ORIGINAL PAPER

GENERALIZATION OF PTOLEMY'S THEOREM

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Abstract. *We establish a generalization of Ptolemy's theorem and we also deduce Ptolemy's inequality and its strengthened version from this generalization.*
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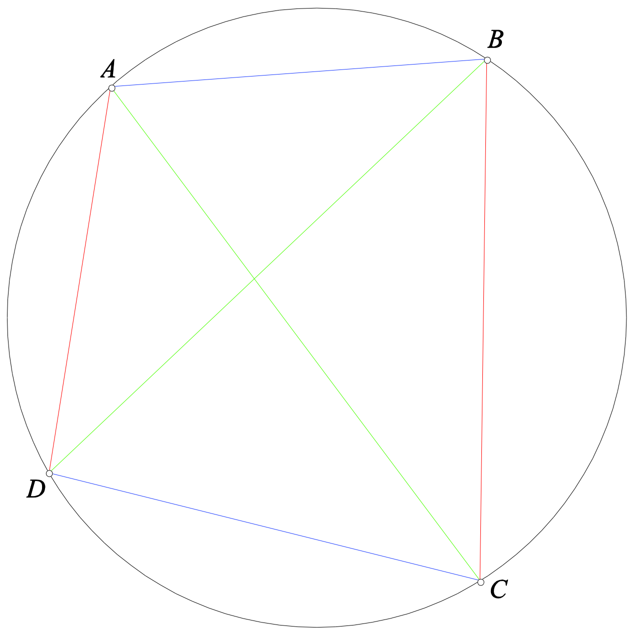
1. INTRODUCTION

Ptolemy's theorem is known as one of the most famous theorems in Euclidean geometry. It also has a lot of application. It usually uses to to prove the important theorems and problems in Euclidean geometry [1-4].

Theorem 1 (Ptolemy's theorem). If the vertices of cyclic quadrilateral are A, B, C, and D in order, then

$$AC \cdot BD = BC \cdot AD + AB \cdot CD.$$



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Theorem 2 (Converse of Ptolemy's theorem). In convex quadrilateral $ABCD$. If

$$AC \cdot BD = BC \cdot AD + AB \cdot CD$$

then $ABCD$ can be inscribed in a circle.

Theorem 3 (Ptolemy's inequality). Given quadrilateral $ABCD$ then

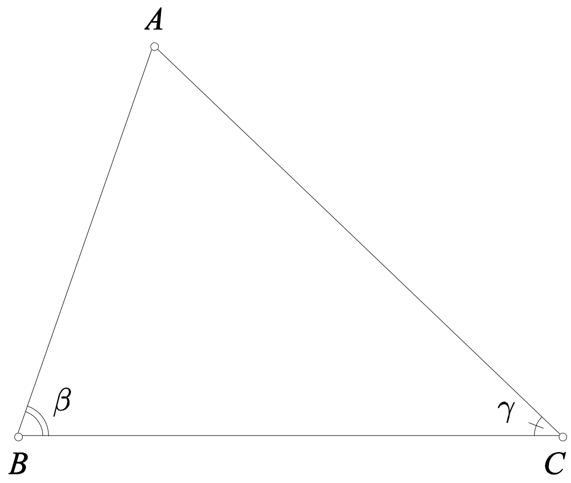
$$AD \cdot BD + AB \cdot CD \geq AB \cdot CD.$$

2. MAIN THEOREM

In this section, we shall give a generalization of Ptolemy's theorem. We also give three consequences of this main theorem. The first, we introduce a lemma.

