ORIGINAL PAPER

SURFACE FAMILY WITH A COMMON NATURAL GEODESIC LIFT OF A TIMELIKE CURVE IN MINKOWSKI 3-SPACE

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Abstract. *In the present paper, we find a surface family possessing the natural lift of a given timelike curve as a geodesic in Minkowski 3-space. We express necessary and sufficient conditions for the given curve such that its natural lift is a geodesic on any member of the surface family. Finally, we illustrate the method with some examples.*
Keywords: *Surface family, natural lift, geodesic.*

1. INTRODUCTION

Minkowski 3-space IR_1^3 is the vector space IR^3 equipped with the Lorentzian inner product g given by

$$g(X,X) = -x_1^2 + x_2^2 + x_3^2$$

where $X = (x_1, x_2, x_3) \in IR^3$. A vector $X = (x_1, x_2, x_3) \in IR^3$ is said to be timelike if $g(X,X) < 0$, spacelike if $g(X,X) > 0$ or $X = 0$ and lightlike (or null) if $g(X,X) = 0$ and $x \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in IR_1^3 can locally be timelike, spacelike or null (lightlike), if all of its velocity vectors $\alpha'(s)$ are respectively timelike, spacelike or null (lightlike), for every $s \in I \subset IR$.

A lightlike vector X is said to be positive (resp. negative) if and only if $x_1 > 0$ (resp. $x_1 < 0$) and a timelike vector X is said to be positive (resp. negative) if and only if $x_1 > 0$ (resp. $x_1 < 0$). The norm of a vector X is defined by $\|X\|_{IL} = \sqrt{|g(X,X)|}$ [1].

The vectors $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3) \in R_1^3$ are orthogonal if and only if $g(X,X) = 0$ [2].

Now let X and Y be two vectors in IR_1^3, then the Lorentzian cross product is given by [3]

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$$X \times Y = \begin{vmatrix} \vec{e_1} & -\vec{e_2} & -\vec{e_3} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \left(x_2y_3 - x_3y_2,\ x_1y_3 - x_3y_1,\ x_2y_1 - x_1y_2 \right).$$

We denote by $\{T(s), N(s), B(s)\}$ the moving Frenet frame along the curve α. Then T, N and B are the tangent, the principal normal and the binormal vector of the curve α, respectively.

Let α be a unit speed timelike curve with curvature κ and torsion τ. So, T is a timelike vector field, N and B are spacelike vector fields. For these vectors, we can write

$$T \times N = -B, \quad N \times B = T, \quad B \times T = -N,$$

where $\times$ is the Lorentzian cross product in IR_1^3. The binormal vector field $B(s)$ is the unique spacelike unit vector fleld perpendicular to the timelike plane $\{T(s), N(s)\}$ at every point $\alpha(s)$ of α, such that $\{T, N, B\}$ has the same orientation as IR_1^3. Then, Frenet formulas are given by [4]

$$T' = \kappa N,\ N' = \kappa T + \tau B,\ B' = -\tau N.$$

Let α be a unit speed spacelike curve with spacelike binormal. Now, T and B are spacelike vector fields and N is a timelike vector field. In this situation,

$$T \times N = -B, \quad N \times B = -T, \quad B \times T = N.$$

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$$T' = \kappa N,\ N' = -\kappa T + \tau B,\ B' = \tau N.$$

Lemma 1. Let X and Y be nonzero Lorentz orthogonal vectors in IR_1^3. If X is timelike, then Y is spacelike [2].

Lemma 2. Let X and Y be positive (negative) timelike vectors in IR_1^3. Then

$$g(X,Y) \leq \|X\|\|Y\|$$

whit equality if and only if X and Y are linearly dependent [2].

Lemma 3.

i) Let X and Y be positive (negative) timelike vectors in IR_1^3. By Lemma 2, there is a unique nonnegative real number $\phi(X,Y)$ such that

$$g(X,Y) = \|X\|\|Y\|\cosh\phi(X,Y).$$

The Lorentzian timelike angle between X and Y is defined to be $\phi(X,Y)$.

ii) Let X and Y be spacelike vectors in IR_1^3 that span a spacelike vector subspace. Then we have

$$|g(X,Y)| \leq \|X\|\|Y\|.$$

Hence, there is a unique real number $\phi(X,Y)$ between 0 and π such that

$$g(X,Y) = \|X\|\|Y\|\cos\phi(X,Y).$$

$\phi(X,Y)$ is defined to be the Lorentzian spacelike angle between X and Y [2].

