# MORGAN-VOYCE MATRIX METHOD FOR GENERALIZED FUNCTIONAL INTEGRO-DIFFERENTIAL EQUATIONS OF VOLTERRA-TYPE 

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#### Abstract

This study deals with the generalized linear Volterra-type functional integro-differential equations with mixed delays. A combination between matrix-collocation method and Morgan-Voyce polynomials is developed to solve these type equations. In addition, an error analysis technique is given to improve the obtained solutions. Numerical examples are performed to confirm the efficiency and validity of the method. The comparisons are made in tables and figures. The discussions show that the method is fast and precise.


Keywords: Morgan-Voyce polynomials; Integro-differential equation; Matrix method; Collocation points; Error Analysis.

## 1. INTRODUCTION

In this study, the Morgan-Voyce matrix-collocation method, which was previously used in [1], is developed to solve the linear Volterra-type functional integro-differential equation with mixed delays and variable bounds in the general form

$$
\begin{equation*}
\sum_{k=0}^{m} P_{k}(x) y^{(k)}(x)+\sum_{i=0}^{m_{1}} \sum_{j=0}^{m_{2}} Q_{i j}(x) y^{(i)}\left(\lambda_{i j} x+\mu_{i j}\right)=g(x)+\sum_{r=0}^{m_{3}} \sum_{s=0}^{m_{4}} \psi_{r s} \int_{u_{r s}(x)}^{v_{r s}(x)} K_{r s}(x, t) y^{(r)}\left(\alpha_{r s} t+\beta_{r s}\right) d t \tag{1}
\end{equation*}
$$

subject to hybrid conditions

$$
\begin{equation*}
\sum_{k=0}^{m-1}\left(a_{l k} y^{(k)}(a)+b_{l k} y^{(k)}(b)\right)=\lambda_{l}, l=0,1, \ldots, m-1, m \geq m_{1}, m_{3} \tag{2}
\end{equation*}
$$

where $P_{k}(x), Q_{i j}(x), g(x), u_{r s}(x), v_{r s}(x)$, and $K_{r s}(x, t)$ are defined on $a \leq x, t \leq b$ $\left(a \leq u_{r s}(x)<v_{r s}(x) \leq b\right) ; a_{l k}, b_{l k}, \lambda_{l}$, and $\psi_{r s}$ are appropriate constants.

[^0]Linear functional integro-differential equation derived from Eq. (1) is a combination of integral, differential, and integro-differential-(difference or delay) equations. These type functional equations govern many physical phenomena arising in applied sciences, such as mathematics, engineering, electrodynamics, heat and mass transfer, mechanics, physics, biology etc. [1-20].

Recently, the mentioned functional equations have been investigated by many authors for their analytical and numerical solutions. Most of these equations have no analytical solution and so numerical methods are required to obtain their approximate solutions. Some of them are Laguerre approach method [8], cubic b-spline scaling function technique [9], Dickson matrix collocation method [10-12], varionational iteration method [13], Taylor polynomial matrix collocation method [14, 21, 22], homotopy analysis method [15], Bessel collocation method [16], Lagrange and Chebyshev interpolation method [17], LegendreGauss collocation method [18] and Morgan-Voyce polynomial method [1].

Our aim in this study is to develop a matrix method based on the first kind MorganVoyce polynomials and to seek the approximate solution of the problem (1)-(2) in the form (see [1])

$$
\begin{equation*}
y(x) \cong y_{N}(x)=\sum_{n=0}^{N} y_{n} b_{n}(x), a \leq x \leq b, \tag{3}
\end{equation*}
$$

where $a_{n}$ are unknown coefficients and $b_{n}(x)$ is Morgan-Voyce polynomials, which are defined to be (see [23-27])

$$
\begin{equation*}
b_{n}(x)=\sum_{j=0}^{n}\binom{n+j}{n-j} x^{j}, \quad n \in \mathbb{N}, a \leq x \leq b \tag{4}
\end{equation*}
$$

Morgan-Voyce polynomials $b_{n}(x)$, which are appeared in electrical ladder networks, contain the following important properties [23-27]:

- The polynomials $b_{n}(x)$ expressed by Eq. (4) are recursively defined by the relation

$$
\begin{equation*}
b_{n}(x)=(x+2) b_{n-1}(x)-b_{n-2}(x), \quad n \geq 2 \tag{5}
\end{equation*}
$$

with $b_{0}(x)=1$ and $b_{1}(x)=x+1$.

- By Eqs. (4) and (5), the polynomials $y(x)=b_{n}(x),(n=0,1, \ldots)$ read the second order differential equation

$$
x(x+4) y^{\prime \prime}(x)+2(x+1) y^{\prime}(x)-n(n+1) y(x)=0 .
$$

- The generating function of $b_{n}(x)$ is of the form

$$
b(x, t)=\sum_{n=0}^{\infty} b_{n}(x) t^{n}=(1-t) B(x, t)
$$

where

$$
B(x, t)=\left[1-\left(x t+2 t-t^{2}\right)\right]^{-1}
$$

## 2. MATERIALS AND METHODS

### 2.1. MAIN MATRIX RELATIONS

In this section, the matrix forms related to Eq. (1) and the conditions (2) are presented. For this purpose, the approximate solution (3) of Eq. (1) can be written as the truncated Morgan-Voyce series form (see [1])

$$
\begin{equation*}
y_{N}(x)=\boldsymbol{b}(x) \boldsymbol{Y} \tag{6}
\end{equation*}
$$

where

$$
\boldsymbol{b}(x)=\left[b_{0}(x), b_{1}(x), \ldots, b_{N}(x)\right] \text { and } \boldsymbol{Y}=\left[y_{0}, y_{1}, \ldots, y_{N}\right]^{T}
$$

