ORIGINAL PAPER

A UNIQUENESS THEOREM FOR DIRAC SYSTEM

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Abstract. *In this paper, we solve inverse problem for Dirac System. Especially, we show that it is possible to obtain the uniqueness of the potential function by using spectrums.*

Keywords: *Dirac Equation, uniqueness theorem, spectrum*

1. INTRODUCTION

The Dirac equation is a modern presentation of the relativistic quantum mechanics of electrons intended to make new mathematical results accessible to a wider audience. It treats in some depth the relativistic invariance of a quantum theory, self-adjointness and spectral theory, qualitative features of relativistic bound and scattering states, and the external field problem in quantum electrodynamics, without neglecting the interpretational difficulties and limitations of the theory. Note that, inverse problems for Dirac system had been investigated by Moses [1], Prats and Toll [2], Verde [3], Gasymov and Levitan [4] and Panakhov [5]. It is well known that two spectra uniquely determine the matrix-potential. Let Q denotes a matrix operator

$$Q=\begin{pmatrix} p_{11}(x) & p_{12}(x) \\ p_{21}(x) & p_{22}(x) \end{pmatrix},\ p_{12}(x)=p_{21}(x)$$

where $p_{ik}(x)$ $(i,k=1,2)$ are real functions which are defined and continuous on the interval $[0,\pi]$. Further, let $\varphi(x)$ denotes a two component vector function

$$\varphi(x,\lambda)=\begin{pmatrix} \varphi_1(x,\lambda) \\ \varphi_2(x,\lambda) \end{pmatrix}.$$

Then the equation

$$\left(B\frac{d}{dx}+Q-\lambda I\right)\varphi=0,$$

where λ is a parameter, and

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$$B=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\ I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$

is equivalent to a system of two simultaneous first-order ordinary differential equations

$$\frac{d\phi_2}{dx}+p_{11}(x)\phi_1+p_{12}(x)\phi_2=\lambda\phi_1,$$
$$-\frac{d\phi_1}{dx}+p_{21}(x)\phi_1+p_{22}(x)\phi_2=\lambda\phi_2. \tag{1}$$

For the case in which $p_{12}(x)=p_{21}(x)=0$, $p_{11}(x)=V(x)+m$, $p_{22}(x)=V(x)-m$, where $V(x)$ is a potential function, and m is the mass of a particle, the system (1) is known in relativistic quantum theory as a stationary one-dimensional Dirac system. For the case in which

$$p_{12}(x)=p_{21}(x)=0$$

$$p_{11}(x)=V(x)+m=p(x)$$

$$p_{22}(x)=V(x)-m=r(x),$$

we obtain the following system called first canonic form of Dirac operator

$$\phi'_2-\{\lambda+p(x)\}\phi_1=0,$$
$$\phi'_1+\{\lambda+r(x)\}\phi_2=0, \tag{2}$$

Let us denote by $\varphi(x,\lambda)$ the solution of the system (2) satisfying the initial conditions

$$\varphi_1(0,\lambda)=\cos\alpha,\ \varphi_2(0,\lambda)=-\sin\alpha. \tag{3}$$

The function $\varphi(x,\lambda)$ obviously satisfies the following condition

$$\varphi_2(0,\lambda)\cos\alpha+\varphi_1(0,\lambda)\sin\alpha=0. \tag{4}$$

Let us consider the problem (2)-(3) for $p(x)=r(x)=0$. As is not difficult to see, in this case

$$\phi_1(x,\lambda)=\cos(\lambda x-\alpha),$$

$$\phi_2(x,\lambda)=\sin(\lambda x-\alpha)$$

We will assume that the functions $p(x)$ and $r(x)$ are continuous on the interval $[0,\pi]$ and λ is an eigenvalue of this problem [6].

Lemma 1. [4] For $|\lambda| \to \infty$ the following estimates hold uniformly with respect to x, $0 \le x \le \pi$:

$$\varphi_1(x,\lambda) = \cos\{\xi(x,\lambda) - \alpha\} + O\left(\frac{1}{\lambda}\right),$$
$$\varphi_2(x,\lambda) = \sin\{\xi(x,\lambda) - \alpha\} + O\left(\frac{1}{\lambda}\right),$$

where $\xi(x,\lambda) = \lambda x - \frac{1}{2}\int_0^x [p(\tau) + r(\tau)]d\tau$ and the sequence $\{\lambda_n\}$ satisfies the classical asymptotic form

$$\lambda_n = n + \frac{\upsilon}{\pi} + O\left(\frac{1}{n}\right), (n = 0, \pm 1, \pm 2, ...),$$

where υ, α and β are defined by

$$\upsilon = \beta - \alpha - \frac{1}{2}\int_0^\pi [p(\tau) + r(\tau)]d\tau,$$

$$\alpha = \cos\left\{\frac{1}{2}\int_0^x [p(\tau) + r(\tau)]d\tau\right\}$$

and

$$\beta = \sin\left\{\frac{1}{2}\int_0^x [p(\tau) + r(\tau)]d\tau\right\}.$$

Remark: [4] For the case of $p(x) = r(x) = 0$, we get $\alpha = 1, \beta = 0$ and $\lambda_n = n + O\left(\frac{1}{n}\right)$.

