# A NEW APPROACH TO BENDING ENERGY OF ELASTICA FOR SPACE CURVES IN DE-SITTER SPACE 

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#### Abstract

In this paper, we firstly introduce kinematics properties of a moving particle lying in De-Sitter space $\mathcal{S}_{1}^{3}$. We assume that the particle corresponds to a different type of space curves such that they are characterized by using the Frenet vector field in De-Sitter spacetime. Based on this assumption, we present geometrical understanding of energy on the particle in each Frenet vector fields depending on being a spacelike or timelike curve in $\mathcal{S}_{1}^{3}$. Then, we also determine the bending elastic energy functional for the same particle in $\mathcal{S}_{1}^{3}$ by assuming the particle has a bending feature of elastica. Finally, we prove that bending energy formula can be represented by the energy on the particle in each Frenet vector field. We conclude our results by providing energy variation sketches with respect to time for different cases.


Keywords: Energy, De-Sitter Space, Frenet Vector Fields, Elastica.
Mathematics Subject Classifications: 53C41, 53A10.

## 1. INTRODUCTION

It is well known that there exist two different space forms having a constant sectional curvature which are Lorentzian space form and Riemannian space form. The Lorentzian space form with the positive constant curvature is called De-Sitter space which is the analog in Minkowski spacetime of a sphere in ordinary Euclidean space.

Constant curvature spacetimes have attracted much notice, from both observational and also theoretical viewpoint of modern cosmology. The major interest in De-Sitter spacetime is recognized by current cosmological observations which shows that our universe is asymptotic. In other words, a solution of the equation of Einstein with a positive cosmological constant is found in De-Sitter metric which models an expanding universe [1]. To obtain these data and facts focusing on kinematics and dynamical aspect of the structure on the corresponding spacetime is crucial.

A search of the literature indicates that there is almost no concrete computations of entropy, laws of horizon dynamics and energy in the case of De-Sitter spacetime. Various attempts are being made to describe the concept of energy using quasi-local or local concepts. However, these definitions of energy do not agree with each other all the time and they are not applicable to the universes of De-Sitter type [2-4]. Therefore, we believe that we should start with using a local approach to make any progress on the notion of energy in this

[^0]spacetime. Thus, we consider that one of the most effective ways to make this approach is to use intrinsic geometrical features of the moving particle in the De-Sitter spacetime.

A motion of a particle in space is important due to the wide range application of the subject. The motion of the particle in absolute space and time was defined firstly by Newtonian dynamics [5, 6]. Then, a geometric generalization of the action which includes terms belonging to curvature of the moving particle's trajectory in different spacetimes is given in [7].

The equations of moving the particle in the certain vector field are obtained by considering its generalized acceleration, velocity, and coordinate. Guided by this, unit vector field's energy on a Riemannian manifold $M$ is described to be equal to the energy of the mapping $M \rightarrow T_{1} M$, where $T_{1} M$ is defined as unit tangent bundle equipped with Sasaki metric [8]. By similar argument volume of a unit vector field $X$ is described as the volume of the submanifold in the unit tangent bundle defined by $X(M)$ [9]. It is also investigated that the energy and volume of vector fields have many similarities [10-15].

This study organizes as follows. We firstly present fundamental definitions of Frenet frame equations for a different type of space curves in De-Sitter space. Then we give a geometrical interpretation of the energy for unit vector fields. In the following section, we set a connection between the physical and geometrical understanding of the energy for a moving particle in De-Sitter spacetime considering the dynamics of different type of space curves. Finally, we give some energy variation sketches for different cases.

## 2. KINEMATICS IN DE-SITTER SPACE

Let $\Gamma$ be a particle moving in a De-Sitter space $\mathcal{S}_{1}^{3}$ such that the precise location of the particle is specified by $\Gamma=\Gamma(t)$, where $t$ is a time parameter. Changing the time parameter describes the motion and trajectory. Thus, the trajectory corresponds to a curve $\zeta$ in the space for a moving particle. It is convenient to remind the arc-length parameter $s$, which is used to compute the distance traveled by a particle along its trajectory. It is defined by