Here, using the first kind Morgan-Voyce polynomials $b_{N}(x)$, the matrix form $\boldsymbol{b}(x)$ can also be written as (see [1])

$$
\begin{equation*}
\boldsymbol{b}(x)=\boldsymbol{X}(x) \boldsymbol{M} \tag{7}
\end{equation*}
$$

where

$$
\boldsymbol{X}(x)=\left[1, x, \ldots, x^{N}\right]
$$

and

$$
\boldsymbol{M}=\left[\begin{array}{cccc}
\binom{0}{0} & \binom{1}{1} & \binom{2}{2} & \cdots
\end{array}\binom{N}{N}\right]
$$

Besides, the relation between the matrix $\boldsymbol{X}(x)$ and its derivative $\boldsymbol{X}^{(k)}(x)$ yields

$$
\begin{equation*}
\boldsymbol{X}^{(k)}(x)=\boldsymbol{X}(x) \boldsymbol{T}^{k}, \quad k=0,1, \ldots, m, \tag{8}
\end{equation*}
$$

where

$$
\boldsymbol{T}=\left[\begin{array}{ccccc}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 2 & \ldots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & \ldots & N \\
0 & 0 & 0 & \ldots & 0
\end{array}\right],
$$

and $\boldsymbol{T}^{0}$ is a identity matrix in the dimension $(N+1) \times(N+1)$. By using the matrix relations (6), (7) and (8), it holds that

$$
\begin{equation*}
y^{(k)}(x) \cong y_{N}^{(k)}(x)=\boldsymbol{b}^{(k)}(x) \boldsymbol{Y}=\boldsymbol{X}^{(k)}(x) \boldsymbol{M} \boldsymbol{Y}=\boldsymbol{X}(x) \boldsymbol{T}^{k} \boldsymbol{M} \boldsymbol{Y} \tag{9}
\end{equation*}
$$

and substituting $x \rightarrow \lambda_{i j} x+\mu_{i j}, k \rightarrow i$ and $x \rightarrow \alpha_{r s} t+\beta_{r s}, k \rightarrow r$ into the matrix relation (9), respectively. Then the matrix relations become

$$
\begin{equation*}
y_{N}^{(i)}\left(\lambda_{i j} x+\mu_{i j}\right)=\boldsymbol{X}(x) \boldsymbol{L}\left(\lambda_{i j}, \mu_{i j}\right) \boldsymbol{T}^{i} \boldsymbol{M} \boldsymbol{Y}, \tag{10}
\end{equation*}
$$

and

$$
\begin{equation*}
y_{N}^{(r)}\left(\alpha_{r s} t+\beta_{r s}\right)=\boldsymbol{X}(x) \boldsymbol{L}\left(\alpha_{r s}, \beta_{r s}\right) \boldsymbol{T}^{r} \boldsymbol{M} \boldsymbol{Y} \tag{11}
\end{equation*}
$$

Note that the matrix $X(\lambda x+\mu)$ can be formed as [8, 11, 12]

$$
\boldsymbol{X}(\lambda x+\mu)=\boldsymbol{X}(x) \boldsymbol{L}(\lambda, \mu)
$$

where

$$
\left.\left.L(\lambda, \mu)=\left[\begin{array}{cccc}
\binom{0}{0} \lambda^{0} \mu^{0} & \binom{1}{0} \lambda^{0} \mu^{1} & \binom{2}{0} \lambda^{0} \mu^{2} & \cdots \\
0 & \binom{1}{1} \lambda^{1} \mu^{0} & \binom{2}{0} \lambda^{1} \mu^{1} & \ldots \\
0 & \binom{N}{1} \lambda^{N} \mu^{N-1} \\
0 & 0 & \binom{2}{2} \lambda^{2} \mu^{0} & \ldots \\
\vdots & \vdots & \vdots & \binom{N}{2} \lambda^{2} \mu^{N-2} \\
0 & 0 & 0 & \cdots
\end{array}\right] \begin{array}{c}
N \\
N
\end{array}\right) \lambda^{N} \mu^{0}\right] .
$$

On the other hand, the kernel functions $K_{r s}(x, t), r=0,1, \ldots, m_{3}$ and $s=0,1, \ldots, m_{4}$ can be approximated by the following series form $[8,11,12,14]$

$$
\begin{equation*}
K_{r s}(x, t)=\sum_{m=0}^{N} \sum_{n=0}^{N} k_{m n}^{r, s} x^{m} t^{n}, \tag{12}
\end{equation*}
$$

where

$$
k_{m n}^{r, s}=\frac{1}{m!n!} \frac{\partial^{m+n} K_{r s}(0,0)}{\partial x^{m} \partial t^{n}} ; m, n=0,1, \ldots, N
$$