2. A UNIQUENESS THEOREM

In this section, we will consider two problems consist of two different potentials and then two spectrums. We will show that two spectrums are enough for obtaining the potentials as uniquely. There are similar type problems in the literature.

Let us consider the problem

$$By' + Q_1(x)y = \lambda y \quad (0 \le x \le \pi) \tag{5}$$

$$y_2(0) - hy_1(0) = 0 \tag{6}$$

$$y_2(\pi)+Hy_1(\pi)=0 \tag{7}$$

where the functions $p_i(x)$, $q_i(x)$ $(i=1,2)$ are real and continuous on $[0,\pi]$ and h, H are and

$$B=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\quad Q_1(x)=\begin{pmatrix}p_1(x) & q_1(x)\\ q_1(x) & -p_1(x)\end{pmatrix},\quad y(x)=\begin{pmatrix}y_1(x)\\ y_2(x)\end{pmatrix}.$$

We will show the spectrum of this problem as $\Lambda(h,H;Q_1)$.
Also let us consider the second problem (6)-(7)

$$By'+Q_2(x)y=\lambda y \quad (0\le x\le \pi) \tag{8}$$

Let's consider the equation. In equation (8)

$$By'+Q_1(x)y=\lambda y$$

$$y_2(0)-hy_1(0)=0$$

$$y_2(\pi)+H_1y_1(\pi)=0 \qquad (H\neq H_1) \tag{9}$$

Spectrum of this problem is $\Lambda(h,H;Q_2)$ and

$$Q_2(x)=\begin{pmatrix}p_2(x) & q_2(x)\\ q_2(x) & -p_2(x)\end{pmatrix}$$

Now, we are ready to give to following uniqueness theorem.

Theorem 2.1. [7] If $H\neq H_1$. Then, $\Lambda(h,H;Q_1)=\Lambda(h,H;Q_2)$ and $\Lambda(h,H_1;Q_1)=\Lambda(h,H_1;Q_2)$ then $Q_1(x)=Q_2(x)$.

Proof: Solution of (5)-(7) $\varphi(x)=\begin{pmatrix}\varphi_1(x,\lambda)\\ \varphi_2(x,\lambda)\end{pmatrix}$ and solution of the problem (6)-(8) $\psi(x)=\begin{pmatrix}\psi_1(x,\lambda)\\ \psi_2(x,\lambda)\end{pmatrix}$ and also spectrums of these problems are $\Lambda(h,H;Q_1)$, $\Lambda(h,H_1;Q_1)$ respectively. Then, we have

$$\varphi_2'(x,\lambda_n)+\{p_1(x)-\lambda_n\}\varphi_1(x,\lambda_n)+q_1(x)\varphi_2(x,\lambda_n)=0 \tag{10}$$

$$-\varphi_1'(x,\lambda_n)+q_1(x)\varphi_1(x,\lambda_n)-\{p_1(x)+\lambda_n\}\varphi_2(x,\lambda_n)=0 \tag{11}$$

$$\varphi_2(0,\lambda_n)-h\varphi_1(0,\lambda_n)=0 \tag{12}$$

$$\varphi_2(\pi,\lambda_n)+H\varphi_1(\pi,\lambda_n)=0 \tag{13}$$

also for $\psi(x,\lambda_n)$ we can write

$$\psi_2'(x,\lambda_n)+\{p_2(x)-\lambda_n\}\psi_1(x,\lambda_n)+q_2(x)\psi_2(x,\lambda_n)=0 \tag{14}$$

$$-\psi_1'(x,\lambda_n)+q_2(x)\psi_1(x,\lambda_n)-\{p_2(x)+\lambda_n\}\psi_2(x,\lambda_n)=0 \tag{15}$$

$$\psi_2(0,\lambda_n)-h\psi_1(0,\lambda_n)=0 \tag{16}$$

$$\psi_2(\pi,\lambda_n)+H\psi_1(\pi,\lambda_n)=0 \tag{17}$$

multiply (10)-(11)-(14)-(15) by $-\psi_1(x,\lambda_n), -\psi_2(x,\lambda_n), \varphi_1(x,\lambda_n), \varphi_2(x,\lambda_n)$ respectively and sum them