$$
\frac{d s}{d t}=\|\mathbf{v}\|
$$

where $\mathbf{v}=\mathbf{v}(t)=\frac{d \zeta}{d t}$ is the velocity vector and $\frac{d \zeta}{d t} \neq 0$. In particle dynamics, the arc-length parameter $s$ is considered as a function of $t$. Thanks to the arc-lenght, it is also determined Serret-Frenet frame, which allows us determining the characterization of the intrinsic geometrical features of the regular curve. This coordinate system is constructed by three orthonormal vectors $\mathbf{e}_{(\alpha)}$ and the curve $\zeta$ itself, assuming the curve is sufficiently smooth at each point. The index within the parenthesis is the tetrad index that describes a particular member of the tetrad. In particular, $\mathbf{e}_{(0)}$ is the unit tangent vector, $\mathbf{e}_{(1)}, \mathbf{e}_{(2)}$ is the unit normal and binormal vector of the curve $\zeta$, respectively. Pseudo-orthonormality conditions are summarized by $\mathbf{e}_{(\alpha)} \mathbf{e}_{(\beta)}=\eta_{\alpha \beta}$, where $\eta_{\alpha \beta}$ is Lorentzian metric such that: $\operatorname{diag}(-1,1,1,1)$ if the curve is timelike, $\operatorname{diag}(1, \pm 1, \pm 1, \pm 1)$ if the curve is spacelike. Thus, we have the following formulas for the Frenet frame equations.

Case 1. Let the moving particle has a unit timelike binormal vector $\mathbf{e}_{(2)}$, then we have

$$
\begin{aligned}
\frac{D \mathbf{e}_{(0)}}{d s} & =\kappa \mathbf{e}_{(1)}-\zeta \\
\frac{D \mathbf{e}_{(1)}}{d s} & =\kappa \delta(\zeta) \mathbf{e}_{(0)}+\tau \mathbf{e}_{(2)} \\
\frac{D \mathbf{e}_{(2)}}{d s} & =\tau \mathbf{e}_{(1)} \\
\frac{D \zeta^{u}}{d s} & =\mathrm{e}_{(0)}
\end{aligned}
$$

where $\delta(\zeta)=-\operatorname{si\rho n}\left(\mathbf{e}_{(1)}\right), \tau, \kappa$ are torsion and curvature of a curve $\zeta$ given by [16].

Case 2. Let the moving particle has a unit timelike normal vector $\mathbf{e}_{(1)}$, then we have

$$
\begin{align*}
& \frac{D \mathbf{e}_{(0)}}{d s}=\kappa \mathbf{e}_{(1)}-\zeta, \\
& \frac{D \mathbf{e}_{(1)}}{d s}=\kappa \mathbf{e}_{(0)}+\sigma \tau \mathbf{e}_{(2)}, \\
& \frac{D \mathbf{e}_{(2)}}{d s}=\tau \mathbf{e}_{(1)}, \tag{2}
\end{align*}
$$

$$
\frac{D \zeta^{u}}{d s}=\mathbf{e}_{(0)}
$$

where $\sigma= \pm 1, \tau, \kappa$ are torsion and curvature of a curve $\zeta$ given by [17].
Case 3. Let the moving particle has a unit timelike tangent vector $\mathbf{e}_{(0)}$, then we have

$$
\begin{align*}
& \frac{D \mathbf{e}_{(0)}}{d s}=\kappa \mathbf{e}_{(1)}+\zeta \\
& \frac{D \mathbf{e}_{(1)}}{d s}=\kappa \mathbf{e}_{(0)}+\tau \mathbf{e}_{(2)},  \tag{3}\\
& \frac{D \mathbf{e}_{(2)}}{d s}=-\tau \mathbf{e}_{(1)},
\end{align*}
$$

$$
\frac{D \zeta^{u}}{d s}=\mathbf{e}_{(0)},
$$

where $\tau, \kappa$ are torsion and curvature of a curve $\zeta$ given by [18].
Since we identify $\mathbf{e}_{(0)}$ as a unit vector which is a tangent to the curve at each point on the curve, we have $\mathbf{e}_{(0)}=d \Gamma^{u} / d s$, where $\Gamma^{u}$ is the point on the trajectory of curve $\zeta$. Thus $\mathbf{e}_{(0)}, \mathbf{e}_{(1)}$ and $\mathbf{e}_{(2)}$ and $\zeta$ generate the Frenet frame for each case.

## 3. ENERGY OF UNIT VECTOR FIELDS

We first give the fundamental definitions and propositions which are used to compute the energy of the unit vector field.

Definition 1 For two Riemannian manifolds $(M, \rho)$ and $(N, h)$ the energy of a differentiable map $f:(M, \rho) \rightarrow(N, h)$ can be defined as

$$
\begin{equation*}
\operatorname{snergy}(f)=\frac{1}{2} \int_{M} \sum_{a=1}^{n} h\left(d f\left(e_{a}\right), d f\left(e_{a}\right)\right) v, \tag{4}
\end{equation*}
$$

where $\left\{e_{a}\right\}$ is a local basis of the tangent space and $v$ is the canonical volume form in $M$ [8,19].