Thus, Eq. (12) can be stated as the matrix form [ $8,11,12,14]$

$$
\begin{equation*}
K_{r s}(x, t)=\boldsymbol{X}(x) \boldsymbol{K}_{r s} \boldsymbol{X}^{T}(t), \tag{13}
\end{equation*}
$$

where

$$
\boldsymbol{K}_{r s}=\left[k_{m n}^{r, s}\right], \quad \boldsymbol{X}(x)=\left[1, x, \ldots, x^{N}\right], \quad \boldsymbol{X}(t)=\left[1, t, \ldots, t^{N}\right]
$$

Inserting the matrix relations (9), (10), (11), and (13) into Eq. (1), then the matrix equation is obtained as

$$
\begin{align*}
\sum_{k=0}^{m} P_{k}(x) \boldsymbol{X}(x) \boldsymbol{T}^{k} \boldsymbol{M} \boldsymbol{Y} & +\sum_{i=0}^{m_{1}} \sum_{j=0}^{m_{2}} Q_{i j}(x) \boldsymbol{X}(x) \boldsymbol{L}\left(\lambda_{i j}, \mu_{i j}\right) \boldsymbol{T}^{i} \boldsymbol{M} \boldsymbol{Y}=g(x)  \tag{14}\\
& +\sum_{r=0}^{m_{s}} \sum_{s=0}^{m_{s}} \psi_{r s} \boldsymbol{X}(x) \boldsymbol{K}_{r s} \boldsymbol{S}_{r s}(x) \boldsymbol{L}\left(\alpha_{r s}, \beta_{i r s}\right) \boldsymbol{T}^{r} \boldsymbol{M} \boldsymbol{Y}
\end{align*}
$$

where

$$
\boldsymbol{S}_{r s}(x)=\left[s_{p q}^{r s}(x)\right]=\int_{u_{s s}(x)}^{v_{r s}(x)} \boldsymbol{X}^{T}(t) \boldsymbol{X}(t) d t=\left[\frac{\left(v_{r s}(x)\right)^{p+q+1}-\left(u_{r s}(x)\right)^{p+q+1}}{p+q+1}\right], p, q=0,1, \ldots, N
$$

### 2.2. MORGAN-VOYCE MATRIX METHOD

In this section, the matrix method is established using matrix equation (14) and the collocation points, which are defined to be

$$
\begin{equation*}
x_{l}=a+\frac{b-a}{N} l, \quad l=0,1, \ldots, N \tag{15}
\end{equation*}
$$

Substituting the collocation points (15) into the matrix equation (14), a system of the matrix relations are obtained as

$$
\binom{\sum_{k=0}^{m} P_{k}\left(x_{l}\right) \boldsymbol{X}\left(x_{l}\right) \boldsymbol{T}^{k}+\sum_{i=0}^{m_{1}} \sum_{j=0}^{m_{2}} Q_{i j}\left(x_{l}\right) \boldsymbol{X}\left(x_{l}\right) \boldsymbol{L}\left(\lambda_{i j}, \mu_{i j}\right) \boldsymbol{T}^{i}}{-\sum_{r=0}^{m_{3}} \sum_{s=0}^{m_{s}} \psi_{r s} \boldsymbol{X}\left(x_{l}\right) \boldsymbol{K}_{r s} \boldsymbol{S}_{r s}\left(x_{l}\right) \boldsymbol{L}\left(\alpha_{r s}, \beta_{i r s}\right) \boldsymbol{T}^{r}} \boldsymbol{M} \boldsymbol{Y}=g\left(x_{l}\right)
$$

Then, the compact form of this system can be expressed as

$$
\begin{equation*}
\left(\sum_{k=0}^{m} \boldsymbol{P}_{k} \boldsymbol{X} \boldsymbol{T}^{k}+\sum_{i=0}^{m_{1}} \sum_{j=0}^{m_{2}} \boldsymbol{Q}_{i j} \boldsymbol{X} \boldsymbol{L}\left(\lambda_{i j}, \mu_{i j}\right) \boldsymbol{T}^{i}-\sum_{r=0}^{m_{3}} \sum_{s=0}^{m_{s}} \psi_{r s} \overline{\boldsymbol{X}} \overline{\boldsymbol{K}}_{r s} \overline{\boldsymbol{S}}_{r s} \overline{\boldsymbol{L}}\left(\alpha_{r s}, \beta_{r s}\right) \overline{\boldsymbol{T}}^{r}\right) \boldsymbol{M} \boldsymbol{Y}=\boldsymbol{G} \tag{16}
\end{equation*}
$$

where

$$
\begin{aligned}
\boldsymbol{P}_{k} & =\operatorname{diag}\left[P_{k}\left(x_{0}\right), P_{k}\left(x_{1}\right), \ldots, P_{k}\left(x_{N}\right)\right]_{(N+1) \times(N+1)} \\
\boldsymbol{Q}_{i j} & =\operatorname{diag}\left[Q_{i j}\left(x_{0}\right), Q_{i j}\left(x_{1}\right), \ldots, Q_{i j}\left(x_{N}\right)\right]_{(N+1) \times(N+1)}
\end{aligned}
$$

