ORIGINAL PAPER

CERTAIN NEW CLASS OF FNITE INTEGRALS INVOLVING A PRODUCT OF GENERALIZED BESSEL FUNCTIONS

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Abstract.*The present paper is influenced by the work of Khan and Ghayasuddin see [1]. In this paper, first we derive two generalized integrals involving the product of n Bessel functions by employing the result of Lavoie and Trottier see [2] and then we obtain a set of several new integral formulae as special cases of our main results.*

Keywords: *Generalized Bessel function, Kampe′ de Fe′riet function, Srivastava and Daoust function and Lavoie and Trottier integral.*

1. INTRODUCTION

As the special functions play an indispensable role in many branches of applied mathematics and sciences, a number of authors see [3-11] have studied their properties in many aspects.

In recent years, we have seen several integral transforms involving various kind of special functions which have many applications in the field of physics and engineering (for example, in the field of plasma physics, radio physics, astro physics etc.). So, in a sequel of such type of works, in this paper we extend the results of Khan and Ghayasuddin see [1].

In order to present our main finding, we recall here the definition of some well known functions as follows:

The generalized Bessel function of first kind of order ν, where $z \in C \setminus \{0\}, b, c, \nu \in C,$ is defined by see [1, 6]

$$w_{\nu,c}^{b}(z) = \sum_{m=0}^{\infty} \frac{(-c)^{m} \left(z/2\right)^{\nu+2m}}{m!\Gamma\left(\nu + m + \frac{1+b}{2}\right)}, \quad (1.1)$$

where C denote the set of complex numbers.

If we consider b = c = 1 in (1.1), then $w_{\nu,c}^{b}(z)$ reduces to the Bessel function of first kind $J_{\nu}(z)$ and if we consider b = 1; c = −1 in (1.1) then $w_{\nu,c}^{b}(z)$ reduces to the modified Bessel function of purely imaginary argument $I_{\nu}(z)$ see [12, 13].

The Srivastava and Daoust multivariable hypergeometric function is given as follows see [9, 13]:

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$$F^{p:q_1;q_2\ldots;q_r}_{l:m_1,m_2\ldots,m_r}\left[\begin{matrix}\left(a_j:\alpha^1_j;..;\alpha^{(r)}_j\right)_{1,p}:(c^1_j,r^1_j)_{1,q_1};\ldots;(c^{(r)}_j,r^{(r)}_j)_{1,q_r}\\ \left(b_j:\beta^1_j;\ldots;\beta^{(r)}_j\right)_{1,l}:(d^1_j,\delta^1_j)_{1,m_1};\ldots;(d^{(r)}_j,\delta^{(r)}_j)_{1,m_r}\end{matrix}\; x_1,x_2\ldots x_r\right]$$

$$=\sum_{n_1,n_2\ldots n_r=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{n_1\alpha^1_j+\ldots n_r\alpha^{(r)}}\prod_{j=1}^{q_1}(c^1_j)_{n_1r^1_j}\cdots\prod_{j=1}^{q_r}(c^{(r)}_j)_{n_{(r)}r^{(r)}_j}}{\prod_{j=1}^{l}(b_j)_{n_1\beta^1_j+\ldots n_r\beta^{(r)}}\prod_{j=1}^{m_1}(d^1_j)_{n_1\delta^1_j}\cdots\prod_{j=1}^{m_r}(d^{(r)}_j)_{n_{(r)}\delta^{(r)}_j}}\frac{x_1^{n_1}\ldots x_r^{n_r}}{n_1!\ldots n_r!},\tag{1.2}$$

where the multiple hypergeometric series converges absolutely under the parametric variable constrains and $(\lambda)_v$ denotes the well known Pochhammer symbol.

Also, we recall here the following most useful result of Lavoie and Trottier [2] by mean of which we have established our main results:

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}dx=\left(\frac{2}{3}\right)^{2\alpha}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},\tag{1.3}$$

where, $\mathrm{Re}(\alpha)\ \&\ \mathrm{Re}(\beta)>0.$

2. MAIN RESULTS

In this section, we evaluate two integrals involving the product of n-generalized Bessel functions, whose explicit representations are given in terms of Srivastava and Daoust functions.