Proposition 2 Let $Q: T\left(T^{1} M\right) \rightarrow T^{1} M$ be the connection map. Then following two conditions hold:
i) $\omega \circ Q=\omega \circ d \omega$ and $\omega \circ Q=\omega \circ \tilde{\omega}$, where $\tilde{\omega}: T\left(T^{1} M\right) \rightarrow T^{1} M$ is the tangent bundle projection;
ii) for $\rho \in T_{x} M$ and a section $\xi: M \rightarrow T^{1} M$; we have

$$
\begin{equation*}
Q(d \xi(\rho))=D_{\rho} \xi \tag{5}
\end{equation*}
$$

where $D$ is the Levi-Civita covariant derivative $[8,19]$.
Definition 3 For $\varsigma_{1}, \varsigma_{2} \in T_{\xi}\left(T^{1} M\right)$, we define

$$
\begin{equation*}
\rho_{S}\left(\varsigma_{1}, \varsigma_{2}\right)=\rho\left(d \omega\left(\varsigma_{1}\right), d \omega\left(\varsigma_{2}\right)\right)+\rho\left(Q\left(\varsigma_{1}\right), Q\left(\varsigma_{2}\right)\right) . \tag{6}
\end{equation*}
$$

This yields a Riemannian metric on $T M$. As known $\rho_{S}$ is called the Sasaki metric that also makes the projection $\omega: T^{1} M \rightarrow M$ a Riemannian submersion.

## 4. ENERGY ON A PARTICLE IN DE-SITTER SPACE $\mathcal{S}_{1}^{3}$

In the theory of relativity, all the energy moving through an object contributes to the body's total mass that measures how much it can resist to acceleration. Each kinetic and potential energy makes a highly proportional contribution to the mass [20,21,22]. In this study not only we compute the energy of space curves in $\mathcal{S}_{1}^{3}$ but we also investigate its close correlation with bending energy of elastica which is a variational problem proposed firstly by Daniel Bernoulli to Leonard Euler in 1744. Euler elastica of bending energy formula for an elastic curve in the De-Sitter space is given by Frenet curvature along the curve

$$
\begin{equation*}
H_{B}=\frac{1}{2} \int\left\|D_{\mathbf{e}_{(0)}} \mathbf{e}_{(0)}\right\|^{2} d s, \tag{7}
\end{equation*}
$$

where $s$ is an arclength, [23].
Case 1. Let the moving particle has a unit timelike binormal vector $\mathbf{e}_{(2)}$.

Theorem 4 Let $\Gamma$ be a moving particle in $\mathcal{S}_{1}^{3}$ such that it corresponds to a curve $\zeta$. Then, energy on the particle in tangent vector field by using Sasaki metric is stated by

$$
\text { Energye }_{(0)}=s+\frac{1}{2} \int_{0}^{s} \kappa^{2} d s .
$$

Proof: From (4) and (5) we know that

$$
\operatorname{snergy}_{(0)}=\frac{1}{2} \int_{0}^{s} \rho_{S}\left(d \mathbf{e}_{(0)}\left(\mathbf{e}_{(0)}\right), d \mathbf{e}_{(0)}\left(\mathbf{e}_{(0)}\right)\right) d s .
$$

Using Eq. (6) we have

$$
\begin{gathered}
\rho_{S}\left(d \mathbf{e}_{(0)}\left(\mathbf{e}_{(0)}\right), d \mathbf{e}_{(0)}\left(\mathbf{e}_{(0)}\right)\right)=\rho\left(d \omega\left(\mathbf{e}_{(0)}\left(\mathbf{e}_{(0)}\right)\right), d \omega\left(\mathbf{e}_{(0)}\left(\mathbf{e}_{(0)}\right)\right)\right) \\
+\rho\left(Q\left(\mathbf{e}_{(0)}\left(\mathbf{e}_{(0)}\right)\right), Q\left(\mathbf{e}_{(0)}\left(\mathbf{e}_{(0)}\right)\right)\right) .
\end{gathered}
$$

Since $\mathbf{e}_{(0)}$ is a section, we get

$$
d(\omega) \circ d\left(\mathbf{e}_{(0)}\right)=d\left(\omega \circ \mathbf{e}_{(0)}\right)=d\left(i d_{C}\right)=i d_{T C} .
$$