iii) Let X and Y be spacelike vectors in IR_1^3 that span a timelike vector subspace. Then, we have

$$g(X,Y) > \|X\|\|Y\|.$$

Hence, there is a unique positive real number $\phi(X,Y)$ between 0 and π such that

$$|g(X,Y)| = \|X\|\|Y\|\cosh\phi(X,Y).$$

$\phi(X,Y)$ is defined to be the Lorentzian timelike angle between X and Y [2].

iv) Let X be a spacelike vector and Y be a positive timelike vector in IR_1^3. Then there is a unique nonnegative real number $\phi(X,Y)$ such that

$$|g(X,Y)| = \|X\|\|Y\|\sinh\phi(X,Y).$$

$\phi(X,Y)$ is defined to be the Lorentzian timelike angle between X and Y [2].

Let P be a surface in IR_1^3 and let $\alpha : I \to P$ be a parametrized curve. α is called an integral curve of X if

$$\frac{d}{ds}\left(\alpha(s)\right) = X\left(\alpha(s)\right), \ \left(\text{for all } t \in I\right),$$

where X is a smooth tangent vector field on P [1]. We have

$$TP = \bigcup_{p\in P} T_pP = \chi(P),$$

where T_pP is the tangent space of P at p and $\chi(P)$ is the space of tangent vector fields on P.

For any parametrized curve $\alpha : I \to P$, $\bar{\alpha} : I \to TP$ is given by

$$\bar{\alpha}(s) = \left(\alpha(s), \alpha'(s)\right) = \alpha'(s)|_{\alpha(s)} \tag{1}$$

is called the natural lift of α on TP [5] .

Let $\alpha(s)$, $L_1 \le s \le L_2$, be an arc length timelike curve. Then, the natural lift $\bar{\alpha}$ of α is a spacelike curve with timelike or spacelike binormal. We have following relations between the Frenet frame $\{T(s), N(s), B(s)\}$ of α and the Frenet frame $\{\bar{T}(s), \bar{N}(s), \bar{B}(s)\}$ of $\bar{\alpha}$.

a) Let the natural lift $\bar{\alpha}$ of α is a spacelike curve with timelikebinormal.

i) If the Darboux vector W of the curve α is a spacelike vector, then we have

$$\begin{pmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \cosh\theta & 0 & \sinh\theta \\ \sinh\theta & 0 & \cosh\theta \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix} \tag{2}$$

ii) If W is a timelike vector, then we have

$$\begin{pmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \sinh\theta & 0 & \cosh\theta \\ \cosh\theta & 0 & \sinh\theta \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix} \tag{3}$$

b) Let the natural lift $\bar{\alpha}$ of α is a spacelike curve with spacelikebinormal.

i) If W is a spacelike vector, then we have

$$\begin{pmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \cosh\theta & 0 & \sinh\theta \\ -\sinh\theta & 0 & -\cosh\theta \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix} \tag{4}$$

ii) If W is a timelike vector, then we have

$$\begin{pmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \sinh\theta & 0 & \cosh\theta \\ -\cosh\theta & 0 & -\sinh\theta \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix} \tag{5}$$

2. SURFACE FAMILY WITH A COMMON NATURAL GEODESIC LIFT OF A TIMELIKE CURVE IN MINKOWSKI 3-SPACE

Suppose we are given a 3-dimensional timelike curve $\alpha(s),\ L_1 \le s \le L_2,$ in which s is the arc length and $\|\alpha''(s)\| \ne 0,\ L_1 \le s \le L_2$ .Let $\bar{\alpha}(s),\ L_1 \le s \le L_2,$ be the natural lift of the given curve $\alpha(s)$. Now, $\bar{\alpha}$ is a spacelike curve with timelike or spacelike binormal.

Surface family that interpolates $\bar{\alpha}(s)$ as a common curve is given in the parametric form as

$$P(s,t) = \bar{\alpha}(s) + u(s,t)\bar{T}(s) + v(s,t)\bar{N}(s) + w(s,t)\bar{B}(s) \tag{6}$$

where $u(s,t),\ v(s,t)$ and $w(s,t)$ are C^1 functions, called *marching-scale functions,* and $\{\bar{T}(s), \bar{N}(s), \bar{B}(s)\}$ is the Frenet frame of the curve $\bar{\alpha}$.