Theorem 2.1. For ρ,σ,ν, b, c ∈ C with $\mathrm{Re}(\rho+\sigma)>0$, $\mathrm{Re}(\rho+\nu_s)>0$ and $\mathrm{Re}(\nu_j)>-\frac{1+b}{2}$; where j = 1,2,…,n, the following integral formula holds true:

$$\int_0^1 x^{\sigma+\rho-1}(1-x)^{(2\rho-1)}\left(1-\frac{x}{3}\right)^{2(\rho+\sigma)-1}\left(1-\frac{x}{4}\right)^{\rho-1}\prod_{j=1}^{n} w^{b_j}_{\nu_j,c_j}\left(y_j\left(1-\frac{x}{4}\right)(1-x)^2\right)dx=\left(\frac{2}{3}\right)^{(\rho+\sigma)}$$

$$\times\prod_{j=1}^{n}\frac{\left(\frac{y_j}{2}\right)^{\nu_j}\Gamma(\rho+\sigma)\Gamma(\rho+\nu_s)}{\Gamma\left(\nu_j+\frac{1+b_j}{2}\right)\Gamma(2\rho+\sigma+\nu_s)}$$

$$\times F^{1:0;0..;0}_{1:1;1;\ldots;1}\left[\begin{matrix}[\rho+\nu_s:2,2,\ldots2]:-;.-;-;\ldots\ldots;-;\\ [2\rho+\sigma+\nu_s:2,2,\ldots,2]:\left[\nu_1+\frac{1+b_1}{2}:1\right];\ldots;\left[\nu_n+\frac{1+b_n}{2}:1\right]\end{matrix};\frac{-c_1y_1^2}{4},\frac{-c_2y_2^2}{4},\ldots,\frac{-c_1y_n^2}{4}\right],\tag{2.1}$$

where $\prod_{j=1}^{n} w^{b_j}_{\nu_j,c_j}=w^{b_1}_{\nu_1,c_1},w^{b_2}_{\nu_2,c_2}\ldots w^{b_n}_{\nu_n,c_n}$ and $\nu_s=\nu_1+\nu_2+\ldots+\nu_n$.

Proof .In order to prove the result (2.1), we assume the the left-hand side of (2.1) by I_1, expanding all the generalized Bessel functions with the help of (1.1), to get

$$I_1=\int_0^1 x^{\sigma+\rho-1}(1-x)^{(2\rho-1)}\left(1-\frac{x}{3}\right)^{2(\rho+\sigma)-1}\left(1-\frac{x}{4}\right)^{\rho-1}\sum_{k_1=0}^{\infty}\frac{(-c_1)^{k_1}\left(\frac{y_1}{2}\left(1-\frac{x}{4}\right)(1-x)^2\right)^{\nu_1+2k_1}}{k_1!\Gamma\left(\nu_1+k_1+\frac{1+b_1}{2}\right)}$$

$$\sum_{k_1=0}^{\infty}\frac{(-c_2)^{k_2}\left(\frac{y_2}{2}\left(1-\frac{x}{4}\right)(1-x)^2\right)^{\nu_2+2k_2}}{k_2!\Gamma\left(\nu_2+k_2+\frac{1+b_2}{2}\right)}\cdots\sum_{k_n=0}^{\infty}\frac{(-c_n)^{k_n}\left(\frac{y_n}{2}\left(1-\frac{x}{4}\right)(1-x)^2\right)^{\nu_n+2k_n}}{k_n!\Gamma\left(\nu_n+k_n+\frac{1+b_n}{2}\right)}\,\mathrm{dx}. \qquad (2.2)$$

Collecting the powers of (1 − x), $\left(1-\frac{x}{4}\right)$, interchanging the order of integration and summation (which is true under the given conditions) and after some simplification, we get

$$I_1=\sum_{k_1,k_2\ldots k_n=0}^{\infty}\frac{(-c_1)^{k_1}(-c_2)^{k_2}\ldots(-c_n)^{k_n}}{k_1k_2\ldots k_n}\left[\frac{\left(\frac{1}{2}\right)^{\nu_s+2k_1+2k_2+\ldots+2k_n}}{\Gamma\left(\nu_1+\frac{1+b_1}{2}\right)\left(\nu_2+\frac{1+b_2}{2}\right)\ldots\left(\nu_n+\frac{1+b_n}{2}\right)}\right.$$

$$\left.\times\frac{y_1^{\nu_1+2k_1}y_2^{\nu_2+2k_2}\ldots y_n^{\nu_n+2k_n}}{\left(\nu_1+\frac{1+b_1}{2}\right)\left(\nu_2+\frac{1+b_2}{2}\right)\ldots\left(\nu_n+\frac{1+b_n}{2}\right)}\right]\int_0^1 x^{\sigma+\rho-1}(1-x)^{(2\rho+2\nu_s+4k_1+4k_2+\ldots4k_n-1)}\left(1-\frac{x}{3}\right)^{2(\rho+\sigma)-1}\left(1-\frac{x}{4}\right)^{\rho+\nu_s+2k_1+2k_2+\ldots2k_n-1)-1}dx.$$

(2.3)

Finally, applying (1.3) in (2.3) and after arranging the resulting expression into Srivastava and Daoust function, we easily seen our required result.