We also know

$$
Q\left(\mathbf{e}_{(0)}\left(\mathbf{e}_{(0)}\right)\right)=D_{\mathbf{e}_{(0)}} \mathbf{e}_{(0)}=\kappa \mathbf{e}_{(1)}-\zeta .
$$

Thus, we find from (1)

$$
\begin{gathered}
\rho_{S}\left(d \mathbf{e}_{(0)}\left(\mathbf{e}_{(0)}\right), d \mathbf{e}_{(0)}\left(\mathbf{e}_{(0)}\right)\right)=\rho\left(\mathbf{e}_{(0)}, \mathbf{e}_{(0)}\right)+\rho\left(D_{\mathbf{e}_{(0)}} \mathbf{e}_{(0)}, D_{\mathbf{e}_{(0)}} \mathbf{e}_{(0)}\right) \\
=2+\kappa^{2} .
\end{gathered}
$$

So we can easily obtain

$$
\text { Energye }_{(0)}=s+\frac{1}{2} \int_{0}^{s} \kappa^{2} d s
$$

This completes the proof.
Theorem 5 Let $\Gamma$ be a moving particle in $\mathcal{S}_{1}^{3}$ such that it corresponds to a curve $\zeta$. Then, energy on the particle in normal vector field by using Sasaki metric is stated by

$$
\text { snergy }_{(1)}=\frac{1}{2}\left(s+\int_{0}^{s}\left(\delta^{2}(\zeta) \kappa^{2}-\tau^{2}\right) d s\right) .
$$

Proof: If we follow similar steps at Theorem 4, the proof is obvious.
Theorem 6 Let $\Gamma$ be a moving particle in $\mathcal{S}_{1}^{3}$ such that it corresponds to a curve $\zeta$. Then, energy on the particle in binormal vector field by using Sasaki metric is stated by

$$
\text { snergy }_{(2)}=\frac{1}{2}\left(s+\int_{0}^{s} \tau^{2} d s\right) .
$$

Proof: If we follow similar steps at Theorem 4, the proof is obvious.
Now, we give highly important result by using the above theorems and definitions. It describes energy on the moving particle in De-Sitter space for the given case.

Theorem 7 Let $\Gamma$ be a moving particle in $\mathcal{S}_{1}^{3}$ such that it corresponds to a curve $\zeta$. Then, energy on the particle $\zeta$ by using Sasaki metric is stated by

$$
\text { snergy } \zeta=s .
$$

Proof: By (4) and (5) we have

$$
\text { Energy } \zeta=\frac{1}{2} \int_{0}^{s} \rho_{S}\left(d \zeta\left(\mathbf{e}_{(0)}\right), d \zeta\left(\mathbf{e}_{(0)}\right)\right) d s
$$

Then using Eq. (6) we get

$$
\begin{aligned}
\rho_{S}\left(d \zeta\left(\mathbf{e}_{(0)}\right)\right. & \left., d \zeta\left(\mathbf{e}_{(0)}\right)\right)=\rho\left(d \omega\left(\zeta\left(\mathbf{e}_{(0)}\right)\right), d \omega\left(\zeta\left(\mathbf{e}_{(0)}\right)\right)\right) \\
& +\rho\left(Q\left(\zeta\left(\mathbf{e}_{(0)}\right)\right), Q\left(\zeta\left(\mathbf{e}_{(0)}\right)\right)\right) .
\end{aligned}
$$

Knowing $\mathbf{e}_{(0)}$ is a section, we obtain

$$
d(\omega) \circ d(\zeta)=d(\omega \circ \zeta)=d\left(i d_{C}\right)=i d_{T C}
$$

and

$$
Q\left(\zeta\left(\mathbf{e}_{(0)}\right)\right)=D_{\mathbf{e}_{(0)}} \zeta=\mathbf{e}_{(0)} .
$$

Thus, we compute the metric by using (1) as

$$
\rho_{S}\left(d \zeta\left(\mathbf{e}_{(0)}\right), d \zeta\left(\mathbf{e}_{(0)}\right)\right)=\rho\left(\mathbf{e}_{(0)}, \mathbf{e}_{(0)}\right)+\rho\left(D_{\mathbf{e}_{(0)}} \zeta, D_{\mathbf{e}_{(0)}} \zeta\right)=2 .
$$