$$-(\phi_2'(x,\lambda_n)\psi_1(x,\lambda_n)+\phi_2(x,\lambda_n)\psi_1'(x,\lambda_n))$$

$$+\{(\phi_1(x,\lambda_n)\psi_1(x,\lambda_n)-\phi_2(x,\lambda_n)\psi_2(x,\lambda_n)\}\left(p_2(x)-p_1(x)\right)$$

$$+\left(\phi_1'(x,\lambda_n)\psi_2(x,\lambda_n)+\phi_1(x,\lambda_n)\psi_2'(x,\lambda_n)\right)$$

$$+\{(\phi_1(x,\lambda_n)\psi_2(x,\lambda_n)+\phi_2(x,\lambda_n)\psi_1(x,\lambda_n)\}\left(q_2(x)-q_1(x)\right)=0$$

After some corrections, we obtain

$$\left(\phi_1'(x,\lambda_n)\psi_2(x,\lambda_n)+\phi_1(x,\lambda_n)\psi_2'(x,\lambda_n)\right)-(\phi_2'(x,\lambda_n)\psi_1(x,\lambda_n)+\phi_2(x,\lambda_n)\psi_1'(x,\lambda_n))$$

$$=\{(\phi_1(x,\lambda_n)\psi_1(x,\lambda_n)-\phi_2(x,\lambda_n)\psi_2(x,\lambda_n)\}\left(p_2(x)-p_1(x)\right)$$

$$+\{(\phi_1(x,\lambda_n)\psi_2(x,\lambda_n)+\phi_2(x,\lambda_n)\psi_1(x,\lambda_n)\}\left(q_2(x)-q_1(x)\right)$$

Integrating the last equality from 0 to, it will be

$$\begin{aligned}&\int_0^\pi \frac{d}{dx}\{\varphi_1(x,\lambda_n)\psi_2(x,\lambda_n)-\varphi_2(x,\lambda_n)\psi_1(x,\lambda_n)\}\\&=\int_0^\pi\{[\varphi_1(x,\lambda_n)\psi_1(x,\lambda_n)-\varphi_2(x,\lambda_n)\psi_2(x,\lambda_n)](p_2(x)-p_1(x))\}dx\\&+\int_0^\pi\{[\varphi_1(x,\lambda_n)\psi_2(x,\lambda_n)+\varphi_2(x,\lambda_n)\psi_1(x,\lambda_n)](q_2(x)-q_1(x))\}dx\end{aligned} \tag{18}$$

with the boundary conditions

$$[\varphi_1(x,\lambda_n)\psi_2(x,\lambda_n)-\varphi_2(x,\lambda_n)\psi_1(x,\lambda_n)]_{x=0}^{x=\pi}=0$$

and then, we obtain

$$\begin{aligned}&\int_0^\pi\{[\varphi_1(x,\lambda_n)\psi_1(x,\lambda_n)-\varphi_2(x,\lambda_n)\psi_2(x,\lambda_n)](p_2(x)-p_1(x))\}dx\\&+\int_0^\pi\{[\varphi_1(x,\lambda_n)\psi_2(x,\lambda_n)+\varphi_2(x,\lambda_n)\psi_1(x,\lambda_n)](q_2(x)-q_1(x))\}dx=0\end{aligned}\tag{19}$$

Similarly, $\Lambda(h,H_1;Q_1)=\Lambda(h,H_1;Q_2)$ such that $\varphi(x,\mu_n)$ satisfies

$$\varphi_2'(x,\mu_n)+\{p_1(x)-\mu_n\}\varphi_1(x,\mu_n)+q_1(x)\varphi_2(x,\mu_n)=0 \tag{10'}$$

$$-\varphi_1'(x,\mu_n)+q_1(x)\varphi_1(x,\mu_n)-\{p_1(x)+\mu_n\}\varphi_2(x,\mu_n)=0 \tag{11'}$$

$$\varphi_2(0,\mu_n)-h\varphi_1(0,\mu_n)=0 \tag{12'}$$

$$\varphi_2(\pi,\mu_n)+H\varphi_1(\pi,\mu_n)=0 \tag{13'}$$

on the other hand we can write also for $\psi(x,\mu_n)$

$$\psi_2'(x,\mu_n)+\{p_2(x)-\mu_n\}\psi_1(x,\mu_n)+q_2(x)\psi_2(x,\mu_n)=0 \tag{14'}$$

$$-\psi_1'(x,\mu_n)+q_2(x)\psi_1(x,\mu_n)-\{p_2(x)+\mu_n\}\psi_2(x,\mu_n)=0 \tag{15'}$$