Theorem 2.2. For ρ,σ,ν , b, c ∈C with $\mathrm{Re}(\rho+\sigma)>0$, $\mathrm{Re}(\rho+\nu_s)>0$ and $\mathrm{Re}(\nu_j)>-\frac{1+b}{2}$; where j = 1,2,…,n, the following integral formula holds true:

$$\int_0^1 x^{\rho-1}(1-x)^{2(\rho+\sigma)-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\rho+\sigma-1}\prod_{j=1}^{n}w_{\nu_j,c_j}^{b_j}\left(xy_j\left(1-\frac{x}{3}\right)^2\right)dx=\left(\frac{2}{3}\right)^{(\rho+\nu_s)}$$

$$\times\prod_{j=1}^{n}\frac{\left(\frac{y_j}{2}\right)^{\nu_j}\Gamma(\rho+\sigma)\Gamma(\rho+\nu_s)}{\Gamma\left(\nu_j+\frac{1+b_j}{2}\right)\Gamma(2\rho+\sigma+\nu_s)}$$

$$\times F_{2:1;1;\ldots;1}^{2:0;0\ldots;0}\left[\begin{matrix}\Delta(2:\rho+\nu_s):-;.-;-;\ldots\ldots;-;\\ \Delta(2:2\rho+\sigma+\nu_s):\left[\nu_1+\frac{1+b_1}{2}:1\right];\ldots;\left[\nu_n+\frac{1+b_n}{2}:1\right]\end{matrix};\frac{-4c_1y_1^2}{81},-\frac{4c_2y_2^2}{81},\ldots,\frac{-4c_1y_n^2}{81}\right], \qquad (2.4)$$

where $\prod_{j=1}^{n} w_{\nu_j,c_j}^{b_j} = w_{\nu_1,c_1}^{b_1}, w_{\nu_2,c_2}^{b_2} \ldots w_{\nu_n,c_n}^{b_n}$ and $\nu_s = \nu_1 + \nu_2 + \ldots + \nu_n$.

Proof. In order to prove the result (2.4), we assume the left-hand side of (2.4) by I_2, expending all the generalized Bessel functions with the help of (1.1), to get

$$I_2 = \int_0^1 x^{\rho-1}(1-x)^{2(\rho+\sigma)-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\rho+\sigma-1}\sum_{k_1=0}^{\infty}\frac{(-c_1)^{k_1}\left(\frac{xy_1}{2}\left(1-\frac{x}{3}\right)^2\right)^{\nu_1+2k_1}}{k_1!\Gamma\left(\nu_1+k_1+\frac{1+b_1}{2}\right)}$$

$$\times\sum_{k_1=0}^{\infty}\frac{(-c_2)^{k_2}\left(\frac{xy_2}{2}\left(1-\frac{x}{3}\right)^2\right)^{\nu_2+2k_2}}{k_2!\Gamma\left(\nu_2+k_2+\frac{1+b_2}{2}\right)}\ldots\sum_{k_n=0}^{\infty}\frac{(-c_n)^{k_n}\left(\frac{xy_n}{2}\left(1-\frac{x}{3}\right)^2\right)^{\nu_n+2k_n}}{k_n!\Gamma\left(\nu_n+k_n+\frac{1+b_n}{2}\right)}\,\mathrm{dx}. \qquad (2.5)$$

Collecting the powers of (1 − x), $\left(1-\frac{x}{3}\right)$, interchanging the order of integration and summation (which is true under the given conditions) and after some simplification, we get

$$I_1 = \sum_{k_1,k_2\ldots k_n=0}^{\infty}\frac{\left(-\frac{c_1y_1^2}{4}\right)^{k_1}\left(-\frac{c_2y_2^2}{4}\right)^{k_{21}}\ldots\left(-\frac{c_ny_n^2}{4}\right)^{k_n}}{k_1k_2\ldots k_n}\left[\frac{\left(\frac{1}{2}\right)^{\nu_s+2k_1+2k_2+\ldots+2k_n}}{\Gamma\left(\nu_1+\frac{1+b_1}{2}\right)\left(\nu_2+\frac{1+b_2}{2}\right)\ldots\left(\nu_n+\frac{1+b_n}{2}\right)}\right.$$

$$\left.\times\frac{y_1^{\nu_1}y_2^{\nu_2}\ldots y_n^{\nu_n}}{\left(\nu_1+\frac{1+b_1}{2}\right)_{k_1}\left(\nu_2+\frac{1+b_2}{2}\right)_{k_2}\ldots\left(\nu_n+\frac{1+b_n}{2}\right)_{k_n}}\right]\int_0^1 x^{\rho+\nu_1+\nu_2+\ldots+\nu_n+2k_1+2k_2+\ldots+2k_n-1}(1-x)^{2(\rho+\sigma)-1}\left(1-\frac{x}{3}\right)^{2(\rho+\nu_1+\nu_2+\ldots+\nu_n+2k_1+2k_2+\ldots+2k_n)-1}\left(1-\frac{x}{4}\right)^{\rho+\sigma-1}dx$$