So we calculate energy on $\zeta$ as

$$
\text { snergy } \zeta=s .
$$

This completes the proof.
Corollary 8 Let the moving particle has a unit timelike binormal vector $\mathbf{e}_{(2)}$. Then we obtain following relation between energy on the moving particle in Frenet vector fields and bending energy of elastica in De-Sitter space.

$$
\begin{aligned}
& H_{B}=\text { हnergy }_{(0)}-\frac{1}{2} s, \\
& H_{B}=\text { हnergy }_{(1)}+\frac{1}{2} \int_{0}^{s} \tau^{2} d s, \\
& H_{B}=\text { Energy }_{(2)}+\frac{1}{2} \int_{0}^{s}\left(\kappa^{2}-\tau^{2}\right) d s, \\
& H_{B}=\text { Energy }^{2}-\frac{1}{2} s+\frac{1}{2} \int_{0}^{s} \kappa^{2} d s .
\end{aligned}
$$

Proof: It is obvious from (7), Theorem 4, Theorem 5, Theorem 6, and Theorem 7.
Case 2. Let the moving particle has a unit timelike normal vector $\mathbf{e}_{(1)}$.
Theorem 9 Let $\Gamma$ be a moving particle in $\mathcal{S}_{1}^{3}$ such that it corresponds to a curve $\zeta$. Then, energy on the particle in tangent vector field by using Sasaki metric is stated by

$$
\text { snergy }_{(0)}=s-\frac{1}{2} \int_{0}^{s} \kappa^{2} d s .
$$

Proof: From (4) and (5) we know

$$
\text { snergy }_{(0)}=\frac{1}{2} \int_{0}^{s} \rho_{S}\left(d \mathbf{e}_{(0)}\left(\mathbf{e}_{(0)}\right), d \mathbf{e}_{(0)}\left(\mathbf{e}_{(0)}\right)\right) d s
$$

Using Eq. (6) we have

$$
\begin{gathered}
\rho_{S}\left(d \mathbf{e}_{(0)}\left(\mathbf{e}_{(0)}\right), d \mathbf{e}_{(0)}\left(\mathbf{e}_{(0)}\right)\right)=\rho\left(d \omega\left(\mathbf{e}_{(0)}\left(\mathbf{e}_{(0)}\right)\right), d \omega\left(\mathbf{e}_{(0)}\left(\mathbf{e}_{(0)}\right)\right)\right) \\
+\rho\left(Q\left(\mathbf{e}_{(0)}\left(\mathbf{e}_{(0)}\right)\right), Q\left(\mathbf{e}_{(0)}\left(\mathbf{e}_{(0)}\right)\right)\right) .
\end{gathered}
$$

Since $\mathbf{e}_{(0)}$ is a section, we get

$$
d(\omega) \circ d\left(\mathbf{e}_{(0)}\right)=d\left(\omega \circ \mathbf{e}_{(0)}\right)=d\left(i d_{C}\right)=i d_{T C} .
$$

Then

$$
Q\left(\mathbf{e}_{(0)}\left(\mathbf{e}_{(0)}\right)\right)=D_{\mathbf{e}_{(0)}} \mathbf{e}_{(0)}=\boldsymbol{k} \mathbf{e}_{(1)}-\zeta .
$$

Thus, we find from (2)

$$
\begin{gathered}
\rho_{S}\left(d \mathbf{e}_{(0)}\left(\mathbf{e}_{(0)}\right), d \mathbf{e}_{(0)}\left(\mathbf{e}_{(0)}\right)\right)=\rho\left(\mathbf{e}_{(0)}, \mathbf{e}_{(0)}\right)+\rho\left(D_{\mathbf{e}_{(0)}} \mathbf{e}_{(0)}, D_{\mathbf{e}_{(0)}} \mathbf{e}_{(0)}\right) \\
=2-\kappa^{2} .
\end{gathered}
$$

So we can easily obtain

$$
\text { Energy }_{(0)}=s-\frac{1}{2} \int_{0}^{s} \kappa^{2} d s
$$

This completes the proof.
Theorem 10 Let $\Gamma$ be a moving particle in $\mathcal{S}_{1}^{3}$ such that it corresponds to a curve $\zeta$. Then, energy on the particle in normal vector field by using Sasaki metric is stated by

$$
\text { Energye }_{(1)}=\frac{1}{2}\left(s+\int_{0}^{s}\left(\kappa^{2}+\sigma^{2} \tau^{2}\right) d s\right) .
$$

Proof: Using similar argument as in Theorem 9 gives the proof easily.
Theorem 11 Let $\Gamma$ be a moving particle in $\mathcal{S}_{1}^{3}$ such that it corresponds to a curve $\zeta$. Then, energy on the particle in binormal vector field by using Sasaki metric is stated by

$$
\text { Energye }_{(2)}=\frac{1}{2}\left(s-\int_{0}^{s} \tau^{2} d s\right) .
$$

Proof: Using similar argument as in Theorem 9 the proof is obtained easily.
Now, we give highly important result by using above theorems and definitions. It describes energy on the moving particle in De-Sitter space for the given case.