$$
\begin{gathered}
\boldsymbol{X}=\left[\begin{array}{c}
X\left(x_{0}\right) \\
X\left(x_{1}\right) \\
\vdots \\
X\left(x_{N}\right)
\end{array}\right]=\left[\begin{array}{ccccc}
1 & x_{0} & x_{0}^{2} & \cdots & x_{0}^{N} \\
1 & x_{1} & x_{1}^{2} & \cdots & x_{1}^{N} \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
1 & x_{N} & x_{N}^{2} & \cdots & x_{N}^{N}
\end{array}\right], \overline{\boldsymbol{X}}=\operatorname{diag}\left[\boldsymbol{X}\left(x_{0}\right), \boldsymbol{X}\left(x_{1}\right), \ldots, \boldsymbol{X}\left(x_{N}\right)\right]_{(N+1) \times(N+1)^{2}}, \\
\overline{\boldsymbol{K}}_{r s}=\operatorname{diag}\left[K_{r s}, K_{r s}, \ldots, K_{r s}\right]_{(N+1)^{2} \times(N+1)^{2}}, \\
\overline{\boldsymbol{S}}_{r s}=\operatorname{diag}\left[\boldsymbol{S}_{r s}\left(x_{0}\right), \boldsymbol{S}_{r s}\left(x_{1}\right), \ldots, \boldsymbol{S}_{r s}\left(x_{N}\right)\right]_{(N+1)^{2} \times(N+1)^{2}}, \\
\overline{\boldsymbol{L}}\left(\alpha_{r s}, \beta_{r s}\right)=\operatorname{diag}\left[\boldsymbol{L}\left(\alpha_{r s}, \boldsymbol{\beta}_{r s}\right), \boldsymbol{L}\left(\alpha_{r s}, \beta_{r s}\right), \ldots, \boldsymbol{L}\left(\alpha_{r s}, \beta_{r s}\right)\right]_{(N+1)^{2} \times(N+1)^{2}}, \\
\overline{\boldsymbol{T}}^{r}=\left[\begin{array}{llll}
\boldsymbol{T}^{r} & \boldsymbol{T}^{r} & \cdots & \boldsymbol{T}^{r}
\end{array}\right]_{(N+1) \times(N+1)^{2}}^{T}, \text { and } \boldsymbol{G}=\left[\begin{array}{lll}
g\left(x_{0}\right) & g\left(x_{1}\right) & \cdots \\
g\left(x_{N}\right)
\end{array}\right]_{1 \times(N+1)}^{T}
\end{gathered}
$$

The fundamental matrix equation (16) of Eq. (1) can be given as

$$
\begin{equation*}
\boldsymbol{W} \boldsymbol{Y}=\boldsymbol{G} \text { or }[\boldsymbol{W}: \boldsymbol{G}], \tag{17}
\end{equation*}
$$

where

$$
\boldsymbol{W}=\left\{\sum_{k=0}^{m} \boldsymbol{P}_{k} \boldsymbol{X} \boldsymbol{T}^{k}+\sum_{i=0}^{m_{1}} \sum_{j=0}^{m_{2}} \boldsymbol{Q}_{i j} \boldsymbol{X} \boldsymbol{L}\left(\lambda_{i j}, \mu_{i j}\right) \boldsymbol{T}^{i}-\sum_{r=0}^{m_{s}} \sum_{s=0}^{m_{4}} \psi_{r s} \overline{\boldsymbol{X}} \overline{\boldsymbol{K}}_{r s} \overline{\boldsymbol{S}}_{r s} \overline{\boldsymbol{L}}\left(\alpha_{r s}, \beta_{r s}\right) \overline{\boldsymbol{T}}^{r}\right\} \boldsymbol{M} .
$$

On the other hand, by the relation (9), we can express the matrix relation of the conditions (2) as

$$
\begin{equation*}
\boldsymbol{U}_{j} \boldsymbol{Y}=\left[\lambda_{j}\right] \text { or }\left[\boldsymbol{U}_{j} ; \lambda_{j}\right] \tag{18}
\end{equation*}
$$

such that

$$
\boldsymbol{U}_{j}=\sum_{k=0}^{m}\left(a_{k j} \boldsymbol{X}(a)+b_{k j} \boldsymbol{X}(b)\right) \boldsymbol{T}^{k} \boldsymbol{M}=\left[u_{j o}, u_{j 1}, \ldots, u_{j N}\right], j=0,1, \ldots, m-1 .
$$

Consequently, in order to obtain the first kind Morgan-Voyce polynomial solution of Eq. (1) subject to the conditions (2), we replace $m$ row(s) matrices in Eq. (18) by any $m$ row(s) of the matrix equation (17). Thereby we obtain an augmented matrix system as

$$
\begin{equation*}
\tilde{\boldsymbol{W}} \boldsymbol{Y}=\tilde{\boldsymbol{G}} \text { or }[\tilde{W}: \tilde{\boldsymbol{G}}] . \tag{19}
\end{equation*}
$$

When the rank of the system (19) yields $N+1$, it holds that $\boldsymbol{Y}=(\tilde{\boldsymbol{W}})^{-1} \tilde{\boldsymbol{G}}$. The approximate solution is obtained after inserting the matrix $\boldsymbol{Y}$ into Eq. (3).

