ORIGINAL PAPER

k-FRACTIONAL INTEGRAL INEQUALITIES OF HADAMARD TYPE FOR DIFFERENTIABLE HARMONICALLY CONVEX FUNCTIONS

RASHIDA HUSSAIN[1], ASGHAR ALI[1], YASMIN BASHIR[1], ASIA LATIF[1]

Manuscript received: 29.02.2020. Accepted paper: 01.04.2020;
Published online: 30.06.2020.

Abstract. *Present article takes into account results on Hadamard type inequality. It intents to generalize some of the existing fractional integral inequalities of Hadamard type inequality for harmonically convex functions to corresponding k-fractional inequalities, by using the classic k-fractional integrals. For this purpose first a significant Hadamard type inequality for harmonically convex functions via k-fractional integrals is procured. Then the differences of middle term with right and left terms of the aforesaid inequalities are estimated.*

Keywords: *Hadamard inequality, harmonically convex function, k-hypergeometric function, k-fractional integrals, power mean inequality, Hölder's inequality.*

1. INTRODUCTION

The realm of convexity is very vast with a rich past, adorable present and we foresee a bright future of it. Extensive literature including number of books and a lots of research papers, are available on inequalities. Convexity blended with other mathematical concepts give rise to many important results, hence making this field more voluminous for further research, see [1-8] and the references therein. The present paper is concern with harmonically convex functions, introduced by İşcan in 2013 [9]. Since its debut, it has grabbed attention of many researchers, as a result many new articles are dedicated to this concept see [10-15].

September 30, 1695 is considered to be the date of birth of fractional calculus. Fractional calculus bridges the gap present in integral calculus, by considering fractional orders of differentiation and integration, consequently, making Mathematics more close to physical reality. For more details see [16-22] and the references therein.

Present article is alliance of concepts of harmonically convex function and k - Riemann-Liouville fractional integrals, in the form of Hadamard's type inequalities. We start with a few basic definitions and results which are to be used in the development of main results of this paper.

The subsequent class of convex functions namely harmonically convex functions followed by the corresponding Hadamard's type inequality is introduced by İşcan [9]:

Definition 1: *A function* $f : I \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ *is said to be harmonically convex if the following inequality*

$$f\left(\frac{ab}{ta+(1-t)b}\right) \leq tf(b) + (1-t)f(a) \tag{1.1}$$

[1] Mirpur University of Science and Technology (MUST), 10250 Mirpur, Pakistan.
E-mail: rashida12@gmail.com; drali.must@gmail.com.

holds for all $a, b \in I$ *and* $t \in [0,1]$.

Theorem 1: *Let* $f: I \subseteq \mathbb{R}\backslash\{0\} \rightarrow \mathbb{R}$ *be a harmonically convex function and* $a, b \epsilon I$ *with* $a < b$. *If* $f \in L[a,b]$, *then the following inequality holds*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(t)}{t^2}dt \leq \frac{f(a)+f(b)}{2} \tag{1.2}$$

Now we recaptulate Pochmer k-symbol, k-gamma function and k-betta function with $k > 0$ described in [23], by Diaz and Pariguan. We also mention here an integral represention of k-hypergeometric function given by Mubeen and Habibullah [23].

i. Pochmer k-symbol

$$(x)_{n,k} = x(x+k)(x+2k)\ldots(x+(n-1)k).$$

ii. k-gamma Function

$$\Gamma_k(x) = \int_0^\infty t^{x-1} \exp\left(-\frac{t^k}{k}\right) dt, \; x > 0.$$

Moreover $\Gamma_k(x+k) = x\Gamma_k(x)$, $(x)_{n,k} = \frac{\Gamma_k(x+n\,k)}{\Gamma_k(x)}$.

iii. k-Beta Function

$$\beta_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)} = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt, \quad x,y > 0.$$

iv. The k-hyper geometric function

$${}_2^kF_1(a,b;c;x) = \frac{1}{k\,\beta_k(b,\ c-b)}\int_0^1 t^{\frac{b}{k}-1}(1-t)^{\frac{c-b}{k}-1}(1-kxt)^{-\frac{a}{k}}dt$$

where $c > b > 0$, $|x| < 1$.

Mubeen and Habibullah [24] introduced an important generalization of Riemann-Liouville fractional integrals called k-Riemann-Liouville fractional integrals where $k > 0$, also see [11, 22].

Definition 2: *Let* $f \in L[a,b]$. *The k-Riemann-Liouville integrals* ${}_kJ_{a+}^{\alpha}f$ *and* ${}_kJ_{b-}^{\alpha}f$ *of order* $\alpha > 0$, *with* $a \geq 0$ *and* $k > 0$ *are defined by:*

$${}_kJ_{a+}^{\alpha}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_a^x (x-t)^{\frac{\alpha}{k}-1}f(t)dt, \quad x > a$$

and

$${}_kJ_{b-}^{\alpha}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_x^b (t-x)^{\frac{\alpha}{k}-1}f(t)dt, \quad x < b,$$

respectively, where $\Gamma_k(.)$ *is the k-gamma function.*

One must note that as $k \rightarrow 1$, (i)-(iv) and Definition 2 turn out to be classical pochmer symbol, gamma function, betta function, hypergeometric function and Riemann-Liouville fractional integrals respectively. For further detail of k-fractional calculus (see [11, 19-21, 23-24]).

Lemma 1: (see [25-26]) *For* $0 < \alpha \leq 1$ *and* $0 \leq x < y$, *we have*

$$|x^{\alpha} - y^{\alpha}| \leq (y-x)^{\alpha}.$$

2. RESULTS AND DISCUSSION

We start this section with the proof of a Hadamard like inequality for harmonically convex function by using k-fractional integrals. The rest of main results comprises of inequalities based on right and left hand side of this inequality. Interesting readers are recommended to also see Hussain et al. [11, 20-22], in which authors have established Hadamard like inequalities for various convex functions by way of the same k-fractional integrals. In the suquel f is taken as real valued function on interval $I \subseteq R^{+}$, where R^{+} is set of all positive real numbers.

Theorem 2: *Let* $f \in L[a,b]$, *where* $a,b \in I$ *with* $a < b$. *If* f *is a harmonically convex function on* $[a,b]$, *then for k-fractional integrals subsequent inequality is satisfied*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{\Gamma_k(\alpha+k)}{2}\left(\frac{ab}{b-a}\right)^{\alpha/k} \left\{ {}_kJ^{\alpha}_{\frac{1}{a}-}(f\circ g)\left(\frac{1}{b}\right) + {}_kJ^{\alpha}_{\frac{1}{b}+}(f\circ g)\left(\frac{1}{a}\right)\right\} \leq \frac{f(a)+f(b)}{2}, \tag{2.1}$$

with $\alpha, k > 0$.