Theorem 12 Let $\Gamma$ be a moving particle in $\mathcal{S}_{1}^{3}$ such that it corresponds to a curve $\zeta$. Then, energy on the particle $\zeta$ by using Sasaki metric is stated by

$$
\text { snergy } \zeta=s .
$$

Proof: By (4) and (5) we have

$$
\varepsilon(\zeta)=\frac{1}{2} \int_{0}^{s} \rho_{S}\left(d \zeta\left(\mathbf{e}_{(0)}\right), d \zeta\left(\mathbf{e}_{(0)}\right)\right) d s
$$

Then using Eq. (6) we get

$$
\begin{aligned}
\rho_{S}\left(d \zeta\left(\mathbf{e}_{(0)}\right)\right. & \left., d \zeta\left(\mathbf{e}_{(0)}\right)\right)=\rho\left(d \omega\left(\zeta\left(\mathbf{e}_{(0)}\right)\right), d \omega\left(\zeta\left(\mathbf{e}_{(0)}\right)\right)\right) \\
& +\rho\left(Q\left(\zeta\left(\mathbf{e}_{(0)}\right)\right), Q\left(\zeta\left(\mathbf{e}_{(0)}\right)\right)\right) .
\end{aligned}
$$

Knowing $\mathbf{e}_{(0)}$ is a section, we obtain

$$
d(\omega) \circ d(\zeta)=d(\omega \circ \zeta)=d\left(i d_{C}\right)=i d_{T C}
$$

and

$$
Q\left(\zeta\left(\mathbf{e}_{(0)}\right)\right)=D_{\mathbf{e}_{(0)}} \zeta=\mathbf{e}_{(0)} .
$$

Thus, we compute the metric by using (2) as

$$
\rho_{S}\left(d \zeta\left(\mathbf{e}_{(0)}\right), d \zeta\left(\mathbf{e}_{(0)}\right)\right)=\rho\left(\mathbf{e}_{(0)}, \mathbf{e}_{(0)}\right)+\rho\left(D_{\mathbf{e}_{(0)}} \zeta, D_{\mathbf{e}_{(0)}} \zeta\right)=2 .
$$

So we calculate energy on $\zeta$ as

$$
\text { Energy }(\zeta)=s
$$

This completes the proof.
Corollary 13 Let the moving particle has a unit timelike normal vector $\mathbf{e}_{(1)}$. Then we obtain following relation between energy on the moving particle in Frenet vector fields and bending energy of elastica in De-Sitter space.

$$
\begin{gathered}
H_{B}=\operatorname{snergy}_{(0)}-\frac{1}{2} s, \\
H_{B}=- \text { Energye }_{(1)}+s+\frac{1}{2} \int_{0}^{s} \tau^{2} d s, \\
H_{B}=\text { snergy }_{(2)}+\frac{1}{2} \int_{0}^{s}\left(-\kappa^{2}+\tau^{2}\right) d s, \\
H_{B}=\text { gnergy }^{2}-\frac{1}{2} s-\frac{1}{2} \int_{0}^{s} \kappa^{2} d s .
\end{gathered}
$$

Proof: It is obvious from (7), Theorem 9, Theorem 10, Theorem 11, and Theorem 12.
Case 3. Let the moving particle has a unit timelike tangent vector $\mathbf{e}_{(0)}$.
Theorem 14 Let $\Gamma$ be a moving particle in $\mathcal{S}_{1}^{3}$ such that it corresponds to a curve $\zeta$. Then, energy on the particle in tangent vector field by using Sasaki metric is stated by

$$
\operatorname{snergy}_{(0)}=\frac{1}{2} \int_{0}^{s} \kappa^{2} d s
$$

Proof: From (4) and (5) we know

$$
\operatorname{snergy}_{(0)}=\frac{1}{2} \int_{0}^{s} \rho_{S}\left(d \mathbf{e}_{(0)}\left(\mathbf{e}_{(0)}\right), d \mathbf{e}_{(0)}\left(\mathbf{e}_{(0)}\right)\right) d s
$$