## 3. ERROR ANALYSIS VIA RESIDUAL FUNCTION

In recent years, an efficient residual error analysis has been developed to correct the approximate solutions for some methods [8, 11, 12, 28]. Here, the present method is applied to this error analysis. Thus, the Morgan-Voyce polynomial solution $y_{N}(x)$ can be improved. By substituting this solution into Eq. (1), the residual function stands for

$$
\begin{equation*}
R_{N}(x)=L\left[y_{N}(x)\right]-g(x), \tag{20}
\end{equation*}
$$

where

$$
L\left[y_{N}(x)\right]=\sum_{k=0}^{m} P_{k}(x) y_{N}^{(k)}(x)+\sum_{i=0}^{m_{1}} \sum_{j=0}^{m_{2}} Q_{i j}(x) y_{N}^{(i)}\left(\lambda_{i j} x+\mu_{i j}\right)-\sum_{r=0}^{m_{3}} \sum_{s=0}^{m_{4}} \psi_{r s} \int_{u_{r s}(x)}^{v_{s x}(x)} K_{r s}(x, t) y_{N}^{(r)}\left(\alpha_{r s} t+\beta_{r s}\right) d t .
$$

By considering the exact solution $y(x)$, the error function can be written as

$$
\begin{equation*}
e_{N}(x)=y(x)-y_{N}(x) \tag{21}
\end{equation*}
$$

By Eqs. (1), (2), (20), and (21), the error equation is

$$
\begin{equation*}
L\left[e_{N}(x)\right]=L[y(x)]-L\left[y_{N}(x)\right]=-R_{N}(x), \tag{22}
\end{equation*}
$$

subject to the homogeneous hybrid conditions

$$
\begin{equation*}
\sum_{k=0}^{m-1}\left[a_{l k} e_{N}^{(k)}(a)+b_{l k} e_{N}^{(k)}(b)\right]=0 \tag{23}
\end{equation*}
$$

Eqs. (22) and (23) constitute an error problem, which can be readily solved via the procedure given in Section 2.2. Thus, the estimated error function is of the form

$$
e_{N, M}(x)=\sum_{n=0}^{M} y_{n}^{*} b_{n}(x),(M>N)
$$

such that the corrected Morgan-Voyce polynomial solution $y_{N, M}(x)=y_{N}(x)+e_{N, M}(x)$ and the corrected error function $E_{N, M}(x)=y(x)-y_{N, M}(x)$ are obtained.

Furthermore, let us give an error computation $\delta_{N}$, which is used to specifically compare the results in this study. $\delta_{N}$ is defined to be (see [20])

$$
\delta_{N}=\sqrt{\frac{1}{N} \sum_{l=0}^{N} e_{N}^{2}\left(x_{l}\right)}
$$

where $x_{l}$ is the collocation points (15). We can also compute the error $\delta_{N, M}$, which is as follows:

$$
\delta_{N, M}=\sqrt{\frac{1}{M} \sum_{l=0}^{M} E_{N, M}^{2}\left(x_{l}\right)},
$$

where $E_{N, M}(x)$ is the corrected error function.

## 4. NUMERICAL EXAMPLES

In this section, five illustrative examples are treated to test the efficiency and validity of the present method. Numerical calculations have been performed using a computer program module on Matlab and Mathematica softwares.

Example 1. [3] Consider the second-order neutral differential equation with proportional delays

$$
\begin{equation*}
y^{\prime \prime}(x)-\frac{3}{4} y(x)-y\left(\frac{x}{2}\right)-y^{\prime}\left(\frac{x}{2}\right)-\frac{1}{2} y^{\prime \prime}\left(\frac{x}{2}\right)=-x^{2}-x+1, \quad x \in[0,1], \tag{24}
\end{equation*}
$$

subject to the initial conditions $y(0)=y^{\prime}(0)=0$. By Eq. (16), the fundamental matrix equation of the problem can be constructed as

$$
\begin{equation*}
\left(\sum_{k=0}^{2} P_{k} \boldsymbol{X} \boldsymbol{T}^{k}+\sum_{i=0}^{2} \sum_{j=0}^{0} \boldsymbol{Q}_{i j} \boldsymbol{X} \boldsymbol{L}\left(\lambda_{i j}, \mu_{i j}\right) \boldsymbol{T}^{i}\right) \boldsymbol{M} \boldsymbol{Y}=\boldsymbol{G} \tag{25}
\end{equation*}
$$

where

$$
\begin{gathered}
\boldsymbol{P}_{0}=\left[\begin{array}{rrr}
-\frac{3}{4} & 0 & 0 \\
0 & -\frac{3}{4} & 0 \\
0 & 0 & -\frac{3}{4}
\end{array}\right], \boldsymbol{P}_{1}=\left[\begin{array}{lll}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right], \boldsymbol{P}_{2}=\left[\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right], \boldsymbol{X}=\left[\begin{array}{c}
\boldsymbol{X}(0) \\
\boldsymbol{X}\left(\frac{1}{2}\right) \\
\boldsymbol{X}(1)
\end{array}\right]=\left[\begin{array}{lll}
1 & 0 & 0 \\
1 & \frac{1}{2} & \frac{1}{4} \\
1 & 1 & 1
\end{array}\right], \\
\boldsymbol{T}^{1}=\left[\begin{array}{lll}
0 & 1 & 0 \\
0 & 0 & 2 \\
0 & 0 & 0
\end{array}\right], \boldsymbol{T}^{2}=\left[\begin{array}{lll}
0 & 0 & 2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right], \boldsymbol{Q}_{00}=\left[\begin{array}{ccc}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{array}\right], \boldsymbol{L}\left(\lambda_{00}, \mu_{00}\right)=\boldsymbol{L}\left(\frac{1}{2}, 0\right)=\left[\begin{array}{lll}
1 & 0 & 0 \\
0 & \frac{1}{2} & 0 \\
0 & 0 & \frac{1}{4}
\end{array}\right], \\
\boldsymbol{Q}_{10}=\left[\begin{array}{ccc}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{array}\right], \boldsymbol{M}=\left[\begin{array}{ccc}
1 & 1 & 1 \\
0 & 1 & 3 \\
0 & 0 & 1
\end{array}\right], \boldsymbol{Q}_{20}=\left[\begin{array}{ccc}
-\frac{1}{2} & 0 & 0 \\
0 & -\frac{1}{2} & 0 \\
0 & 0 & -\frac{1}{2}
\end{array}\right], \\
\boldsymbol{L}\left(\lambda_{10}, \mu_{10}\right)=\boldsymbol{L}\left(\boldsymbol{\lambda}_{20}, \mu_{20}\right)=\boldsymbol{L}\left(\frac{1}{2}, 0\right)=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \frac{1}{2} & 0 \\
0 & 0 & \frac{1}{4}
\end{array}\right] .
\end{gathered}
$$