(2.6)

Finally, applying (1.3) in (2.6) and after arranging the resulting expression into Srivastava and Daoust function, we easily seen our required result.

Remark: On setting n = 1 in (2.1) and (2.4), respectively, we immediately get the known result of Khan and Ghayasuddin see [1, Eqs.(2.1) and (2.4)]. Also, it is noticed that for n = 1 and $b_1 = c_1 = 1$, our main results (2.1) and (2.4) reduce to the work of Agarwal et al. see [11, Eqs.(2.1) and (2.3)].

3. SPECIAL CASES

This section deals with some special cases of our present investigation.

Corollary 3.1. On assuming $b_j= 1$; $c_j= 1$, (j = 1,2,…,n) in (2.1), we get the under mentioned integral formula (involving product of n Bessel functions):

$$\int_0^1 x^{\sigma+\rho-1}(1-x)^{(2\rho-1)}\left(1-\frac{x}{3}\right)^{2(\rho+\sigma)-1}\left(1-\frac{x}{4}\right)^{\rho-1}\prod_{j=1}^{n} J_{\nu_j}\left(y_j\left(1-\frac{x}{4}\right)(1-x)^2\right)dx=\left(\frac{2}{3}\right)^{(\rho+\sigma)}$$
$$\times\prod_{j=1}^{n}\frac{\left(\frac{y_j}{2}\right)^{\nu_j}\Gamma(\rho+\sigma)\Gamma(\rho+\nu_s)}{\Gamma(\nu_j+1)\Gamma(2\rho+\sigma+\nu_s)}$$
$$\times F^{1:0;0..;0}_{1:1;1;...;1}\left[\begin{matrix}[\rho+\nu_s:2,2,...2]:-;.-;-;..;-; \\ [2\rho+\sigma+\nu_s:2,2,...,2]:[\nu_1+1];...;[\nu_n+1]\end{matrix};\frac{-y_1^2}{4},\frac{-y_2^2}{4},...,\frac{-y_n^2}{4}\right], \tag{3.1}$$

where $\nu_s=\nu_1+\nu_2+...+\nu_n, \mathrm{Re}(\rho+\sigma)>0, \mathrm{Re}(\rho+\nu_s)>0$ and $\mathrm{Re}(\nu_j)>-1$ (j=1,2,3,...,n).

Corollary 3.2. On assuming $b_j= 1$; $c_j= -1$, (j = 1,2,3,…,n) in (2.1), we get the under mentioned integral formula (involving product of modified Bessel functions):

$$\int_0^1 x^{\sigma+\rho-1}(1-x)^{(2\rho-1)}\left(1-\frac{x}{3}\right)^{2(\rho+\sigma)-1}\left(1-\frac{x}{4}\right)^{\rho-1}\prod_{j=1}^{n} I_{\nu_j}\left(y_j\left(1-\frac{x}{4}\right)(1-x)^2\right)dx=\left(\frac{2}{3}\right)^{(\rho+\sigma)}$$
$$\times\prod_{j=1}^{n}\frac{\left(\frac{y_j}{2}\right)^{\nu_j}\Gamma(\rho+\sigma)\Gamma(\rho+\nu_s)}{\Gamma(\nu_j+1)\Gamma(2\rho+\sigma+\nu_s)}$$
$$\times F^{1:0;0..;0}_{1:1;1;...;1}\left[\begin{matrix}[\rho+\nu_s:2,2,...2]:-;.-;-;..;-; \\ [2\rho+\sigma+\nu_s:2,2,...,2]:[\nu_1+1];...;[\nu_n+1]\end{matrix};\frac{y_1^2}{4},\frac{y_2^2}{4},...,\frac{y_n^2}{4}\right], \tag{3.2}$$

where $\nu_s=\nu_1+\nu_2+...+\nu_n; \mathrm{Re}(\rho+\sigma)>0; \mathrm{Re}(\rho+\nu_s)>0$ and $\mathrm{Re}(\nu_j)>-1$ (j=1,2,3,...,n).