Remark 4. Observe that choosing different marching-scale functions yields different surfaces possessing $\bar{\alpha}(s)$ as a common curve.

Our goal is to find the necessary and sufficient conditions for which the curve $\bar{\alpha}(s)$ is isoparametric and geodesic on the surface $P(s,t)$. Firstly, as $\bar{\alpha}(s)$ is an isoparametric curve on the surface $P(s,t)$, there exists a parameter $t_0 \in [T_1, T_2]$ such that

$$u(s,t_0) = v(s,t_0) = w(s,t_0) \equiv 0,\ L_1 \le s \le L_2,\ T_1 \le t_0 \le T_2 \tag{7}$$

Secondly the curve $\bar{\alpha}$ is a geodesic on the surface $P(s,t)$ if and only if along the curve the normal vector field $n(s,t_0)$ of the surface is parallel to the principal normal vector field $\bar{N}$ of the curve $\bar{\alpha}$. The normal vector of $P(s,t)$ can be written as

$$n(s,t) = \frac{\partial P(s,t)}{\partial s} \times \frac{\partial P(s,t)}{\partial t}.$$

Along the curve $\bar{\alpha},$ one can obtain the normal vector $n(s,t_0)$ using Eqns. (6-7) with an appropriate equation in Eqns. (2-5). It has one of the following forms:

i) if $\bar{\alpha}$ is a spacelike curve with timelikebinormal and Darboux vector W is spacelike or timelike, then

$$n(s,t_0)=\kappa\left[\frac{\partial w}{\partial t}(s,t_0)\bar{N}(s)+\frac{\partial v}{\partial t}(s,t_0)\bar{B}(s)\right], \tag{8}$$

ii) if $\bar{\alpha}$ is a spacelike curve with spacelike binormal and Darboux vector W is spacelike, then

$$n(s,t_0)=-\kappa\left[\frac{\partial w}{\partial t}(s,t_0)\bar{N}(s)+\frac{\partial v}{\partial t}(s,t_0)\bar{B}(s)\right], \tag{9}$$

iii) if $\bar{\alpha}$ is a spacelike curve with spacelikebinormal and Darboux vector W timelike, then

$$n(s,t_0)=\kappa\left[\frac{\partial w}{\partial t}(s,t_0)\bar{N}(s)-\frac{\partial v}{\partial t}(s,t_0)\bar{B}(s)\right], \tag{10}$$

where κ is the curvature of the curve α.

Since $\kappa(s)\neq 0,\ L_1\leq s\leq L_2,$ the curve $\bar{\alpha}$ is a geodesic on the surface $P(s,t)$ if and only if
$\frac{\partial w}{\partial t}(s,t_0)\neq 0,\ \frac{\partial v}{\partial t}(s,t_0)=0.$

So, we can present:

Theorem 5: Let $\alpha(s),\ L_1\leq s\leq L_2,$ be a unit speed timelike curve with nonvanishing curvature and $\bar{\alpha}(s),\ L_1\leq s\leq L_2,$ be its natural lift. $\bar{\alpha}$ is a geodesic on the surface (6) if and only if

$$\begin{cases} u(s,t_0)=v(s,t_0)=w(s,t_0)=\frac{\partial v}{\partial t}(s,t_0)\equiv 0, \\ \qquad\qquad \frac{\partial w}{\partial t}(s,t_0)\neq 0, \end{cases} \tag{11}$$

where $L_1\leq s\leq L_2,\ T_1\leq t,\ t_0\leq T_2\ \ (t_0$ fixed).

Corollary 6: Let $\alpha(s),\ L_1\leq s\leq L_2,$ be a unit speed timelike curve with nonvanishing curvature and $\bar{\alpha}(s),\ L_1\leq s\leq L_2,$ be its natural lift. If

$$\begin{cases} u(s,t)=w(s,t)=t-t_0,\ v(s,t)\equiv 0 \\ \qquad\qquad or \\ u(s,t)=v(s,t)\equiv 0,\ w(s,t)=t-t_0, \end{cases} \tag{12}$$

where $L_1\leq s\leq L_2,\ T_1\leq t,\ t_0\leq T_2\ (t_0 \text{ fixed})$ then (6) is a ruled surface possessing $\bar{\alpha}$ as a geodesic.