The matrix system of Eq. (25) is computed as

$$
[\boldsymbol{W} ; \boldsymbol{G}]=\left[\begin{array}{rrl}
-\frac{7}{4} & -\frac{11}{4} & -\frac{15}{4}: 1  \tag{26}\\
-\frac{7}{4} & -\frac{27}{8} & -\frac{51}{8}: \frac{1}{4} \\
-\frac{7}{4} & -4 & -\frac{19}{2}:-1
\end{array}\right]
$$

and the matrix forms for the initial conditions are obtained as

$$
\left[\boldsymbol{U}_{0} ; \lambda_{0}\right]=\left[\begin{array}{lllll}
1 & 1 & 1 & : & 0
\end{array}\right] \text { and }\left[\boldsymbol{U}_{1} ; \lambda_{1}\right]=\left[\begin{array}{lllll}
0 & 1 & 3 & : & 0 \tag{27}
\end{array}\right] .
$$

By using the system (26) and the condition matrices (27), we obtain the solution $y_{2}(x)=x^{2}$, which is the exact solution. While the present method directly reaches the exact solution, a direct method based on Chebyshev cardinal functions [3] has yielded the error $6.16 e-64$ on avarege.

Example 2. Consider the Volterra type functional integral equation with variable bounds

$$
\begin{equation*}
y(x)=g(x)+\int_{0}^{0.05 x} \cos (x-t) y(t) d t+\int_{0}^{0.95 x} \sin (x-t) y(t) d t, x \in[0,1], \tag{28}
\end{equation*}
$$

where

$$
g(x)=-0.25 \cos (0.9 x)+(1.25-0.025 x) \cos (x)+0.25 \sin (0.9 x)-(0.25+0.475 x) \sin (x)
$$

The exact solution of Eq. (28) is $y(x)=\cos (x)$. The fundamental matrix equation yields

$$
\left\{\boldsymbol{P}_{0} \boldsymbol{X}-\psi_{00} \overline{\boldsymbol{X}} \overline{\boldsymbol{K}}_{00} \overline{\boldsymbol{S}}_{00} \overline{\boldsymbol{L}}\left(\alpha_{00}, \beta_{00}\right) \overline{\boldsymbol{T}}^{0}-\psi_{01} \overline{\boldsymbol{X}} \overline{\boldsymbol{K}}_{01} \overline{\boldsymbol{S}}_{01} \overline{\boldsymbol{L}}\left(\alpha_{01}, \beta_{01}\right) \overline{\boldsymbol{T}}^{0}\right\} \boldsymbol{M} \boldsymbol{Y}=\boldsymbol{G} .
$$

After following the procedure of our method, we can solve the augmented matrix and thus the Morgan-Voyce polynomial solutions are obtained in the form (3) for $N=2,3,5,6$ on $[0,1]$. The comparison of the exact and Morgan-Voyce polynomial solutions are given in Fig. 1 and Table 1. The error computations are simulated in Fig. 2.


Figure 1. Comparison of the exact solution and Morgan-Voyce polynomial solution of Example 2.


Figure 2. Comparison of the actual absolute errors of Example 2.

Table 1. Numerical results of the exact solution, Morgan-Voyce polynomial solution, and mean absolute errors for Example 2.

| x | $y\left(x_{i}\right)=\cos \left(x_{i}\right)$ | $N=2$ | $N=3$ | $N=5$ | $N=6$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | 1 |  | 1 | 1 | 1 |
| 0.1 | 0.995 | 0.98613 | 0.99554 | 0.995 | 0.995 |
| 0.2 | 0.98007 | 0.96759 | 0.98053 | 0.98007 | 0.98007 |
| 0.3 | 0.95534 | 0.94438 | 0.95547 | 0.95534 | 0.95534 |
| 0.4 | 0.92106 | 0.9165 | 0.9209 | 0.92106 | 0.92106 |
| 0.5 | 0.87758 | 0.88395 | 0.87732 | 0.87758 | 0.87758 |
| 0.6 | 0.82534 | 0.84673 | 0.82526 | 0.82533 | 0.82534 |
| 0.7 | 0.76484 | 0.80484 | 0.76524 | 0.76483 | 0.76485 |
| 0.8 | 0.69671 | 0.75828 | 0.69778 | 0.69668 | 0.69673 |
| 0.9 | 0.62161 | 0.70705 | 0.6234 | 0.62153 | 0.62167 |
| 1 | 0.5403 | 0.65115 | 0.54262 | 0.54013 | 0.54044 |
| Mean abs.error | n.a. | 0.03295 | $6.58 e-04$ | $2.74 e-05$ | $2.20 e-05$ |