Proof: Since f is harmonically convex on $[a,b]$, therefore by the definition of harmonically convex function, we have

$$f\left(\frac{xy}{rx+(1-r)y}\right) \leq rf(y) + (1-r)f(x), \text{ where } r \in [0,1] \tag{2.2}$$

Let $x = \frac{ab}{ta+(1-t)b}$, $y = \frac{ab}{tb+(1-t)a}$ and $r = \frac{1}{2}$, then (2.2) becomes

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{1}{2}\left[f\left(\frac{ab}{tb+(1-t)a}\right) + f\left(\frac{ab}{ta+(1-t)b}\right)\right] \tag{2.3}$$

now multiplying both sides of (2.3) by $t^{\alpha/k\ -1}$ and then integrating w.r.t t on $[0,1]$, we have

$$\int_0^1 f\left(\frac{2ab}{a+b}\right) t^{\alpha/k-1}dt \leq \frac{1}{2}\left[\int_0^1 f\left(\frac{ab}{tb+(1-t)a}\right) t^{\alpha/k-1}dt + \int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right) t^{\alpha/k-1}dt\right], \tag{2.4}$$

using the change of variables $u=\frac{tb+(1-t)a}{ab}$ and $v=\frac{ta+(1-t)b}{ab}$ in (2.4), we have

$$\frac{k}{\alpha} f\left(\frac{2ab}{a+b}\right) \leq \frac{1}{2}\left(\frac{ab}{b-a}\right)^{\alpha/k}\left[\int_{\frac{1}{b}}^{\frac{1}{a}} f\left(\frac{1}{u}\right)\left(u-\frac{1}{b}\right)^{\alpha/k-1} du + \int_{\frac{1}{b}}^{\frac{1}{a}} f\left(\frac{1}{v}\right)\left(\frac{1}{a}-v\right)^{\alpha/k-1} dv\right]$$

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{\alpha}{2k}\left(\frac{ab}{b-a}\right)^{\alpha/k}\left[\int_{\frac{1}{b}}^{\frac{1}{a}} (fog)(u)\left(u-\frac{1}{b}\right)^{\alpha/k-1} du + \int_{\frac{1}{b}}^{\frac{1}{a}} (fog)(v)\left(\frac{1}{a}-v\right)^{\alpha/k-1} dv\right] \tag{2.5}$$

multiplying both sides of (2.5) by $\Gamma_k(\alpha)$ and using the definition of k-Riemann-Liouville fractional integrals, we have

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{\Gamma_k(\alpha+k)}{2}\left(\frac{ab}{b-a}\right)^{\alpha/k}\left\{ {}_k J^{\alpha}_{\frac{1}{a}-}(f\circ g)\left(\frac{1}{b}\right) + {}_k J^{\alpha}_{\frac{1}{b}+}(f\circ g)\left(\frac{1}{a}\right)\right\}. \tag{2.6}$$

using the definition of harmonically convex function again, we have

$$f\left(\frac{ab}{ta+(1-t)b}\right) \leq tf(b) + (1-t)f(a),$$

also

$$f\left(\frac{ab}{tb+(1-t)a}\right) \leq tf(a) + (1-t)f(b),$$

so that we have

$$f\left(\frac{ab}{ta+(1-t)b}\right) + f\left(\frac{ab}{tb+(1-t)a}\right) \leq f(a)+f(b) \tag{2.7}$$

multiplying both sides of (2.7) by $t^{\alpha/k-1}$, integrating w.r.t. t on $[0,1]$ and then using the definition of k-Riemann-Liouville fractional integrals, we have

$$\frac{\Gamma_k(\alpha+k)}{2}\left(\frac{ab}{b-a}\right)^{\alpha/k}\left\{{}_{k}J^{\alpha}_{\frac{1}{a}-}(f\circ g)\left(\frac{1}{b}\right)+{}_{k}J^{\alpha}_{\frac{1}{b}+}(f\circ g)\left(\frac{1}{a}\right)\right\}\le\frac{f(a)+f(b)}{2}, \tag{2.8}$$

combining (2.6) and (2.8) we get inequality (2.1).

2.1. RIGHT HANDED INEQUALITIES

In this section we establish Hadamard type inequalities that give estimates of the difference in the right and middle terms of inequality (2.1), for harmonically convex functions. Throughout this section we take $f: I \subseteq (0,\infty) \to \mathbb{R}$ as a differentiable function on I^o, the intterior of I, the following notation

$$R_f(g;\ \alpha,a,b,k)=\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2}\left(\frac{ab}{b-a}\right)^{\alpha/k}\left\{{}_{k}J^{\alpha}_{\frac{1}{a}-}(fog)\left(\frac{1}{b}\right)+{}_{k}J^{\alpha}_{\frac{1}{b}+}(fog)\left(\frac{1}{a}\right)\right\}$$

and $A_t = ta + (1-t)b$, where $a,b \in I$, with $a<b$, $\alpha,k>0$, $g(t)=\frac{1}{t}$ and Γ_k is k-gamma function. We first prove following lemma which helps in establishing theorems of this section.

Lemma 2: *Let $f' \in L[a,b]$, where $a,b \in I$ with $a<b$. Then the following identity for fractional integrals is satisfied*

$$R_f(g;\ \alpha,a,b,k)=\frac{ab(b-a)}{2}\int_0^1\frac{\left[t^{\alpha/k}-(1-t)^{\alpha/k}\right]}{(A_t)^2}f'\left(\frac{ab}{A_t}\right)dt,$$

with $\alpha,k>0$, $g(t)=\frac{1}{t}$ and Γ_k is k-gamma function.