Using Eq. (6) we have

$$
\begin{gathered}
\rho_{S}\left(d \mathbf{e}_{(0)}\left(\mathbf{e}_{(0)}\right), d \mathbf{e}_{(0)}\left(\mathbf{e}_{(0)}\right)\right)=\rho\left(d \omega\left(\mathbf{e}_{(0)}\left(\mathbf{e}_{(0)}\right)\right), d \omega\left(\mathbf{e}_{(0)}\left(\mathbf{e}_{(0)}\right)\right)\right) \\
+\rho\left(Q\left(\mathbf{e}_{(0)}\left(\mathbf{e}_{(0)}\right)\right), Q\left(\mathbf{e}_{(0)}\left(\mathbf{e}_{(0)}\right)\right)\right) .
\end{gathered}
$$

Since $\mathbf{e}_{(0)}$ is a section, we get

$$
d(\omega) \circ d\left(\mathbf{e}_{(0)}\right)=d\left(\omega \circ \mathbf{e}_{(0)}\right)=d\left(i d_{C}\right)=i d_{T C} .
$$

Then

$$
Q\left(\mathbf{e}_{(0)}\left(\mathbf{e}_{(0)}\right)\right)=D_{\mathbf{e}_{(0)}} \mathbf{e}_{(0)}=\kappa(s) \mathbf{e}_{(1)}(s)+\zeta .
$$

Thus, we find from (3)

$$
\rho_{S}\left(d \mathbf{e}_{(0)}\left(\mathbf{e}_{(0)}\right), d \mathbf{e}_{(0)}\left(\mathbf{e}_{(0)}\right)\right)=\rho\left(\mathbf{e}_{(0)}, \mathbf{e}_{(0)}\right)+\rho\left(D_{\mathbf{e}_{(0)}} \mathbf{e}_{(0)}, D_{\mathbf{e}_{(0)}} \mathbf{e}_{(0)}\right)=\kappa^{2} .
$$

So we can easily obtain

$$
\text { snergye }_{(0)}=\frac{1}{2} \int_{0}^{s} \kappa^{2} d s .
$$

This completes the proof.
Theorem 15 Let $\Gamma$ be a moving particle in $\mathcal{S}_{1}^{3}$ such that it corresponds to a curve $\zeta$. Then, energy on the particle in normal vector field by using Sasaki metric is stated by

$$
\text { snergy }_{(1)}=\frac{1}{2}\left(-s+\int_{0}^{s}\left(-\kappa^{2}+\tau^{2}\right) d s\right) .
$$

Proof: It is obvious if we use the similar argument as in Theorem 14.

Theorem 16 Let $\Gamma$ be a moving particle in $\mathcal{S}_{1}^{3}$ such that it corresponds to a curve $\zeta$. Then, energy on the particle in binormal vector field by using Sasaki metric is stated by

$$
\text { snergy }_{(2)}=\frac{1}{2}\left(-s+\int_{0}^{s} \tau^{2} d s\right) .
$$

Proof: It is obvious if we use the similar argument as in Theorem 14.
Now, we give highly important result by using above theorems and definitions. It describes energy on the moving particle in De-Sitter space for the given case.

Theorem 17 Let $\Gamma$ be a moving particle in $\mathcal{S}_{1}^{3}$ such that it corresponds to a curve $\zeta$. Then, energy on the particle $\zeta$ by using Sasaki metric is stated by

$$
\text { snergy } 5=-s
$$

Proof: By (4) and (5) we have

$$
\text { Energy } \zeta=\frac{1}{2} \int_{0}^{s} \rho_{S}\left(d \zeta\left(\mathbf{e}_{(0)}\right), d \zeta\left(\mathbf{e}_{(0)}\right)\right) d s
$$

Then using Eq. (6) we have

$$
\begin{aligned}
\rho_{S}\left(d \zeta\left(\mathbf{e}_{(0)}\right)\right. & \left., d \zeta\left(\mathbf{e}_{(0)}\right)\right)=\rho\left(d \omega\left(\zeta\left(\mathbf{e}_{(0)}\right)\right), d \omega\left(\zeta\left(\mathbf{e}_{(0)}\right)\right)\right) \\
& +\rho\left(Q\left(\zeta\left(\mathbf{e}_{(0)}\right)\right), Q\left(\zeta\left(\mathbf{e}_{(0)}\right)\right)\right) .
\end{aligned}
$$