Corollary 3.3. Assuming n = 2 in (2.1), we obtain the following integral (which is given in terms of Kampe' de Fe'riet function):

$$\int_0^1 x^{\sigma+\rho-1}(1-x)^{(2\rho-1)}\left(1-\frac{x}{3}\right)^{2(\rho+\sigma)-1}\left(1-\frac{x}{4}\right)^{\rho-1} w^{b_1}_{\nu_1,c_1}\left(y_1\left(1-\frac{x}{4}\right)(1-x)^2\right)w^{b_2}_{\nu_2,c_2}\left(y_2\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$

$$=\left(\frac{2}{3}\right)^{(\rho+\sigma)}\prod_{j=1}^{2}\frac{\left(\frac{y_j}{2}\right)^{\nu_j}\Gamma(\rho+\sigma)\Gamma(\rho+\nu_s')}{\Gamma\left(\nu_j+\frac{1+b_j}{2}\right)\Gamma(2\rho+\sigma+\nu_s')}$$

$$\times F^{1:0;0}_{2:1;1}\left[\begin{matrix}\frac{\rho+\nu_s'}{2}:\frac{\rho+\nu_s'+1}{2};.-;-;...;-;\\ \frac{2\rho+\sigma+\nu_s'}{2}:\frac{2\rho+\sigma+\nu_s'+1}{2};\left[\nu_1+\frac{1+b_1}{2}\right];\left[\nu_2+\frac{1+b_2}{2}\right]\end{matrix};\frac{-c_1y_1^2}{4},\frac{-c_2y_2^2}{4}\right], \quad (3.3)$$

where $\nu_s'=\nu_1+\nu_2;\mathrm{Re}(\rho+\sigma)>0;\mathrm{Re}(\rho+\nu_s')>0$ and $\mathrm{Re}(\nu_j)>-\frac{1+b_j}{2}$, (j=1,2) and $F^{p:q;k}_{l:m;n}$ is Kampe′ de Fe′riet function see [13, eq.(16)].

Corollary 3.4. On making $b_j = c_j = 1$, (j=1,2), (3.3) reduces to the following result:

$$\int_0^1 x^{\sigma+\rho-1}(1-x)^{(2\rho-1)}\left(1-\frac{x}{3}\right)^{2(\rho+\sigma)-1}\left(1-\frac{x}{4}\right)^{\rho-1}J_{\nu_1}\left(y_1\left(1-\frac{x}{4}\right)(1-x)^2\right)J_{\nu_2}\left(y_2\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$

$$=\left(\frac{2}{3}\right)^{(\rho+\sigma)}\prod_{j=1}^{2}\frac{\left(\frac{y_j}{2}\right)^{\nu_j}\Gamma(\rho+\sigma)\Gamma(\rho+\nu_s')}{\Gamma(\nu_j+1)\Gamma(2\rho+\sigma+\nu_s')}$$

$$\times F^{1:0;0}_{2:1;1}\left[\begin{matrix}\frac{\rho+\nu_s'}{2}:\frac{\rho+\nu_s'+1}{2};.-;-;...;-;\\ \frac{2\rho+\sigma+\nu_s'}{2}:\frac{2\rho+\sigma+\nu_s'+1}{2};\left[\nu_1+\frac{1+b_1}{2}\right];\left[\nu_2+\frac{1+b_2}{2}\right]\end{matrix};-\frac{y_1^2}{4},-\frac{y_2^2}{4}\right], \quad (3.4)$$

where $\nu_s'=\nu_1+\nu_2;\mathrm{Re}(\rho+\sigma)>0;\mathrm{Re}(\rho+\nu_s')>0$ and $\mathrm{Re}(\nu_j)>-1$, (j=1,2).

Corollary 3.5. On assuming $b_j = 1$ and $c_j = -1$; (j = 1; 2) in (3.3), we arrive at the under mentioned integral formula:

$$\int_0^1 x^{\sigma+\rho-1}(1-x)^{(2\rho-1)}\left(1-\frac{x}{3}\right)^{2(\rho+\sigma)-1}\left(1-\frac{x}{4}\right)^{\rho-1}I_{\nu_1}\left(y_1\left(1-\frac{x}{4}\right)(1-x)^2\right)I_{\nu_2}\left(y_2\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$

$$=\left(\frac{2}{3}\right)^{(\rho+\sigma)}\prod_{j=1}^{2}\frac{\left(\frac{y_j}{2}\right)^{\nu_j}\Gamma(\rho+\sigma)\Gamma(\rho+\nu_s')}{\Gamma(\nu_j+1)\Gamma(2\rho+\sigma+\nu_s')}$$

