax}{b} - \frac{2ax^2}{b} \\ B_1 &= y \\ B_2 &= ey \end{aligned} \tag{6}$$

Theorem 1: In case of $\tau = 0$ and $A_1 = 2abe - a$, $A_2 = 0$, $B_1 = a - abe$, $B_2 = e(a - abe)$, if $A_1 + B_1 > 0$ and $A_2 + B_2 > 0$, then the equilibrium point E_2 is locally asymptotic stable.

Proof: If the state $\tau = 0$ in the equation (5) is considered, then we get that

$$\lambda^2 + (A_1 + B_1)\lambda + A_2 + B_2 = 0 \tag{7}$$

such that

$$\begin{aligned} A_1 &= 2abe - a \\ A_2 &= 0 \\ B_1 &= a - abe \\ B_2 &= e(a - abe) \end{aligned} \tag{8}$$

are the coefficients evaluated at the equilibrium point $E_2 = (be,\ a - abe)$. Also, if certain coefficient totals are calculated, then we have

$$\begin{aligned} A_1 + B_1 &= abe \\ A_2 + B_2 &= ea(1 - be) \end{aligned} \tag{9}$$

From Routh-Hurwitz criterion (see [16-18]), if $A_1 + B_1 > 0$ and $A_2 + B_2 > 0$, then all the roots of the equation (7) are complex conjugate or negative real numbers with negative real part. So, E_2 is locally asymptotic stable.

Teorem 2: In case of $\tau > 0$ and $A_1 = 2abe - a$, $A_2 = 0$, $B_1 = a - abe$, $B_2 = e(a - abe)$, if $A_1^2 - 2A_2 - B_1^2 > 0$ *ve* $A_2^2 - B_2^2 > 0$, then E_2 is locally asymptotic stable.

Proof: Let be $\tau > 0$ and $A_1 = 2abe - a$, $A_2 = 0$, $B_1 = a - abe$, $B_2 = e(a - abe)$. In (7), for $\lambda = iw$, we get

$$-w^2 + A_1 wi + A_2 + (B_1 wi + B_2)[\text{coswt} - \text{isinwt}] = 0. \tag{10}$$

By seperating the real and imaginary parts, we can write

$$\begin{aligned} w^2 - A_2 &= B_1 w \sin wt + B_2 w \cos wt \\ -A_1 w &= -B_2 \sin wt + B_1 w \cos wt \end{aligned} \tag{11}$$

and it is equivalent to

$$w^4 + (A_1^2 - 2A_2 - B_1^2)w^2 + (\text{A}_2^2 - B_2^2) = 0 \tag{12}$$

If $A_1^2 - 2A_2 - B_1^2 > 0$ and $\text{A}_2^2 - B_2^2 > 0$, then the equation (7) can not have a purely imaginary root iw. Therefore, the real part of all the eigenvalues of the equation (7) is negative for all $\tau > 0$. Consequently, E_2 is locally asymptotic stable.

Remark 1. For all $\tau \geq 0$ and values given in (8), if $A_1 + B_1 > 0,\ A_2 + B_2 > 0$, $A_1^2 - 2A_2 - B_1^2 > 0$ and $A_2^2 - B_2^2 > 0$, then E_2 is locally asymptotic stable.

Remark 2. For all $\tau \geq 0$ and values given in (8), if any one of the expressions $A_1^2 - 2A_2 - B_1^2$ or $A_2^2 - B_2^2$ are negative such that $A_1 + B_1 > 0,\ A_2 + B_2 > 0$, then there is a single pair of purely imaginary root $\mp i\, w_0$, for the equation (7).

Remark 3. τ_k corresponding to w_0 can be found as follows:

$$\tau_k = \frac{1}{w_0}\arccos\left[\frac{(B_2 - A_1B_1)w^2 - A_2B_2}{B_1^2w^2 + B_2^2}\right] + \frac{2k\pi}{w_0} \quad , \quad k = 0,1,2,\ ... \tag{13}$$

from (11).

Teorem 3. If $A_1^2 - 2A_2 - B_1^2 > 0$ and $A_2^2 - B_2^2 < 0$ such that $A_1 + B_1 > 0,\ A_2 + B_2 > 0$, then Hopf bifurcation is available at $\tau = \tau_0$.