Knowing $\zeta$ is a section, we get

$$
d(\omega) \circ d(\zeta)=d(\omega \circ \zeta)=d\left(i d_{C}\right)=i d_{T C}
$$

and

$$
Q\left(\zeta\left(\mathbf{e}_{(0)}\right)\right)=D_{\mathbf{e}_{(0)}} \zeta=\mathbf{e}_{(0)} .
$$

Thus, we compute the metric by using (3) as

$$
\rho_{S}\left(d \zeta\left(\mathbf{e}_{(0)}\right), d \zeta\left(\mathbf{e}_{(0)}\right)\right)=\rho\left(\mathbf{e}_{(0)}, \mathbf{e}_{(0)}\right)+\rho\left(D_{\mathbf{e}_{(0)}} \zeta, D_{\mathbf{e}_{(0)}} \zeta\right)=-2 .
$$

So we calculate energy on $\zeta$ as

$$
\text { snergy }(\zeta)=-s
$$

This completes the proof.

Corollary 18 Let the moving particle has a unit timelike tangent vector $\mathbf{e}_{(0)}$. Then we obtain following relation between energy on the moving particle in Frenet vector fields and bending energy of elastica in De-Sitter space.

$$
\begin{gathered}
H_{B}=\operatorname{snergy}_{(0)}+\frac{1}{2} s, \\
H_{B}=- \text { Energy }_{(1)}+\frac{1}{2} \int_{0}^{s} \tau^{2} d s, \\
H_{B}=- \text { Energye }_{(2)}+\frac{1}{2} \int_{0}^{s}\left(\kappa^{2}+\tau^{2}\right) d s, \\
H_{B}=- \text { Energy }^{2}-\frac{1}{2} s+\frac{1}{2} \int_{0}^{s} \kappa^{2} d s .
\end{gathered}
$$

## 5. APPLICATION TO THE ARGUMENT

In this section, we transmit our geometric understanding of the energy for a particle in $\mathcal{S}_{1}^{3}$ to the physical example by drawing its graph for different cases. By doing this practice, we have a chance to observe the variation of the energy on the particle with respect to time and different characterizations of the curve. Let us choose $\kappa(s)=s^{2}, \tau(s)=\sqrt{s^{3}}$ for simplicity and convenience for all cases.

Case 1. Let the moving particle has a unit timelike binormal vector $\mathbf{e}_{(2)}$ with the given characterization in (1). Then, we have the following demonstration the variation of the energy for different Frenet vector fields with respect to time.

$$
\text { (s from }-1.9 \text { to } 2.8 \text { ) }
$$



Figure 1. Energy on the particle in tangent, normal, binormal vector field and energy its own vector field in $\mathcal{S}_{1}^{3}$ is pictured, respectively.

Case 2. Let the moving particle has a unit timelike normal vector $\mathbf{e}_{(1)}$ with the given characterization in (2). Then, Then, we have the following demonstration the variation of the energy for different Frenet vector fields with respect to time.


Figure 2. Energy on the particle in tangent, normal, binormal vector field and energy its own vector field in $\mathcal{S}_{1}^{3}$ is pictured, respectively.

Case 3. Let the moving particle has a unit timelike tangent vector $\mathbf{e}_{(0)}$ with the given characterization in (3). Then, we have the following demonstration the variation of the energy for different Frenet vector fields with respect to time.


Figure 3. Energy on the particle in tangent, normal, binormal vector field and energy its own vector field in $\mathcal{S}_{1}^{3}$ is pictured, respectively.

## 6. CONCLUSIONS

In this study, we studied energy on the particle in the Frenet vector fields in De-Sitter spacetime considering kinematics of the particle. Furthermore, we set a connection between energy on the particle in these vector fields and elastica of bending energy functional. This is important for our future work since a simple characterization on the energy of a vector field can be described as it is up to constants, in other words, it is square $L_{2}$ norm of the vector field's covariant derivative. Thanks to this definition, not only we will correlate the concept of the energy with a volume for the moving particle in these vector fields in space.

As is known, elastic energy may occur by applying different forces besides bending such as twisting and stretching. In our next studies, we also determine the correlation between energy on the particle in each Frenet vector field in case of the particle has the feature of stretching or twisting.

Computing the energy on the moving particle has a wide range of application in the theoretical and applied physics. In the future studies, it will also be investigated the energy on the moving particle in different force fields by obtaining dynamics of the particle in space including work done and force acting on the particle. We believe that this study also will lead
up to further research on the relativistic dynamics of the particle in different spacetimes in terms of computing the energy on a particle in different force fields.

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