$$\times F^{1:0;0}_{2:1;1}\left[\begin{matrix}\frac{\rho+\nu_s'}{2}:\frac{\rho+\nu_s'+1}{2};.-;-;...;-;\\ \frac{2\rho+\sigma+\nu_s'}{2}:\frac{2\rho+\sigma+\nu_s'+1}{2};\left[\nu_1+\frac{1+b_1}{2}\right];\left[\nu_2+\frac{1+b_2}{2}\right]\end{matrix};\frac{y_1^2}{4},\frac{y_2^2}{4}\right], \quad (3.5)$$

where $\nu_s'=\nu_1+\nu_2;\mathrm{Re}(\rho+\sigma)>0;\mathrm{Re}(\rho+\nu_s')>0$ and $\mathrm{Re}(\nu_j)>-1$, (j=1,2).

Corollary 3.6. If we assign n = 3 in (2.1), then we have the following integral (which is given in the form of Srivastava triple hypergeometric series):

$$\int_0^1 x^{\sigma+\rho-1}(1-x)^{(2\rho-1)}\left(1-\frac{x}{3}\right)^{2(\rho+\sigma)-1}\left(1-\frac{x}{4}\right)^{\rho-1}\prod_{j=1}^{3} w_{\nu_j,c_j}^{b_j}\left(y_1\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$

$$=\left(\frac{2}{3}\right)^{(\rho+\sigma)}\prod_{j=1}^{3}\frac{\left(\frac{y_j}{2}\right)^{\nu_j}\Gamma(\rho+\sigma)\Gamma(\rho+\nu_s'')}{\Gamma\left(\nu_j+\frac{1+b_j}{2}\right)\Gamma(2\rho+\sigma+\nu_s'')}$$

$$\times F^{(3)}\left[\begin{array}{c}\frac{\rho+\nu_s''}{2}:\frac{\rho+\nu_s''+1}{2}::\_;\_;\_;\_;\_;\_;\_;\_; \\ \frac{2\rho+\sigma+\nu_s''}{2}:\frac{2\rho+\sigma+\nu_s''+1}{2}::\left[\nu_1+\frac{1+b_1}{2}\right];\left[\nu_2+\frac{1+b_2}{2}\right];\left[\nu_{32}+\frac{1+b_3}{2}\right];\_;\_;\_;\_;\_;\_;\end{array}\frac{-c_1y_1^2}{4},\frac{-c_2y_2^2}{4},\frac{-c_3y_3^2}{4}\right],$$

(3.6)

where $\nu_s''=\nu_1+\nu_2+\nu_3; \mathrm{Re}(\rho+\sigma)>0; \mathrm{Re}(\rho+\nu_s')>0$ and $\mathrm{Re}(\nu_j)>-\frac{1+b_j}{2}$, (j=1,2,3) and $F^{(3)}[x,y,z]$ is Srivastava triple hypergeometric series see [13, eq.(39)].

Corollary 3.7. If we place $b_j = c_j = 1$; (j = 1,2,3) in (3.6), we have

$$\int_0^1 x^{\sigma+\rho-1}(1-x)^{(2\rho-1)}\left(1-\frac{x}{3}\right)^{2(\rho+\sigma)-1}\left(1-\frac{x}{4}\right)^{\rho-1}\prod_{j=1}^{3} J_{\nu_{j_j}}\left(y_1\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$

$$=\left(\frac{2}{3}\right)^{(\rho+\sigma)}\prod_{j=1}^{3}\frac{\left(\frac{y_j}{2}\right)^{\nu_j}\Gamma(\rho+\sigma)\Gamma(\rho+\nu_s'')}{\Gamma(\nu_j+1)\Gamma(2\rho+\sigma+\nu_s'')}$$

$$\times F^{(3)}\left[\begin{array}{c}\frac{\rho+\nu_s''}{2}:\frac{\rho+\nu_s''+1}{2}::\_;\_;\_;\_;\_;\_;\_;\_; \\ \frac{2\rho+\sigma+\nu_s''}{2}:\frac{2\rho+\sigma+\nu_s''+1}{2}::[\nu_1+1];[\nu_2+1];[\nu_3+1];\_;\_;\_;\_;\_;\_;\end{array}-\frac{y_1^2}{4},-\frac{y_2^2}{4},-\frac{y_3^2}{4}\right],$$

(3.7)

where $\nu_s''=\nu_1+\nu_2+\nu_3; \mathrm{Re}(\rho+\sigma)>0; \mathrm{Re}(\rho+\nu_s')>0$ and $\mathrm{Re}(\nu_j)>-1$, (j=1,2,3).