Proof: By taking marching scale functions as $\mathrm{u}(\mathrm{s},\mathrm{t})=\mathrm{w}(\mathrm{s},\mathrm{t})=\mathrm{t}-\mathrm{t}_0,\ \mathrm{v}(\mathrm{s},\mathrm{t})\equiv 0$ or $u(s,t)=v(s,t)\equiv 0,\ w(s,t)=t-t_0,$ the surface (6) takes the form

$$P(s,t)=\bar{\alpha}(s)+(t-t_0)\left[\bar{T}(s)+\bar{B}(s)\right]$$

or

$$P(s,t)=\bar{\alpha}(s)+(t-t_0)\bar{B}(s),$$

which is a ruled surface satisfying Eqn. (11).

3. EXAMPLES

Example 1

Let $\alpha(s)=(\sinh s,0,\cosh s)$ be a timelike curve. It is easy to show that

$$T(s)=(\cosh s,0,\sinh s),$$
$$N(s)=(\sinh s,0,\cosh s),$$
$$B(s)=(0,-1,0).$$

The natural lift of the curve α is $\bar{\alpha}(s)=(\cosh s,0,\sinh s)$ and its Frenet vectors

$$\bar{T}(s)=(\sinh s,0,\cosh s),$$
$$\bar{N}(s)=(\cosh s,0,\sinh s),$$
$$\bar{B}(s)=(0,1,0).$$

Choosing marching scale functions as $u(s,t)=t,\ v(s,t)\equiv 0,\ w(s,t)=\sinh t$ Eqn. (11) is satisfied and we obtain the surface

$$P_1(s,t)=(\cosh s+t\sinh s,\sinh t,t\cosh s+\sinh s).$$

$-1\le s\le 1,\ -1\le t\le 0,$ possessing $\bar{\alpha}$ as a common natural geodesic lift (Fig. 1).

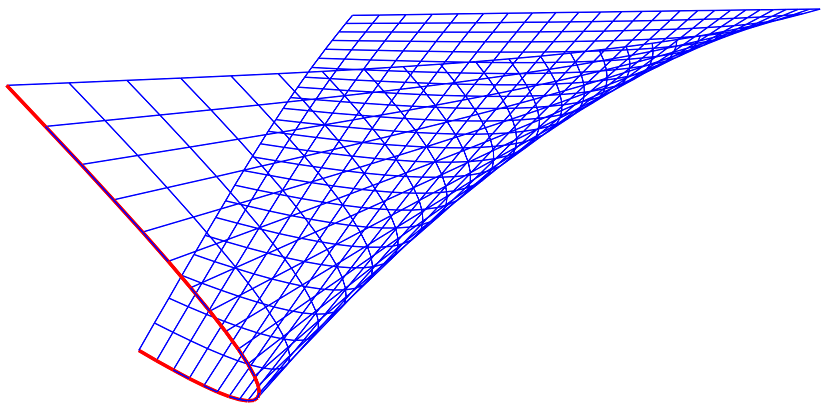

Figure 1. $P_1(s,t)$ **as a member of the surface family with a common natural geodesic lift** $\bar{\alpha}$ **.**

For the same curve, if we choose $u(s,t)\equiv 0,\ v(s,t)=t-\sinh t,\ w(s,t)=(\sinh s)\sinh t$ we get the surface

$$P_2(s,t)=\left(\cosh s+(t-\sinh t)\cosh s,(\sinh s)\sinh t,(t-\sinh t)\sinh s+\sinh s\right),$$

$0<s\le 1,\ -1\le t\le 1,$ satisfying Eqn. (11) and accepting $\bar{\alpha}$ as a common natural geodesic lift (Fig. 2).

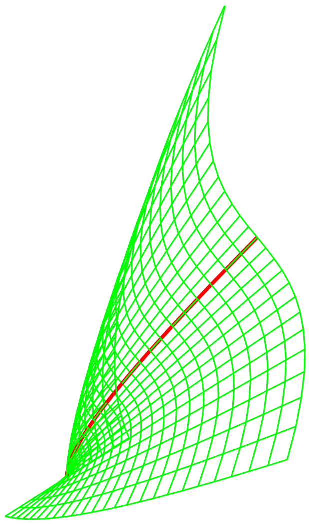

Figure 2. $P_2(s,t)$ **as a member of the surface family with a common natural geodesic lift** $\bar{\alpha}$ **.**