Lemma. Let A, B, and C be three points on plane. Denote by $\angle ABC = \beta$ and $\angle ACB = \gamma$ then

$$BC = AB \cos\beta + AC \cos\gamma.$$



Proof. Apply law of cosine, we have

$$\cos\beta = \frac{BA^2 + BC^2 - CA^2}{2 \cdot BA \cdot BC}$$

and

$$\cos\gamma = \frac{CA^2 + CB^2 - AB^2}{2 \cdot CA \cdot CB}.$$

From this, we deduce

$$
\begin{aligned}
AB\cos\beta + AC\cos\gamma &= AB\cdot\frac{BA^2+BC^2-CA^2}{2\cdot BA\cdot BC} + AC\cdot\frac{CA^2+CB^2-AB^2}{2\cdot CA\cdot CB}\\
&= \frac{BA^2+BC^2-CA^2+CA^2+CB^2-AB^2}{2\cdot BC}\\
&= BC.
\end{aligned}
$$

We complete the proof.

Theorem 4 (Generalization of Ptolemy's theorem). Let $ABCD$ be a convex quadrilateral. Denote the absolute difference of angles by

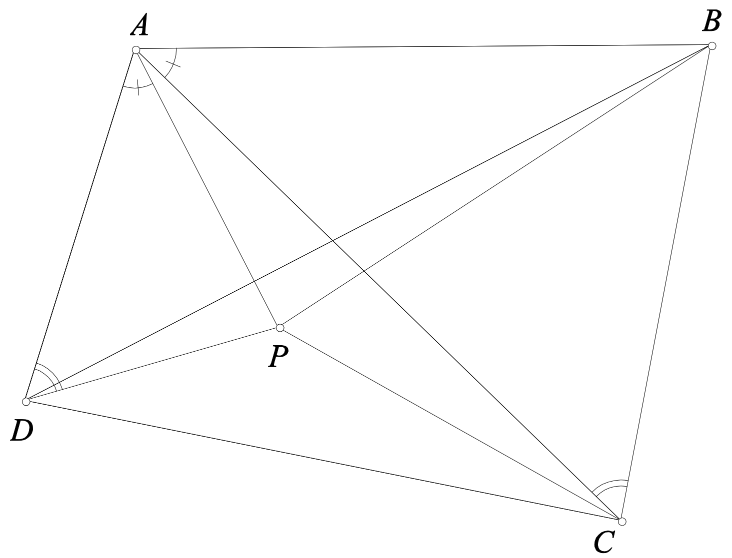
$$\psi = |\angle ADB - \angle ACB| = |\angle DAC - \angle DBC|$$

and

$$\phi = |\angle ABD - \angle ACD| = |\angle BAC - \angle BDC|.$$

Then

$$AC\cdot BD = AD\cdot BC\cdot\cos\psi + AB\cdot CD\cdot\cos\phi.$$



Proof. Let P is the point interior quadrilateral $ABCD$ such that ΔAPD is similar to ΔABC. Thus, also ΔAPB is simlar to ΔADC. We have the product of lengths

$$AD\cdot BC = PD\cdot AC \Rightarrow PD = \frac{AD\cdot BC}{AC} \quad (1)$$

and

$$AB \cdot CD = PB \cdot AC \Rightarrow PB = \frac{AB \cdot CD}{AC}. \quad (2)$$

Note that,

$$\angle PBD = |\angle PBA - \angle ABD| = |\angle ACD - \angle ABD| = \phi$$

and

$$\angle PDB = |\angle PDA - \angle ADB| = |\angle ACB - \angle ADB| = \psi.$$

Now apply lemma for three points P, B, and D with $\angle PBD = \phi$ and $\angle PDB = \psi$, we have

$$BD = PD \cdot \cos\psi + PB \cdot \cos\phi. \quad (3)$$

From (1), (2), and (3), we obtain

$$BD = \frac{AD \cdot BC}{AC} \cdot \cos\psi + \frac{AB \cdot CD}{AC} \cdot \cos\phi$$

or

$$AC \cdot BD = AD \cdot BC \cdot \cos\psi + AB \cdot CD \cdot \cos\phi.$$

We complete our proof.

3. SOME CONSEQUENCES

In this section, we give some consequences of the thereom 4.