Corollary 3.8. On setting $b_j = 1$ and $c_j = -1$; (j = 1,2,3) in (3.6), we have

$$\int_0^1 x^{\sigma+\rho-1}(1-x)^{(2\rho-1)}\left(1-\frac{x}{3}\right)^{2(\rho+\sigma)-1}\left(1-\frac{x}{4}\right)^{\rho-1}\prod_{j=1}^{3} I_{\nu_{j_j}}\left(y_1\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$

$$=\left(\frac{2}{3}\right)^{(\rho+\sigma)}\prod_{j=1}^{3}\frac{\left(\frac{y_j}{2}\right)^{\nu_j}\Gamma(\rho+\sigma)\Gamma(\rho+\nu_s'')}{\Gamma(\nu_j+1)\Gamma(2\rho+\sigma+\nu_s'')}$$

$$\times F^{(3)}\left[\begin{array}{c}\frac{\rho+\nu_s''}{2}:\frac{\rho+\nu_s''+1}{2}::\_;\_;\_;\_;\_;\_;\_;\_; \\ \frac{2\rho+\sigma+\nu_s''}{2}:\frac{2\rho+\sigma+\nu_s''+1}{2}::[\nu_1+1];[\nu_2+1];[\nu_3+1];\_;\_;\_;\_;\_;\_;\end{array}\frac{y_1^2}{4},\frac{y_2^2}{4},\frac{y_3^2}{4}\right],$$

(3.8)

where $\nu''_s = \nu_1 + \nu_2 + \nu_3; \mathrm{Re}(\rho+\sigma) > 0; \mathrm{Re}(\rho+\nu'_s) > 0$ and $\mathrm{Re}(\nu_j) > -1$, (j=1,2,3).

Corollary 3.9. On assuming $b_j = 1$; $c_j = 1$, (j=1,2,…,n) in (2.4), we get the under mentioned integral formula (involving a product of n Bessel function):

$$\int_0^1 x^{\rho-1}(1-x)^{2(\rho+\sigma)-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\rho+\sigma-1}\prod_{j=1}^{n} J_{\nu_j}\left(xy_j\left(1-\frac{x}{3}\right)^2\right)dx = \left(\frac{2}{3}\right)^{(\rho+\nu_s)}$$

$$\times\prod_{j=1}^{n}\frac{\left(\frac{y_j}{2}\right)^{\nu_j}\Gamma(\rho+\sigma)\Gamma(\rho+\nu_s)}{\Gamma(\nu_j+1)\Gamma(2\rho+\sigma+\nu_s)}$$

$$\times F^{2:0;0..;0}_{2:1;1;...;1}\left[\begin{matrix}\Delta(2:\rho+\nu_s):-;.-;-;..;-; \\ \Delta(2:2\rho+\sigma+\nu_s):[\nu_1+1];...;[\nu_n+1]\end{matrix};-\frac{4y_1^2}{81},-\frac{4y_2^2}{81},...,-\frac{4y_n^2}{81}\right],$$

(3.9)

where $\nu_s = \nu_1 + \nu_2 + ... + \nu_n$; $\mathrm{Re}(\rho+\sigma) > 0; \mathrm{Re}(\rho+\nu_s) > 0$ and $\mathrm{Re}(\nu_j) > -1$, (j=1,2,3).

Corollary 3.10. On assuming $b_j = 1$; $c_j = -1$, (j=1,2,...,n) in (2.4), we get the under mentioned integral formula (involving a product of modified Bessel function):

$$\int_0^1 x^{\rho-1}(1-x)^{2(\rho+\sigma)-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\rho+\sigma-1}\prod_{j=1}^{n} I_{\nu_j}\left(xy_j\left(1-\frac{x}{3}\right)^2\right)dx = \left(\frac{2}{3}\right)^{(\rho+\nu_s)}$$

$$\times\prod_{j=1}^{n}\frac{\left(\frac{y_j}{2}\right)^{\nu_j}\Gamma(\rho+\sigma)\Gamma(\rho+\nu_s)}{\Gamma(\nu_j+1)\Gamma(2\rho+\sigma+\nu_s)}$$

$$\times F^{2:0;0..;0}_{2:1;1;...;1}\left[\begin{matrix}\Delta(2:\rho+\nu_s):-;.-;-;..;-; \\ \Delta(2:2\rho+\sigma+\nu_s):[\nu_1+1];...;[\nu_n+1]\end{matrix};\frac{4y_1^2}{81},\frac{4y_2^2}{81},...,\frac{4y_n^2}{81}\right], \quad (3.10)$$

where $\nu_s = \nu_1 + \nu_2 + ... + \nu_n$; $\mathrm{Re}(\rho+\sigma) > 0; \mathrm{Re}(\rho+\nu_s) > 0$ and $\mathrm{Re}(\nu_j) > -1$, (j=1,2,3).