Example 2

Given the arclength timelike curve $\alpha(s)=\left(\tfrac{5}{3}s,\tfrac{4}{9}\cos 3s,\tfrac{4}{9}\sin 3s\right)$ its Frenet apparatus are

$$T(s)=\left(\frac{5}{3},-\frac{4}{3}\sin 3s,\frac{4}{3}\cos 3s\right),$$
$$N(s)=(0,-\cos 3s,-\sin 3s),$$
$$B(s)=\left(-\frac{4}{3},\frac{5}{3}\sin 3s,-\frac{5}{3}\cos 3s\right).$$

The natural lift of the curve α is $\bar{\alpha}(s)=\left(\tfrac{5}{3},-\tfrac{4}{3}\sin 3s,\tfrac{4}{3}\cos 3s\right)$ and its Frenet vectors

$$\bar{T}(s)=(0,-\cos 3s,-\sin 3s),$$
$$\bar{N}(s)=(0,\sin 3s,-\cos 3s),$$
$$\bar{B}(s)=(-1,0,0).$$

If we let marching scale functions as $u(s,t)=v(s,t)\equiv 0,\ w(s,t)=t,$ we get the ruled surface

$$P_3(s,t)=\left(\frac{5}{3}-t,-\frac{4}{3}\sin(3s),\frac{4}{3}\cos(3s)\right),$$

$-1.1\le s\le 1,\ -1\le t\le 1,$ satisfying Eqn. (12) and passing through $\bar{\alpha}$ as a common natural geodesic lift (Fig. 3).

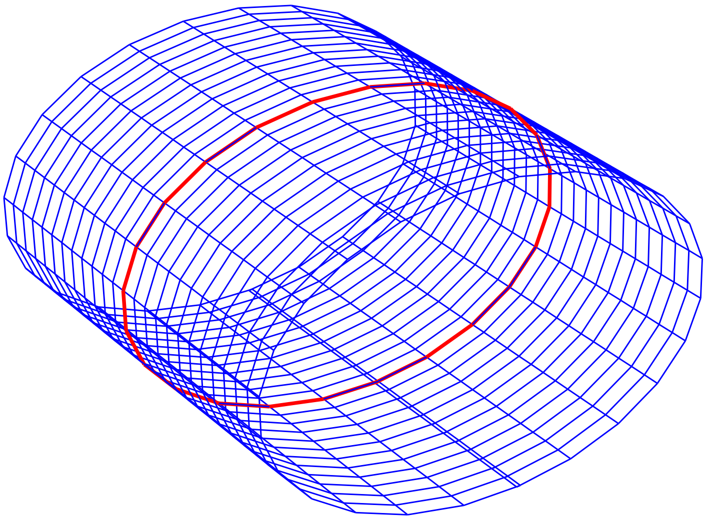

Figure 3. Ruled surface $P_3(s,t)$ as a member of the surface family with a common natural geodesic lift $\bar{\alpha}$.

For the same curve, if we choose $u(s,t)\equiv 0$, $v(s,t)=t^2e^s$, $w(s,t)=t\ln s$ we obtain the surface

$$P_4(s,t)=\left(\frac{5}{3}-t\ln s,-\frac{4}{3}\sin(3s)+t^2e^s\sin(3s),\frac{4}{3}\cos(3s)-t^2e^s\cos(3s)\right),$$

$1<s\leq 2,\ 0\leq t\leq 1,$ satisfying Eqn. (11) and possessing $\bar{\alpha}$ as a common natural geodesic lift (Fig. 4).

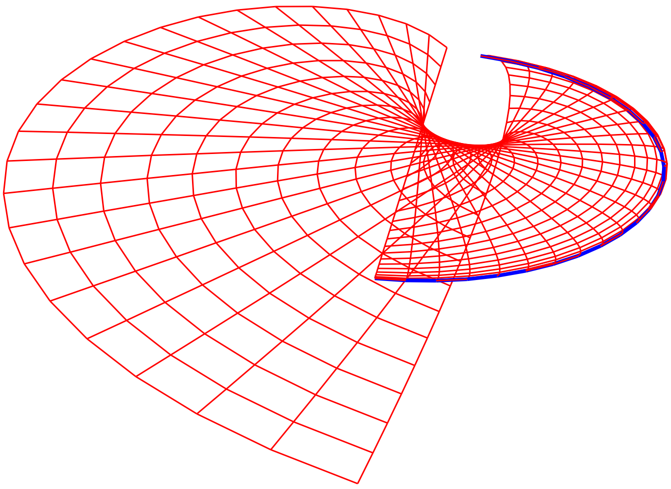

Figure 4. $P_4(s,t)$ as a member of the surface family with a common natural geodesic lift $\bar{\alpha}$.

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