Proof: Consider the following integral, integrating by parts, and using change of variable $u=\frac{A_t}{ab}$ and by definition of k-fractional integrals we have

$$\begin{aligned}
&\frac{ab(b-a)}{2}\int_0^1\frac{\left[t^{\alpha/k}-(1-t)^{\alpha/k}\right]}{(A_t)^2}f'\left(\frac{ab}{A_t}\right)dt\\
&=\frac{1}{2}\left[\left[t^{\alpha/k}-(1-t)^{\alpha/k}\right]f\left(\frac{ab}{A_t}\right)\Big|_0^1-\frac{\alpha}{k}\int_0^1\left[t^{\alpha/k}+(1-t)^{\alpha/k-1}\right]f\left(\frac{ab}{A_t}\right)dt\right]\\
&=\frac{1}{2}\left[f(a)+f(b)-\frac{\alpha}{k}\left(\int_0^1 t^{\alpha/k-1}f\left(\frac{ab}{A_t}\right)dt+\int_0^1(1-t)^{\alpha/k-1}f\left(\frac{ab}{A_t}\right)dt\right)\right]\\
&=\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{2}\left(\frac{ab}{b-a}\right)^{\alpha/k}\left\{{}_{k}J^{\alpha}_{\frac{1}{a}-}(fog)\left(\frac{1}{b}\right)+{}_{k}J^{\alpha}_{\frac{1}{b}+}(fog)\left(\frac{1}{a}\right)\right\}\\
&=R_f(g;\ \alpha,a,b,k).
\end{aligned}$$

Hence proved.

Theorem 3: *Let $f' \in L[a,b]$, where $a,b \in I^o$ with $a<b$. If $|f'|^q$ is harmonically convex on $[a,b]$ for some fixed $q \ge 1$, then for k-fractional integrals subsequent inequality is satisfied*

$$\left|R_f(g;\ \alpha,a,b,k)\right|$$
$$\leq \frac{ab(b-a)}{2} {P_1}^{1-1/q}(\alpha,a,b,k)(P_2(\alpha,a,b,k)\,|f'(b)|^q + P_3(\alpha,a,b,k)|f'(a)|^q)^{1/q},$$

where

$$P_1(\alpha,a,b,k) = \frac{k}{b^2(\alpha+k)}\left[{}_2^kF_1\left(2k,k;\alpha+2k;\frac{1}{k}\left(1-\frac{a}{b}\right)\right) + {}_2^kF_1\left(2k,\alpha+k;\alpha+2k;\frac{1}{k}\left(1-\frac{a}{b}\right)\right)\right]$$

$$P_2(\alpha,a,b,k) = \frac{k}{b^2(\alpha+2k)}\left[\frac{k}{\alpha+k}\,{}_2^kF_1\left(2k,2k;\alpha+3k;\frac{1}{k}\left(1-\frac{a}{b}\right)\right) + {}_2^kF_1\left(2k,\alpha+2k;\alpha+3k;\frac{1}{k}\left(1-\frac{a}{b}\right)\right)\right]$$

$$P_2(\alpha,a,b,k) = \frac{k}{b^2(\alpha+2k)}\left[{}_2^kF_1\left(2k,k;\alpha+3k;\frac{1}{k}\left(1-\frac{a}{b}\right)\right) + \frac{k}{\alpha+k}\,{}_2^kF_1\left(2k,\alpha+k;\alpha+3k;\frac{1}{k}\left(1-\frac{a}{b}\right)\right)\right].$$

and $0 < \alpha \leq k$.

Proof: From Lemma 2, using the properties of modulus, the power mean inequality and the harmonically convexity of $|f'|^q$ respectively, we have

$$\left|R_f(g;\ \alpha,a,b,k)\right|$$
$$\leq \frac{ab(b-a)}{2}\int_0^1 \frac{\left|t^{\alpha/k}-(1-t)^{\alpha/k}\right|}{(A_t)^2}\left|f'\left(\frac{ab}{A_t}\right)\right|dt$$
$$\leq \frac{ab(b-a)}{2}\left(\int_0^1 \frac{\left|t^{\alpha/k}-(1-t)^{\alpha/k}\right|}{(A_t)^2}dt\right)^{1-1/q}\left(\int_0^1 \frac{\left|t^{\alpha/k}-(1-t)^{\alpha/k}\right|}{(A_t)^2}\left|f'\left(\frac{ab}{A_t}\right)\right|^q dt\right)^{1/q}$$
$$\leq \frac{ab(b-a)}{2}\left(\int_0^1 \frac{\left[t^{\alpha/k}+(1-t)^{\alpha/k}\right]}{(A_t)^2}dt\right)^{1-1/q}$$
$$\left(\int_0^1 \frac{t^{\alpha/k}+(1-t)^{\alpha/k}}{(A_t)^2}\left[t|f'(b)|^q+(1-t)|f'(a)|^q\right]dt\right)^{1/q}$$

$$= \frac{ab(b-a)}{2}{T_1}^{1-1/q}(T_2\,|f'(b)|^q + T_3|f'(a)|^q)^{1/q} \tag{2.9}$$

Consider

$$T_1 = \int_0^1 \frac{\left[t^{\alpha/k}+(1-t)^{\alpha/k}\right]}{(A_t)^2}dt$$
$$= \int_0^1 t^{\alpha/k}(A_t)^{-2}\,dt + \int_0^1 (1-t)^{\alpha/k}(A_t)^{-2}\,dt$$

$$= \int_0^1 t^{\alpha/k}\left[b\left(1-\left(1-\frac{a}{b}\right)t\right)\right]^{-2}dt + \int_0^1 (1-t)^{\alpha/k}\left[b\left(1-\left(1-\frac{a}{b}\right)t\right)\right]^{-2}dt$$
$$= \frac{k}{b^2(\alpha+k)}\left[{}_2^kF_1\left(2k,k;\alpha+2k;\frac{1}{k}\left(1-\frac{a}{b}\right)\right) + {}_2^kF_1\left(2k,\alpha+k;\alpha+2k;\frac{1}{k}\left(1-\frac{a}{b}\right)\right)\right]$$
$$= P_1(\alpha,a,b,k).$$

Similarly solving the integrals T_2 and T_3, then using the values of T_1, T_2 and T_3 in (2.9) we get the required result.

Theorem 4: *Let* $f' \in L[a,b]$, *where* $a,b \in I^o$ *with* $a<b$. *If* $|f'|^q$ *is harmonically convex on* $[a,b]$ *for some fixed* $q\geq 1$, *then for k-fractional integrals subsequent inequality is satisfied*

$$\left|R_f(g;\ \alpha,a,b,k)\right| \leq \frac{ab(b-a)}{2}P_1{}^{1-1/q}(\alpha,a,b,k)\left(P_2(\alpha,a,b,k)\,|f'(b)|^q + P_3(\alpha,a,b,k)|f'(a)|^q\right)^{1/q},$$

where

$$P_1(\alpha,a,b,k) = \frac{k}{b^2(\alpha+k)}\left[{}_2^kH_1\left(2k,\alpha+k;\alpha+2k;\frac{1}{k}\left(1-\frac{a}{b}\right)\right) - {}_2^kH_1\left(2k,k;\alpha+2k;\frac{1}{k}\left(1-\frac{a}{b}\right)\right) + {}_2^kH_1\left(2k,k;\alpha+2k;\frac{1}{2k}\left(1-\frac{a}{b}\right)\right)\right]$$