Example 3. [19] Consider the second-order neutral Volterra integro-differential equation

$$
y^{\prime \prime}(x)-(x+1) y^{\prime}(x)+y(x)=(x+1)(\sin (x)-\sin 1)+\int_{-1}^{x}\left[x y(t)+y^{\prime}(t)+t y^{\prime \prime}(t)\right] d t, x \in[-1,1]
$$

subject to the initial conditions $y(-1)=\cos 1$ and $y^{\prime}(-1)=\sin 1$. The exact solution of this problem is $y(x)=\cos x$. This problem is solved for $N=10,11,14$ and $M=11,12,15$ on $[-1,1]$. In Table 2, the absolute and corrected absolute errors of the present method are compared with $L^{2}$ and $L^{\infty}$ errors founds in [19]. The actual and estimated absolute errors are also illustrated in Fig. 5. It is clearly seen from Figs. 3-4 and Table 2 that the Morgan-Voyce polynomial solutions are very close to the exact solution. In addition, when CPU time in Table 3 is investigated, the well-equipped computer is not required to employ the present method.

Table 2. Comparison of the present numerical results with other existing ones for Example 3.

| $x_{i}$ | $\left\|e_{14}\left(x_{i}\right)\right\|$ | $\left\|E_{14,15}\left(x_{i}\right)\right\|$ | $\left\|e_{16}\left(x_{i}\right)\right\|$ | $L^{2}-L^{\infty}$ <br> errors $[19]$ |
| ---: | :---: | :--- | :--- | :---: |
| -1.0 | $3.33 e-016$ | $2.22 e-016$ | $2.22 e-016$ | $4.21 e-014\left(L^{2}, N=14\right)$ |
| -0.5 | $3.33 e-016$ | $5.55 e-016$ | $4.44 e-016$ | $8.96 e-014\left(L^{\infty}, N=14\right)$ |
| 0.0 | $6.66 e-016$ | $4.44 e-016$ | $4.44 e-016$ | $4.19 e-014\left(L^{2}, N=16\right)$ |
| 0.5 | $7.77 e-016$ | 0 | $4.44 e-016$ | $9.30 e-014\left(L^{\infty}, N=16\right)$ |
| 1.0 | $2.43 e-014$ | $1.02 e-014$ | $1.55 e-014$ | - |

Table 3. CPU time(s) of the present method in terms of the different $\boldsymbol{N}$ for Example 3.

| $y_{N}(x)$ | $y_{4}(x)$ | $y_{10}(x)$ | $y_{14}(x)$ | $y_{16}(x)$ |
| :---: | :---: | :---: | :---: | :---: |
| Time (sec.) | 0.24960 | 0.49920 | 1.24801 | 1.88761 |



Figure 3. Comparison of the exact solution and Morgan-Voyce polynomial solution on $[-1,1]$ for Example 3.


Figure 4. Oscillation of the exact solution and Morgan-Voyce polynomial solution on $[-1,11]$ for Example 3.


Figure 5. Comparison of the actual and estimated absolute errors for Example 3.
Example 4. [20] Consider the Fredholm-Volterra integral equation

$$
y(x)=e^{-x}-e^{x}(h(x)-1)-\int_{0}^{1} e^{x+h(t)} y(h(t)) d t+\int_{0}^{h(x)} e^{x+t} y(t) d t, x \in[0,1]
$$

where $h(x)=\{x / 2, x\}$. The exact solution of this equation is $y(x)=e^{-x}$. The equation is treated for $N=5,8$ and $M=9$. In Table 4, the present values of the error $\delta_{N}$ are compared with other existing methods [20]. The Morgan-Voyce polynomial solutions are obtained by the present method in seconds as seen in Table 5. As seen from Tables 6 and 7, the better numerical results are obtained, as both $N$ and $M$ are increased. In addition, the comparisons of the residual functions are plotted in Fig. 6.

Table 4. Comparison of the errors $\delta_{N}$ for Example 4.

| $h(x)$ | $x / 2$ |  |  | $x$ |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| N, M | Present meth. | Lagrange meth. [20] | Taylor meth. [20] | Present meth. | Lagrange meth. [20] | Taylor meth. [20] |
| 5 | $3.78 e-004$ | -00 | 3.36e-004 | $5.93 e-005$ | $4.03 e-007$ |  |
| 8 | $5.72 e-007$ | $6.74 e-007$ | 5.77e-007 | $7.80 e-008$ | $9.50 e-007$ | $2.53 e-007$ |
| 8,9 | $1.27 e-013$ | n.a. | n.a. | $1.73 e-013$ | n.a. | n.a. |

Table 5. CPU time(s) of the present method in terms of different $N$ for Example 4.

| $h(x)$ | $x / 2$ | $x$ |  |  |
| :---: | :---: | :---: | :---: | :---: |
| $y_{N}(x)$ | $y_{5}(x)$ | $y_{8}(x)$ | $y_{5}(x)$ | $y_{8}(x)$ |
| Time (sec.) | 0.17160 | 0.65520 | 0.14040 | 0.53040 |