Ispat: It is possible to calculate the following the value for $\tau = 0$

$$\tau_0 = \frac{1}{w_0}\arccos\left[\frac{(B_2 - A_1B_1)w^2 - A_2B_2}{B_1^2w^2 + B_2^2}\right]$$

from (13). The E_2 is local asymptotic stable since $A_1 + B_1 > 0,\ A_2 + B_2 > 0$ is valid. If $\left.\frac{d\,\mathrm{Re}(\lambda)}{d\tau}\right|_{\lambda = i w_0} > 0$, then E_2 will remain stable for $\tau < \tau_0$. Differentiating (5) according to τ, we get

$$\frac{d\lambda}{d\tau}\left[2\lambda + A_1 + B_1e^{-\lambda\tau} - (B_1\lambda + B_2)\tau e^{-\lambda\tau}\right] = \lambda(B_1\lambda + B_2)e^{-\lambda\tau}. \tag{14}$$

With the necessary arrangements, we have

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda + A_1 + B_1e^{-\lambda\tau} - (B_1\lambda + B_2)\tau e^{-\lambda\tau}}{\lambda(B_1\lambda + B_2)e^{-\lambda\tau}}$$

$$\Rightarrow \left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda + A_1}{\lambda(B_1\lambda + B_2)e^{-\lambda\tau}} + \frac{B_1}{\lambda(B_1\lambda + B_2)} - \frac{\tau}{\lambda}. \tag{15}$$

By using $\left.\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right|_{\lambda = i w_0} = \left.\frac{d\,\mathrm{Re}(\lambda)}{d\tau}\right|_{\lambda = i w_0}$ with $e^{-\lambda\tau}(B_1\lambda + B_2) = -(\lambda^2 + A_1\lambda + A_2)$ from (5), we write

$$\frac{d\,\mathrm{Re}(\lambda)}{d\tau} = \mathrm{Re}\left[\frac{2i\,w_0 + A_1}{-i\,w_0(-w_0^2 + A_1\,i\,w_0 + A_2)} + \frac{B_1}{iw_0(B_1\,i\,w_0 + B_2)} - \frac{\tau}{i\,w_0}\right]$$

$$= \mathrm{Re}\left[\frac{1}{w_0}\left(\frac{2i\,w_0 + A_1}{A_1\,w_0 + (w_0^2 - A_2)\,i} + \frac{B_1}{(-B_1\,w_0^2 + B_2\,i)} + \tau i\right)\right]$$

$$= \frac{1}{w_0}\left(\frac{2w_0(w_0^2 - A_2) + A_1^2\,w_0}{A_1^2\,w_0 + (w_0^2 - A_2)^2} - \frac{B_1^{\,2}w_0}{B_1^2w_0^2 + B_1^2}\right)$$

$$= \frac{2w_0^2 + A_1^2 - 2A_2 - B_1^2}{B_1^2w_0^2 + B_2^2}$$

From there, we have $\frac{d\,\mathrm{Re}(\lambda)}{d\tau}\Big|_{\lambda = i w_0} > 0$ (transversality condition), since $A_1^2 - 2A_2 - B_1^2 > 0$. Consequently, Hopf bifurcation occurs with transversality condition at $w = w_0$, $\tau = \tau_0$.

3. RESULTS AND DISCUSSION

Now, we can give the following example to support the theoretical results obtained.

Example 1: Let's consider the delay prey-predator system as follows:

$$\frac{dx}{dt} = 0.7x(1-x) - x(t-\tau)y(t-\tau)$$
$$\frac{dy}{dt} = 0.667xy - 0.5y$$

with parameter $e = 0,5$, $b = 1.5$ and $a = 0,7$. In this continuous population model, let's find the positive coexistence equilibrium point and the bifurcation value τ_0.

Solution: $(0.75, 0.175)$ is the coexistence equilibrium point of the system. We get

$$\begin{aligned} A_1 + B_1 &= abe \\ &= 0,525 \\ A_2 + B_2 &= e(a - abe) \\ &= 0,0875 \end{aligned}$$

and

$$A_1^2 - 2A_2 - B_1^2 = 0,091875 > 0$$
$$A_2^2 - B_2^2 = -0,00765625 < 0$$

from (8) and (9). Also from (12), we have

$$w^4 + 0,091875\,w^2 - 0,00765625 = 0.$$

From there, if the roots are calculated, then we get

$$w_{1+} = 0,22997,\ w_{2-} = -0,22997 \quad i\,w_{1+} = 0,38048i \text{ and } i\,w_{1-} = -0,38048i$$

Here, we calculate as

$$B_2 - A_1B_1 = 0,02625, \quad A_2B_2 = 0, \quad B_1^2 = 0,030625, \quad B_2^2 = 0,00765625$$

and from (13), we get for $k = 0$,

$$\tau_0 = \frac{1}{0,38048} \arccos\left[\frac{(0,02625)(0,38048)^2}{(0,030625)(0,38048)^2 + 0,00765625}\right]$$

$$\Rightarrow \tau_0 = \frac{1}{0,38048} \arccos(0.314324475)$$

$$\Rightarrow \tau_0 = \frac{1.25105}{0,38048} = 3,288083474$$

So, we say that E_2 locally asymptotic stability at $\tau \in [0\ ,\ 3.288083474)$ and Hopf bifurcation occurs $\tau = \tau_0 = 3,288083474$.

4. CONCLUSION

In this study, the hopf bifurcation analysis of the coexistence equilibrium point of a delay prey-predator model is presented. We conclude that the behavior the coexistence equilibrium point of the system (1) varies from stability to instability at $\tau = \tau_0 = 3,288083474$.

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