$$P_2(\alpha,a,b,k) = \frac{k}{b^2(\alpha+2k)}\left[{}_2^kH_1\left(2k,\alpha+2k;\alpha+3k;\frac{1}{k}\left(1-\frac{a}{b}\right)\right) - \frac{k}{\alpha+k}\,{}_2^kH_1\left(2k,2k;\alpha+3k;\frac{1}{k}\left(1-\frac{a}{b}\right)\right) + \frac{k}{2(\alpha+k)}\,{}_2^kH_1\left(2k,2k;\alpha+3k;\frac{1}{2k}\left(1-\frac{a}{b}\right)\right)\right]$$

$$P_2(\alpha,a,b,k) = \frac{k}{b^2(\alpha+2k)}\left[\frac{k}{\alpha+k}\,{}_2^kH_1\left(2k,\alpha+k;\alpha+3k;\frac{1}{k}\left(1-\frac{a}{b}\right)\right) - {}_2^kH_1\left(2k,k;\alpha+3k;\frac{1}{k}\left(1-\frac{a}{b}\right)\right) + {}_2^kH_1\left(2k,k;\alpha+3k;\frac{1}{2k}\left(1-\frac{a}{b}\right)\right)\right],$$

and $0<\alpha\leq k$.

Proof: From Lemma 2, using the properties of modulus, the power mean inequality and the harmonically convexity of $|f'|^q$ respectively, we have

$$\left|R_f(g;\ \alpha,a,b,k)\right|$$
$$\leq \frac{ab(b-a)}{2}\int_0^1 \frac{\left|t^{\alpha/k}-(1-t)^{\alpha/k}\right|}{(A_t)^2}\left|f'\left(\frac{ab}{A_t}\right)\right|dt$$
$$\leq \frac{ab(b-a)}{2}\left(\int_0^1 \frac{\left|t^{\alpha/k}-(1-t)^{\alpha/k}\right|}{(A_t)^2}dt\right)^{1-1/q}\left(\int_0^1 \frac{\left|t^{\alpha/k}-(1-t)^{\alpha/k}\right|}{(A_t)^2}\left|f'\left(\frac{ab}{A_t}\right)\right|^q dt\right)^{1/q}$$

$$\leq \frac{ab(b-a)}{2} {T_4}^{1-1/q} \left(\int_0^1 \frac{\left|t^{\alpha/k} - (1-t)^{\alpha/k}\right|}{(A_t)^2} [t|f'(b)|^q + (1-t)|f'(a)|^q] dt \right)^{1/q}$$
$$= \frac{ab(b-a)}{2} {T_4}^{1-1/q} (T_5 \, |f'(b)|^q + T_6 |f'(a)|^q)^{1/q}. \quad (2.10)$$

Considering T_4 and using Lemma 1, we have

$$T_4 = \int_0^1 \frac{\left|t^{\alpha/k} - (1-t)^{\alpha/k}\right|}{(A_t)^2} dt$$
$$= \int_0^{1/2} \frac{(1-t)^{\alpha/k} - t^{\alpha/k}}{(A_t)^2} dt + \int_{1/2}^1 \frac{t^{\alpha/k} - (1-t)^{\alpha/k}}{(A_t)^2} dt$$
$$= \int_0^1 \frac{t^{\alpha/k} - (1-t)^{\alpha/k}}{(A_t)^2} dt + 2\int_0^{1/2} \frac{(1-t)^{\alpha/k} - t^{\alpha/k}}{(A_t)^2} dt$$
$$\leq \int_0^1 t^{\alpha/k} (A_t)^{-2} \, dt - \int_0^1 (1-t)^{\alpha/k} (A_t)^{-2} \, dt + 2\int_0^{1/2} (1-2t)^{\alpha/k} (A_t)^{-2} \, dt$$
$$= \int_0^1 t^{\alpha/k} (A_t)^{-2} \, dt - \int_0^1 (1-t)^{\alpha/k} (A_t)^{-2} \, dt + \frac{1}{b^2} \int_0^1 (1-s)^{\alpha/k} \left(1 - \frac{1}{2}\left(1 - \frac{a}{b}\right) s\right)^{-2} ds$$
$$= \frac{k}{b^2(\alpha+k)} \left[{}_2^kH_1 \left(2k, \alpha+k; \alpha+2k; \frac{1}{k}\left(1 - \frac{a}{b}\right)\right) - {}_2^kH_1 \left(2k, k; \alpha+2k; \frac{1}{k}\left(1 - \frac{a}{b}\right)\right) + \right.$$
*2kH12k,k;*α*+2k;12k1−ab.*

$$= P_1(\alpha, a, b, k).$$

Similarly solving T_5 and T_6. Then by using values of T_4, T_5 and T_6 in (2.10) we get the required result.

Theorem 5: *Let $f' \in L[a,b]$, where $a, b \in I^o$ with $a < b$. If $|f'|^q$ is harmonically convex on $[a,b]$ for some fixed $q \geq 1$, then for k-fractional integrals subsequent inequality is satisfied*

$$\left|R_f(g;\ \alpha, a, b, k)\right| \leq \frac{a(b-a)}{2b} \left(\frac{1}{p\frac{\alpha}{k}+1}\right)^{1/p} \left(\frac{\left|f'(b)\right|^q + \left|f'(a)\right|^q}{2}\right)^{1/q}$$
$$\times \left({}_2^kF_1^{1/p} \left(2pk, k; \alpha p + 2k; \frac{1}{k}\left(1 - \frac{a}{b}\right)\right) + {}_2^kF_1^{1/p} \left(2pk, \alpha p + k; \alpha p + 2k; \frac{1}{k}\left(1 - \frac{a}{b}\right)\right)\right),$$

where $\frac{1}{p} + \frac{1}{q} = 1.$

Proof: From Lemma 2, using the properties of modulus, the Hölder's inequality and the harmonically convexity of $|f'|^q$ respectively, we have