Table 6. Comparison of the actual, estimated and corrected absolute errors for Example 4 with $\boldsymbol{h}(\boldsymbol{x})=\boldsymbol{x} / \mathbf{2}$.

| $x_{i}$ | $\left\|e_{5}\left(x_{i}\right)\right\|$ | $\left\|e_{8,9}\left(x_{i}\right)\right\|$ | $\left\|E_{8,9}\left(x_{i}\right)\right\|$ |
| :--- | :--- | :--- | :--- |
| 0 | $6.494 e-007$ | $1.241 e-010$ | $1.513 e-013$ |
| 0.2 | $7.710 e-007$ | $1.779 e-010$ | $1.433 e-013$ |
| 0.4 | $3.661 e-006$ | $3.850 e-010$ | $1.242 e-013$ |
| 0.6 | $4.795 e-005$ | $2.038 e-008$ | $1.401 e-013$ |
| 0.8 | $2.442 e-004$ | $2.407 e-007$ | $2.548 e-013$ |
| 1.0 | $8.067 e-004$ | $1.529 e-006$ | $2.031 e-013$ |

Table 7. Comparison of the actual, estimated and corrected absolute errors for Example 4 with $\boldsymbol{h}(\boldsymbol{x})=\boldsymbol{x}$.

| $x_{i}$ | $\left\|e_{5}\left(x_{i}\right)\right\|$ | $\left\|e_{8,9}\left(x_{i}\right)\right\|$ | $\left\|E_{8,9}\left(x_{i}\right)\right\|$ |
| :--- | :---: | :---: | :---: |
| 0 | $2.269 e-006$ | $2.907 e-009$ | $1.426 e-013$ |
| 0.2 | $1.933 e-006$ | $4.393 e-009$ | $1.476 e-013$ |
| 0.4 | $8.056 e-006$ | $8.834 e-009$ | $8.615 e-014$ |
| 0.6 | $3.880 e-005$ | $3.056 e-008$ | $1.178 e-013$ |
| 0.8 | $1.266 e-004$ | $1.393 e-007$ | $2.914 e-013$ |
| 1.0 | $2.514 e-013$ | $5.155 e-011$ | $4.441 e-016$ |



Figure 6. Comparison of the residual functions for Example 4 with $\boldsymbol{h}(\boldsymbol{x})=\boldsymbol{x} / 2$.

Example 5. Consider the second order Volterra type integro-differential-difference equation with variable bounds

$$
x y^{\prime \prime}(x+1)=g(x)+\int_{x-1}^{x}(x-t) y^{\prime}(t-1) d t, x \in[0,1]
$$

where

$$
g(x)=\cos (1-x)-\cos (2-x)-\sin (2-x)-x \sin (x+1)
$$

subject to the initial conditions $y(0)=0$ and $y^{\prime}(0)=1$. The exact solution is $y(x)=\sin (x)$.
All comparison results for the relations between exact and approximate solutions are illustrated in Fig. 7. It is easily seen from Tables 8 and 9 that the approximate Morgan-Voyce polynomial solutions are improved via the residual error analysis.


Figure 7. Comparison of the exact solution and Morgan-Voyce polynomial solution for Example 5.
Table 8. Comparison of the actual and corrected absolute errors for Example 5.

| $x_{i}$ | $\left\|e_{5}\left(x_{i}\right)\right\|$ | $\left\|e_{9}\left(x_{i}\right)\right\|$ | $\left\|E_{9,10}\left(x_{i}\right)\right\|$ |
| :--- | :--- | :--- | :--- |
| 0 | 0 | 0 | 0 |
| 0.2 | $7.52 e-005$ | $3.50 e-005$ | $3.30 e-006$ |
| 0.4 | $7.90 e-005$ | $1.40 e-004$ | $2.28 e-005$ |
| 0.6 | $9.93 e-004$ | $3.10 e-004$ | $7.00 e-005$ |
| 0.8 | $3.08 e-003$ | $5.33 e-004$ | $1.52 e-004$ |
| 1.0 | $6.57 e-003$ | $7.92 e-004$ | $2.71 e-004$ |

Table 9. Comparison of the errors $\delta_{N}$ for Example 5.

| $\delta_{N}$ | $\delta_{4}$ | $\delta_{5}$ | $\delta_{7}$ | $\delta_{9}$ | $\delta_{9,10}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Errors | $1.03 e-02$ | $3.28 e-03$ | $1.17 e-03$ | $4.16 e-04$ | $1.26 e-04$ |

## 5. CONCLUSIONS

In this study, a matrix-collocation method based on Morgan-Voyce polynomials has been developed to obtain the approximate solution of Volterra type functional integrodifferential equations of Eq. (1) type. To explain the efficiency and validity of the present method, five examples have been treated. The obtained results have been demonstrated in the tables and figures. It is clearly observed from the comparisons of the exact solution and Morgan-Voyce polynomial solution (Figs. 1, 3, 4 and 7, and Table 1) together with the errors (Tables 1, 2, 4, 6-9, Figs. 2 and 5) that the present method is an effective, fast and accurate numerical technique to handle Eq. (1). Numerical values have been immediately obtained, as seen from Tables 3 and 5 . The fact that when the computation limit $N$ is taken sufficiently large then the precision of the method increases. The present method also contains a simple algorithm, which leads to easily encode its computer program routine. It is evident that the present method can be readily modified to deal with partial differential equations.

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