$$\psi_2(0,\mu_n)-h\psi_1(0,\mu_n)=0 \tag{16'}$$

$$\psi_2(\pi,\mu_n)+H\psi_1(\pi,\mu_n)=0\,. \tag{17'}$$

Multiply the equalities by (10’), (11’), (14’), (15’) $-\psi_1(x,\mu_n),-\psi_2(x,\mu_n)$, $\varphi_1(x,\mu_n),\varphi_2(x,\mu_n)$, respectively and sum them we obtain

$$-(\phi_2'(x,\mu_n)\psi_1(x,\mu_n)+\phi_2(x,\mu_n)\psi_1'(x,\mu_n))$$

$$+\{(\phi_1(x,\mu_n)\psi_1(x,\mu_n)-\phi_2(x,\mu_n)\psi_2(x,\mu_n)\}\left(p_2(x)-p_1(x)\right)$$

$$+\left(\phi_1'(x,\mu_n)\psi_2(x,\mu_n)+\phi_1(x,\mu_n)\psi_2'(x,\mu_n)\right)$$

$$+\{(\phi_1(x,\mu_n)\psi_2(x,\mu_n)+\phi_2(x,\mu_n)\psi_1(x,\mu_n)\}\left(q_2(x)-q_1(x)\right)=0$$

after some corrections, we get

$$\begin{aligned}&(\varphi_1'(x,\mu_n)\psi_2(x,\mu_n)+\varphi_1(x,\mu_n)\psi_2'(x,\mu_n))-(\varphi_2'(x,\mu_n)\psi_1(x,\mu_n)+\varphi_2(x,\mu_n)\psi_1'(x,\mu_n))\\&=\{(\varphi_1(x,\mu_n)\psi_1(x,\mu_n)-\varphi_2(x,\mu_n)\psi_2(x,\mu_n)\}(p_2(x)-p_1(x))\\&+\{(\varphi_1(x,\mu_n)\psi_2(x,\mu_n)+\varphi_2(x,\mu_n)\psi_1(x,\mu_n)\}(q_2(x)-q_1(x))\end{aligned}$$

integrating last equality from 0 to π we can easily obtain

$$\begin{aligned}&\int_0^\pi \frac{d}{dx}\{\varphi_1(x,\mu_n)\psi_2(x,\mu_n)-\varphi_2(x,\mu_n)\psi_1(x,\mu_n)\}\\&=\int_0^\pi\{[\varphi_1(x,\mu_n)\psi_1(x,\mu_n)-\varphi_2(x,\mu_n)\psi_2(x,\mu_n)](p_2(x)-p_1(x))\}dx\\&+\int_0^\pi\{[\varphi_1(x,\mu_n)\psi_2(x,\mu_n)+\varphi_2(x,\mu_n)\psi_1(x,\mu_n)](q_2(x)-q_1(x))\}dx\end{aligned}\tag{18'}$$

by the conditions for the solutions $\varphi_1(x,\mu_n),\varphi_2(x,\mu_n)$ and $\psi_1(x,\mu_n),\psi_2(x,\mu_n)$ we yield

$$[\varphi_1(x,\mu_n)\psi_2(x,\mu_n)-\varphi_2(x,\mu_n)\psi_1(x,\mu_n)]_{x=0}^{x=\pi}=0.$$

So,

$$\begin{aligned}&\int_0^\pi\{[\varphi_1(x,\mu_n)\psi_1(x,\mu_n)-\varphi_2(x,\mu_n)\psi_2(x,\mu_n)](p_2(x)-p_1(x))\}dx\\&+\int_0^\pi\{[\varphi_1(x,\mu_n)\psi_2(x,\mu_n)+\varphi_2(x,\mu_n)\psi_1(x,\mu_n)](q_2(x)-q_1(x))\}dx=0\end{aligned}\tag{19'}$$

eventually,

$$\begin{aligned}&\int_0^\pi\{[\varphi_1(x,\mu_n)\psi_1(x,\mu_n)-\varphi_2(x,\mu_n)\psi_2(x,\mu_n)](p_2(x)-p_1(x))\}dx\\&+\int_0^\pi\{[\varphi_1(x,\mu_n)\psi_2(x,\mu_n)+\varphi_2(x,\mu_n)\psi_1(x,\mu_n)](q_2(x)-q_1(x))\}dx=0\end{aligned}\tag{20}$$

and we can obtain that

$$p_2(x) - p_1(x) = 0 \quad , \quad q_2(x) - q_1(x) = 0$$

or,

$$Q_1(x) = Q_2(x)$$

almost everywhere. This completes the proof.

3. CONCLUSION

We considered that two problems consist of two different potentials and then two spectra. We have shown that the two spectra are sufficient to achieve the potentials uniquely. We have shown that it is possible to obtain the uniqueness of the potential function using spectra.

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