$$\left|R_f(g;\ \alpha, a, b, k)\right|$$
$$\leq \frac{ab(b-a)}{2} \int_0^1 \frac{\left|t^{\alpha/k} - (1-t)^{\alpha/k}\right|}{(A_t)^2} \left|f'\left(\frac{ab}{A_t}\right)\right| dt$$
$$\leq \frac{ab(b-a)}{2} \int_0^1 \frac{t^{\alpha/k} + (1-t)^{\alpha/k}}{(A_t)^2} \left|f'\left(\frac{ab}{A_t}\right)\right| dt$$
$$= \frac{ab(b-a)}{2} \left[\int_0^1 \frac{t^{\alpha/k}}{(A_t)^2} \left|f'\left(\frac{ab}{A_t}\right)\right| dt + \int_0^1 \frac{(1-t)^{\alpha/k}}{(A_t)^2} \left|f'\left(\frac{ab}{A_t}\right)\right| dt \right]$$

$$\leq \frac{ab(b-a)}{2}\left[\left(\int_0^1 \frac{t^{\mathrm{p}\,\alpha/k}}{(A_t)^{2p}}dt\right)^{1/p}\left(\int_0^1 \left|f'\left(\frac{ab}{A_t}\right)\right|^q dt\right)^{1/q} + \left(\int_0^1 \frac{(1-t)^{\mathrm{p}\,\alpha/k}}{(A_t)^{2p}}dt\right)^{1/p}\left(\int_0^1 \left|f'\left(\frac{ab}{A_t}\right)\right|^q dt\right)^{1/q}\right]$$

$$= \frac{ab(b-a)}{2}\left[(T_7)^{1/p} + (T_8)^{1/p}\right]\left(\int_0^1 [t|f'(b)|^q + (1-t)|f'(a)|^q]\,dt\right)^{1/q}$$

$$= \frac{ab(b-a)}{2}\left[(T_7)^{1/p} + (T_8)^{1/p}\right]\left(\frac{|f'(b)|^q + |f'(a)|^q}{2}\right)^{1/q} \quad (2.11)$$

and $T_7 = \int_0^1 t^{\mathrm{p}\,\alpha/k}(A_t)^{-2p}\,dt = \frac{k}{b^{2p}(\alpha\mathrm{p}+k)}\,{}_2^kF_1\left(2pk, 2k; \alpha p + 2k; \frac{1}{k}\left(1-\frac{a}{b}\right)\right)$.

Similarly solving the integral T_8, then using the value of T_7 and T_8 in (2.11), we get the required result.

Theorem 6: *Let $f' \in L[a,b]$, where $a, b \in I^o$ with $a < b$. If $|f'|^q$ is harmonically convex on $[a,b]$ for some fixed $q > 1$, then for k-fractional integrals subsequent inequality is satisfied*

$$\left|R_f(g;\ \alpha, a, b, k)\right| \leq \frac{b-a}{2(ab)^{1-1/p}} L_{2p-2}^{2-2/p}(a,b)\left(\frac{1}{q\frac{\alpha}{k}+1}\right)^{1/q}\left(\frac{|f'(b)|^q + |f'(a)|^q}{2}\right)^{1/q},$$

where $\frac{1}{p}+\frac{1}{q} = 1$ and $L_{2p-2}^{2-2/p}(a,b)$ is $(2p-2)$-Logarithmic mean.

Proof: From Lemma 2, Lemma 1, using the properties of modulus, the Hölder's inequality and the harmonically convexity of $|f'|^q$ respectively, we have

$$\left|R_f(g;\ \alpha, a, b, k)\right|$$

$$\leq \frac{ab(b-a)}{2}\int_0^1 \frac{\left|t^{\alpha/k} - (1-t)^{\alpha/k}\right|}{(A_t)^2}\left|f'\left(\frac{ab}{A_t}\right)\right| dt$$

$$\leq \frac{ab(b-a)}{2}\left(\int_0^1 \frac{1}{(A_t)^{2p}}\,dt\right)^{1/p}\left(\int_0^1 \left|t^{\alpha/k} - (1-t)^{\alpha/k}\right|^q \left|f'\left(\frac{ab}{A_t}\right)\right|^q dt\right)^{1/q}$$

$$\leq \frac{ab(b-a)}{2}\left(\int_0^1 \frac{1}{(A_t)^{2p}}\,dt\right)^{1/p}\left(\int_0^1 |1-2t|^{q\,\alpha/k}\,[t\,|f'(b)|^q + (1-t)\,|f'(a)|^q]dt\right)^{1/q}$$

$$\leq \frac{ab(b-a)}{2}\,(T_9)^{1/p}(T_{10}\,|f'(b)|^q + T_{11}\,|f'(a)|^q)^{1/q} \quad (2.12)$$

Consider $T_9 = \int_0^1 \frac{1}{(A_t)^{2p}}\,dt$

$$T_9 = \int_0^1 (A_t)^{-2p}\,dt = \frac{1}{b^{2p}}\,{}_2^kF_1\left(2pk, k; 2k; \frac{1}{k}\left(1-\frac{a}{b}\right)\right) = \frac{1}{(ab)^{2p-1}}L_{2p-2}^{2p-2}(a,b),$$

$T_{10} = \int_0^1 |1-2t|^{q\alpha/k}\, t\, dt = \int_0^{1/2} (1-2t)^{q\alpha/k}\, t\, dt + \int_{1/2}^1 (2t-1)^{q^{\alpha}/k}\, t\, dt = \frac{1}{2\left(q\frac{\alpha}{k}+1\right)}$,

Similarly $T_{11} = \int_0^1 |1-2t|^{q\alpha/k}\,(1-t)dt = \frac{1}{2\left(q\frac{\alpha}{k}+1\right)}$.

Using values of T_9, T_{10} and T_{11} in (2.12) we get the required result.

Theorem 7: *Let $f' \in L[a,b]$, where $a,b \in I$ with $a<b$. If $|f'|^q$ is harmonically convex on $[a,b]$ for some fixed $q>1$, then for k-fractional integrals subsequent inequality is satisfied*

$$\left|R_f(g;\ \lambda, a, b, k)\right| \le \frac{a(b-a)}{2b}\left(\frac{1}{p\frac{\alpha}{k}+1}\right)^{1/p}$$
$$\times\left(\frac{{}_2^kH_1\left(2qk, 2k; 3k; \frac{1}{k}\left(1-\frac{a}{b}\right)\right)|f'(b)|^q + {}_2^kH_1\left(2qk, k; 3k; \frac{1}{k}\left(1-\frac{a}{b}\right)\right)|f'(a)|^q}{2}\right)^{1/q},$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof: From Lemma 2, Lemma 1, using the properties of modulus, the Hölder's inequality and the harmonically convexity of $|f'|^q$ respectively, we have