Consequence 1 (Ptolemy's inequality). Let $ABCD$ be a quarilateral then

$$AC \cdot BD \leq AD \cdot BC + AB \cdot CD.$$

Proof. Because $\cos\psi \leq 1$ and $\cos\phi \leq 1$, from Theorem 4, we have

$$AC \cdot BD = AD \cdot BC \cdot \cos\psi + AB \cdot CD \cdot \cos\phi \leq AD \cdot BC + AB \cdot CD.$$

We complete the proof.

Consequence 2 (Ptolemy's theorem and its converse). Let $ABCD$ be a quarilateral then $ABCD$ is cyclic if only if

$$AC \cdot BD = AD \cdot BC + AB \cdot CD.$$

Proof. If $ABCD$ is cyclic then

$$\psi = |\angle ADB - \angle ACB| = |\angle DAC - \angle DBC| = 0$$

and

$$\phi = |\angle ABD - \angle ACD| = |\angle BAC - \angle BDC| = 0,$$

thus $\cos\psi = \cos\phi = 1$. From Theorem 4, we have

$$AC \cdot BD = AD \cdot BC \cdot \cos\psi + AB \cdot CD \cdot \cos\phi = AD \cdot BC + AB \cdot CD.$$

If $AC \cdot BD = AD \cdot BD + AB \cdot CD$, from Theorem 4, we have

$$AD \cdot BC \cdot \cos\psi + AB \cdot CD \cdot \cos\phi = AC \cdot BD = AD \cdot BC + AB \cdot CD \geq AD \cdot BC \cdot \cos\psi + AB \cdot CD \cdot \cos\phi.$$

Equality occurs iff $\cos\psi = \cos\phi = 1$, thus $\psi = \phi = 0$. We deduce

$$|\angle ADB - \angle ACB| = |\angle DAC - \angle DBC| = 0$$

and

$$|\angle ABD - \angle ACD| = |\angle BAC - \angle BDC| = 0$$

so $ABCD$ is cyclic. We complete the proof.

Consequence 3 (Strengthened version of the Ptolemy's inequality using secant function). Let $ABCD$ be a convex quadrilateral. Denote the absolute difference of angles by

$$\psi = |\angle ADB - \angle ACB| = |\angle DAC - \angle DBC|$$

and

$$\phi = |\angle ABD - \angle ACD| = |\angle BAC - \angle BDC|.$$

Let $k = \min\{\sec\psi, \sec\phi\}$ then

$$AC \cdot BD \leq k \cdot AC \cdot BC \leq AD \cdot BD + AB \cdot CD.$$

Proof. Obvious $k = \min\{\sec\psi, \sec\phi\} \geq 1$ so $AC \cdot BD \leq k \cdot AC \cdot BC$.
Note that, $AD \cdot BC > 0$ and $AB \cdot CD > 0$, so from Theorem 4, we have

$$AC \cdot BD = AD \cdot BC \cdot \cos\psi + AB \cdot CD \cdot \cos\phi \leq (AD \cdot BC + AB \cdot CD) \cdot \max\{\cos\psi, \cos\phi\},$$

thus

$$\begin{aligned} AD\cdot BC + AB\cdot CD &\geq \frac{AC\cdot BD}{\max\{\cos\psi, \cos\phi\}} \\ &= AC\cdot BD\cdot \min\{\frac{1}{\cos\psi}, \frac{1}{\cos\phi}\} \\ &= AC\cdot BD\cdot \min\{\sec\psi, \sec\phi\} \\ &= k\cdot AC\cdot BD. \end{aligned}$$

We complete the proof.

4. CONCLUSIONS

The paper has proved the theorem 4 which can be consider as a generalization of Ptolemy's theorem. We also point out that Ptolemy's inequality, Ptolemy's theorem and its converse and the strengthened version of Ptolemy's inequality as the consequences of our general theorem.

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