$$\begin{aligned}
&\left|R_f(g;\ \alpha, a, b, k)\right| \\
&\le \frac{ab(b-a)}{2}\int_0^1 \frac{\left|t^{\alpha/k}-(1-t)^{\alpha/k}\right|}{(A_t)^2}\left|f'\left(\frac{ab}{A_t}\right)\right| dt \\
&\le \frac{ab(b-a)}{2}\left(\int_0^1 \left|t^{\alpha/k}-(1-t)^{\alpha/k}\right|^p\ dt\right)^{1/p}\left(\int_0^1 \frac{1}{(A_t)^{2q}}\left|f'\left(\frac{ab}{A_t}\right)\right|^q dt\right)^{1/q} \\
&\le \frac{ab(b-a)}{2}\left(\int_0^1 |1-2t|^{p\,\alpha/k}\ dt\right)^{1/p}\left(\int_0^1 \frac{1}{(A_t)^{2q}}\left[t\,|f'(b)|^q+(1-t)\,|f'(a)|^q\right]dt\right)^{1/q} \\
&\le \frac{ab(b-a)}{2}\,(T_{12})^{1/p}\left(T_{13}\,|f'(b)|^q + T_{14}\,|f'(a)|^q\right)^{1/q} \qquad (2.13)
\end{aligned}$$

Consider $T_{12} = \int_0^1 |1-2t|^{p\,\alpha/k}\ dt = \frac{1}{p\frac{\alpha}{k}+1}$,

$$T_{13} = \int_0^1 \frac{1}{(A_t)^{2q}}\, t\, dt = \frac{1}{2b^{2q}}\, {}_2^kH_1\left(2qk, 2k; 3k; \frac{1}{k}\left(1-\frac{a}{b}\right)\right),$$

$$T_{14} = \int_0^1 \frac{1}{(A_t)^{2q}}(1-t)\, dt = \frac{1}{2b^{2q}}\, {}_2^kH_1\left(2qk, k; 3k; \frac{1}{k}\left(1-\frac{a}{b}\right)\right).$$

Using the values of T_{12}, T_{13} and T_{14} in (2.13) we get the required result.

2.2. LEFT HANDED INEQUALITIES

In this section we establish Hadamard type inequalities which give estimates of the difference in the left and middle terms of inequality (2.1), for harmonically convex functions. Throughout this section we take $f: I \subseteq (0,\infty) \to \mathbb{R}$ as a differentiable function on I^o, the following notation

$$L_f(g;\ \alpha, a, b, k) = f\left(\frac{2ab}{a+b}\right) - \frac{\Gamma_k(\alpha+k)}{2}\left(\frac{ab}{b-a}\right)^{\alpha/k}\left\{ {}_kJ^{\alpha}_{\frac{1}{a}-}(f\circ g)\left(\frac{1}{b}\right) + {}_kJ^{\alpha}_{\frac{1}{b}+}(f\circ g)\left(\frac{1}{a}\right)\right\}$$

and $A_t = ta + (1-t)b$, where $a, b \in I$, with $a < b$, $\alpha, k > 0$, $g(t) = \frac{1}{t}$ and Γ_k is k-gamma function. We first prove the following lemma which helps in establishing theorems of this section.

Lemma 3: *Let $f' \in [a,b]$ where $a, b \in I$ with $a < b$. Then for k-fractional we have*

$$L_f(g;\ \alpha, a, b, k) = \frac{1}{2}(I_1 + I_2 + I_3).$$

where

$$I_1 = ab(b-a)\int_0^{1/2} f'\left(\frac{ab}{A_t}\right)\frac{dt}{{A_t}^2},$$

$$I_2 = -ab(b-a)\int_{1/2}^{1} f'\left(\frac{ab}{A_t}\right)\frac{dt}{{A_t}^2}$$

$$I_3 = -ab(b-a)\int_0^1 \left[(1-t)^{\alpha/k} - t^{\alpha/k}\right] f'\left(\frac{ab}{A_t}\right)\frac{dt}{{A_t}^2}$$

where $\alpha, k > 0$ and $g(t) = \frac{1}{t}$

Proof: Considering the following sum and solving third integral as solved in Lemma 2.

$$\frac{1}{2}(I_1 + I_2 + I_3)$$
$$= \frac{ab(b-a)}{2}\left[\int_0^{1/2} f'\left(\frac{ab}{A_t}\right)\frac{dt}{{A_t}^2} - \int_{1/2}^{1} f'\left(\frac{ab}{A_t}\right)\frac{dt}{{A_t}^2} - \int_0^1 \left[(1-t)^{\alpha/k} - t^{\alpha/k}\right] f'\left(\frac{ab}{A_t}\right)\frac{dt}{{A_t}^2}\right]$$
$$= \frac{ab(b-a)}{2}\left[\int_0^{1/2} f'\left(\frac{ab}{A_t}\right)\frac{dt}{{A_t}^2} + \int_{1/2}^{1} f'\left(\frac{ab}{A_t}\right)\frac{dt}{{A_t}^2} + I_3\right].$$
$$= \frac{1}{2}\left[f\left(\frac{ab}{A_t}\right)\Big|_0^{1/2} - f\left(\frac{ab}{A_t}\right)\Big|_{1/2}^{1} - \Gamma_k(\lambda+k)\left(\frac{ab}{b-a}\right)^{\alpha/k}\left\{ {}_kJ^{\alpha}_{\frac{1}{a}-}(f\circ g)\left(\frac{1}{b}\right) + {}_kJ^{\alpha}_{\frac{1}{b}+}(f\circ g)\left(\frac{1}{a}\right)\right\}\right]$$
$$= f\left(\frac{2ab}{a+b}\right) - \frac{\Gamma_k(\alpha+k)}{2}\left(\frac{ab}{b-a}\right)^{\alpha/k}\left\{ {}_kJ^{\alpha}_{\frac{1}{a}-}(f\circ g)\left(\frac{1}{b}\right) + {}_kJ^{\alpha}_{\frac{1}{b}+}(f\circ g)\left(\frac{1}{a}\right)\right\},$$
$$= L_f(g;\ \alpha, a, b, k).$$

Hence proved.

Theorem 8 : *Let* $f' \in L[a,b]$*, where* $a, b \in I^o$ *with* $a < b$*. If* $(f')^q$ *is harmonically convex on* $[a,b]$ *for some fixed* $q \geq 1$*, then for k-fractional integrals subsequent inequality is satisfied*

$$\left|L_f(g;\ \alpha,a,b,k)\right| \leq \frac{f(b)-f(a)}{2} + \frac{ab(b-a)}{2} P_1^{1-1/q}(\alpha,a,b,k)(P_2(\alpha,a,b,k)\,(f'(b))^q + P_3(\alpha,a,b,k)(f'(a))^q)^{1/q},$$

where

$$P_1(\alpha,a,b,k) = \frac{k}{b^2(\alpha+k)}\left[{}_2^kF_1\left(2k,k;\alpha+2k;\frac{1}{k}\left(1-\frac{a}{b}\right)\right) + {}_2^kF_1\left(2k,\alpha+k;\alpha+2k;\frac{1}{k}\left(1-\frac{a}{b}\right)\right)\right]$$

$$P_2(\alpha,a,b,k) = \frac{k}{b^2(\alpha+2k)}\left[\frac{k}{\alpha+k}\,{}_2^kF_1\left(2k,2k;\alpha+3k;\frac{1}{k}\left(1-\frac{a}{b}\right)\right) + {}_2^kF_1\left(2k,\alpha+2k;\alpha+3k;\frac{1}{k}\left(1-\frac{a}{b}\right)\right)\right]$$

$$P_3(\alpha,a,b,k) = \frac{k}{b^2(\alpha+2k)}\left[{}_2^kF_1\left(2k,k;\alpha+3k;\frac{1}{k}\left(1-\frac{a}{b}\right)\right) + \frac{k}{\alpha+k}\,{}_2^kF_1\left(2k,\alpha+k;\alpha+3k;\frac{1}{k}\left(1-\frac{a}{b}\right)\right)\right].$$

Proof: From Lemma 3 and using the properties of modulus, we have

$$\left|L_f(g;\ \lambda,m,n,k)\right| \leq \frac{1}{2}(|I_1| + |I_2| + |I_3|).$$

Solving $|I_1|$ and $|I_2|$ as solved in Lemma 3 and solving $|I_3|$ as solved in Theorem 3 by using; Lemma 2, properties of modulus, power mean inequality and the fact that $(f')^q$ is harmonically convex on $[a,b]$ respectively, we get the required resut.

Theorem 9: *Let* $f' \in L[a,b]$*, where* $a, b \in I^o$ *with* $a < b$*. If* $(f')^q$ *is harmonically convex on* $[a,b]$ *for some fixed* $q \geq 1$*, then for k-fractional integrals subsequent inequality is satisfied*

$$\left|L_f(g;\ \alpha,a,b,k)\right| \leq \frac{f(b)-f(a)}{2} + \frac{ab(b-a)}{2} P_1^{1-1/q}(\alpha,a,b,k)(P_2(\alpha,a,b,k)\,(f'(b))^q + P_3(\alpha,a,b,k)(f'(a))^q)^{1/q},$$

where

$$P_1(\alpha, a, b, k) = \frac{k}{b^2(\alpha+k)}\left[{}_2^kF_1\left(2k, \alpha+k; \alpha+2k; \frac{1}{k}\left(1-\frac{a}{b}\right)\right) - {}_2^kF_1\left(2k, k; \alpha+2k; \frac{1}{k}\left(1-\frac{a}{b}\right)\right) + {}_2^kF_1\left(2k, k; \alpha+2k; \frac{1}{2k}\left(1-\frac{a}{b}\right)\right)\right],$$

$$P_2(\alpha, a, b, k) = \frac{k}{b^2(\alpha+2k)}\left[{}_2^kF_1\left(2k, \alpha+2k; \lambda+3k; \frac{1}{k}\left(1-\frac{a}{b}\right)\right) - \frac{k}{\alpha+k}\,{}_2^kF_1\left(2k, 2k; \alpha+3k; \frac{1}{k}\left(1-\frac{a}{b}\right)\right) + \frac{k}{2(\alpha+k)}\,{}_2^kF_1\left(2k, 2k; \alpha+3k; \frac{1}{2k}\left(1-\frac{a}{b}\right)\right)\right],$$

$$P_2(\alpha, a, b, k) = \frac{k}{b^2(\alpha+2k)}\left[\frac{k}{\alpha+k}\,{}_2^kF_1\left(2k, \alpha+k; \alpha+3k; \frac{1}{k}\left(1-\frac{a}{b}\right)\right) - {}_2^kF_1\left(2k, k; \alpha+3k; \frac{1}{k}\left(1-\frac{a}{b}\right)\right) + {}_2^kF_1\left(2k, k; \alpha+3k; \frac{1}{2k}\left(1-\frac{a}{b}\right)\right)\right],$$

where $0 < \alpha \leq k$.

Proof: From Lemma 3 and using the properties of modulus, we have

$$|L_{\mathcal{F}}(g; \alpha, a, b, k)| \leq \frac{1}{2}(|I_1| + |I_2| + |I_3|).$$

Solving $|I_1|$ and $|I_2|$ as solved in Lemma 3 and solving $|I_3|$ as solved in Theorem 4 by using; Lemma 2, properties of modulus, power mean inequality and the fact that $(f')^q$ is harmonically convex on $[a,b]$, we get the required resut.

Theorem 10: *Let $f' \in L[a,b]$, where $a, b \in I^o$ with $a < b$. If $(f')^q$ is harmonically convex on $[a,b]$ for some fixed $q > 1$, then for k-fractional integrals subsequent inequality is satisfied*

$$\left|L_f(g;\ \alpha, a, b, k)\right| \leq \frac{f(b)-f(a)}{2} + \frac{a(b-a)}{2b}\left(\frac{1}{p\frac{\alpha}{k}+1}\right)^{1/p}\left(\frac{(f'(b))^q+(f'(a))^q}{2}\right)^{1/q} \times\left({}_2^kF_1{}^{1/p}\left(2pk, k; \alpha p+2k; \frac{1}{k}\left(1-\frac{a}{b}\right)\right) + {}_2^kF_1{}^{1/p}\left(2pk, \alpha p+k; \alpha p+2k; \frac{1}{k}\left(1-\frac{a}{b}\right)\right)\right),$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof: From Lemma 3 and using the properties of modulus, we have

$$\left|L_f(g;\ \alpha, a, b, k)\right| \leq \frac{1}{2}(|I_1| + |I_2| + |I_3|).$$

Solving $|I_1|$ and $|I_2|$ as solved in Lemma 3 and solving $|I_3|$ as solved in Theorem 5 by using; Lemma 2, properties of modulus, Hölder's inequality and the fact that $(f')^q$ is harmonically convex on $[a,b]$ respectively, we get the required result.

Theorem 11: *Let $f' \in L[a,b]$, where $a,b \in I^o$ with $a < b$. If $(f')^q$ is harmonically convex on $[a,b]$ for some fixed $q > 1$, then for k-fractional integrals subsequent inequality is satisfied*

$$\left|L_f(g;\ \alpha,a,b,k)\right| \leq \frac{f(b)-f(a)}{2} + \frac{b-a}{2(ab)^{1-1/p}} L_{2p-2}^{2-2/p}(a,b)\left(\frac{1}{q\frac{\alpha}{k}+1}\right)^{1/q}\left(\frac{(f'(b))^q+(f'(a))^q}{2}\right)^{1/q},$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and* $L_{2p-2}^{2-2/p}(a,b)$ *is* $(2p-2)$*-Logarithmic mean.*

Proof: From Lemma 3 and using the properties of modulus

$$\left|L_f(g;\ \alpha,a,b,k)\right| \leq \frac{1}{2}(|I_1|+|I_2|+|I_3|).$$

Solving $|I_1|$ and $|I_2|$ as solved in Lemma 3 and solving $|I_3|$ as solved in Theorem 6 by using; Lemma 2, properties of modulus, Lemma 1, Hölder's inequality and the fact that $(f')^q$ is harmonically convex on $[a,b]$, we get the required resut.

Theorem 12. *Let $f' \in L[a,b]$, where $a,b \in I^o$ with $a < b$. If $(f')^q$ is harmonically convex on $[a,b]$ for some fixed $q > 1$, then for k- fractional integrals holds*

$$\left|L_f(g;\ \alpha,a,b,k)\right| \leq \frac{f(b)-f(a)}{2} + \frac{a(b-a)}{2b}\left(\frac{1}{p\frac{\alpha}{k}+1}\right)^{1/p}$$

$$\times\left(\frac{{}_2^kF_1\left(2qk,2k;3k;\frac{1}{k}\left(1-\frac{a}{b}\right)\right)(f'(b))^q + {}_2^kF_1\left(2qk,k;3k;\frac{1}{k}\left(1-\frac{a}{b}\right)\right)(f'(a))^q}{2}\right)^{1/q},$$

where $\frac{1}{p}+\frac{1}{q}=1.$

Proof: From Lemma 3 and using the properties of modulus

$$\left|L_f(g;\ \alpha,a,b,k)\right| \leq \frac{1}{2}(|I_1|+|I_2|+|I_3|).$$

Solving $|I_1|$ and $|I_2|$ as solved in Lemma 3 and solving $|I_3|$ as solved in Theorem 7 by using; Lemma 2, Lemma 1, properties of modulus, Hölder's inequality, the fact that $(f')^q$ is harmonically convex on $[a,b]$ respectively, we get the required resut.

3. CONCLUSIONS

We have established Hadamard type inequality (2.1) for harmonically convex functions by way of k-Riemann-Liouville fractional integrals. Besides this inequality, estimates of the difference in the right & middle terms of (2.1) and estimates of the difference in the left & middle terms of (2.1), are also presented.

One must note that the right handed inequalities are generalizations of those of [13] for k-fractional integrals, i.e. we get results of article [13] as $k \to 1$ in Lemma 2 to Theorem 7. Also the left handed inequalities are generalizations of those of [14] for k- fractional integrals, i.e. we get results of article [14] as $k \to 1$ in Lemma 3 to Theorem 12.

REFERENCES

[1] Alomari, M., and Darus, M., *J. Ineq. Appl.*, **2009**, 1, 2009.
[2] Chen, F., and Wu, S., *The Scientific World Journal*, **2014**, 1, 2014.
[3] Dahmani, Z., Tabharit, L. and Taf, S., *Bull. Math. Anal. Appl.*, **2**, 93, 2010.
[4] Dragomir, S. S., and Pearce, C. E. M., *RGMIA Monographs*, Victoria University Melbourne City, 2000, 345.
[5] İşcan, İ., *Konuralp Journal of Mathematics*, **1**, 67, 2013.
[6] İşcan, İ., *Contemporary Analysis and Applied Mathematics*, **1**, 253, **2**013.
[7] Ozdemir, M. E., Set E., and Sarikaya, M. Z., *Hacet. J. Math. Stat*. **40**, 219, 2011.
[8] Wang, J., Zhu, C. and Zhou, Y., *J. Ineq. Appl*. **325**, 15, 2013.
[9] İşcan, İ., *Hacet J Math Stat*, **43**, 935, 2014.
[10] İşcan, İ., *Journal of Mathematics*, **2014**, 1, 2014.
[11] Ali, A., Ghulshan, G., Hussain, R., Latif, A. and Muddassar, M., *Journal of Computational Analysis & Applications*, **22**, 1208, 2017.
[12] İşcan, İ., Kunt, M., and Yazici, N., *NTMSCI*, **4**, 239, 2016.
[13] İşcan, İ., and Wu, S., *Appl. Math. Compt.* **238**, 237, 2014.
[14] Set, E., Iscan, I. and Zehirn, F., *Konuralp J. Maths*, **3**, 42, 2015.
[15] Gorenflo, R., and Mainardi, F., *Integral and differential equations of fractional order*, Springer Verlag Wien, 1997, 223.
[16] Loverro, A., *Fractional Calculus: history, definitions and applications for the engineer, Department of Aerospace and Mechanical Engineering*, University of Notre Dame, 28, 2004.
[17] Miller, K., and Ross, B., *An introduction to the fractional calculus and fractional differential equations*, NewYork, Wiley, 363, 1993.
[18] Oldham, K. B., and Spanier, J., *The fractional calculus.* Academic Press. New York London, 225, 1974.
[19] Diaz, R. ,and Pariguan, E., *Divulgaciones Matemáticas*, **15**, 179, 2007.
[20] Hussain, R., Ali, A., Ghulshan, G., and Latif, A., *The Australian Journal of Mathematical Analysis and Applications*, **13**, 1, 2016.
[21] Hussain, R., Ali, A., Latif, A., and Ghulshan, G., *J. of Fractional Differential Calculus*, **7**, 301, 2017.
[22] Hussain, R., Kehkashan, B., Ali, A., Latif, A., and Rauf, K., *Niger. J. Math. Appl.*, **25**, 8, 2016.

[23] Mubeen, S., and Habibullah, G. M., *Int. Math. Forum*, **7,** 203, 2012.
[24] Mubeen, S., and Habibullah, G. M., *Int. J. Contemp. Math. Sci.*, **7**, 89, 2012.
[25] Prudnikov, A. P., Brychkov, Y.A., and Maric, O. I., *Integral and series in elementary functions,* Moscow Russian Fedration, 798, 1981.
[26] Sarikaya, M. Z., and Yaldiz, H., *Konuralp J. Math*, **1**, 33, 2013.