Corollary 3.11. Making n = 2 in (2.4), we obtain the following integral (which is given in terms of Kampe' de Fe'riet function):

$$\int_0^1 x^{\rho-1}(1-x)^{2(\rho+\sigma)-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\rho+\sigma-1}\prod_{j=1}^{2} w^{b_j}_{\nu_j,c_j}\left(xy_j\left(1-\frac{x}{3}\right)^2\right)dx = \left(\frac{2}{3}\right)^{(\rho+\nu'_s)}$$

$$\times\prod_{j=1}^{n}\frac{\left(\frac{y_j}{2}\right)^{\nu_j}\Gamma(\rho+\sigma)\Gamma(\rho+\nu'_s)}{\Gamma\left(\nu_j+\frac{1+b_j}{2}\right)\Gamma(2\rho+\sigma+\nu'_s)}$$

$$\times F^{2:0;0}_{2:1;1}\left[\begin{array}{c}\frac{(\rho+\nu'_s)}{2},\frac{(\rho+\nu'_s+1)}{2}:-;.-;\\ \frac{2\rho+\sigma+\nu'_s}{2},\frac{2\rho+\sigma+\nu'_s+1}{2}:\left[\nu_1+\frac{1+b_1}{2}\right];\left[\nu_2+\frac{1+b_2}{2}\right]\end{array};\frac{-4c_1y_1^2}{81},-\frac{4c_2y_2^2}{81}\right],$$

(3.11)

where $\nu'_s=\nu_1+\nu_2;\mathrm{Re}(\rho+\sigma)>0;\mathrm{Re}(\rho+\nu'_s)>0$ and $\mathrm{Re}(\nu_j)>-\frac{1+b_j}{2}$, (j=1,2) and $F^{p:q;k}_{l:m;n}$ is Kampe$'$ de Fe$'$riet function [13, p.63, eq.(16)].

Corollary 3.12. On setting $b_j= c_j= 1$; (j = 1,2), (3.11) reduces to the following integral (in terms of Kampe$'$ de Fe$'$riet function):

$$\int_0^1 x^{\rho-1}(1-x)^{2(\rho+\sigma)-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\rho+\sigma-1}\prod_{j=1}^{2}J_{\nu_j}\left(xy_j\left(1-\frac{x}{3}\right)^2\right)dx=\left(\frac{2}{3}\right)^{(\rho+\nu'_s)}$$

$$\times\prod_{j=1}^{n}\frac{\left(\frac{y_j}{2}\right)^{\nu_j}\Gamma(\rho+\sigma)\Gamma(\rho+\nu'_s)}{\Gamma(\nu_j+1)\Gamma(2\rho+\sigma+\nu'_s)}$$

$$\times F^{2:0;0}_{2:1;1}\left[\begin{array}{c}\frac{(\rho+\nu'_s)}{2},\frac{(\rho+\nu'_s+1)}{2}:-;.-;\\ \frac{2\rho+\sigma+\nu'_s}{2},\frac{2\rho+\sigma+\nu'_s+1}{2}:[\nu_1+1];[\nu_2+1]\end{array};-\frac{4y_1^2}{81},-\frac{4y_2^2}{81}\right],$$

(3.12)

where $\nu'_s=\nu_1+\nu_2;\mathrm{Re}(\rho+\sigma)>0;\mathrm{Re}(\rho+\nu'_s)>0$ and $\mathrm{Re}(\nu_j)>-1$, (j=1,2).

Corollary 3.13. On setting $b_j= 1$ and $c_j= -1$; (j = 1; 2) in (3.11), we arrive at the under mentioned integral formula:

$$\int_0^1 x^{\rho-1}(1-x)^{2(\rho+\sigma)-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\rho+\sigma-1}\prod_{j=1}^{2}I_{\nu_j}\left(xy_j\left(1-\frac{x}{3}\right)^2\right)dx=\left(\frac{2}{3}\right)^{(\rho+\nu'_s)}$$

$$\times\prod_{j=1}^{n}\frac{\left(\frac{y_j}{2}\right)^{\nu_j}\Gamma(\rho+\sigma)\Gamma(\rho+\nu'_s)}{\Gamma(\nu_j+1)\Gamma(2\rho+\sigma+\nu'_s)}$$

$$\times F^{2:0;0}_{2:1;1}\left[\begin{array}{c}\frac{(\rho+\nu'_s)}{2},\frac{(\rho+\nu'_s+1)}{2}:-;.-;\\ \frac{2\rho+\sigma+\nu'_s}{2},\frac{2\rho+\sigma+\nu'_s+1}{2}:[\nu_1+1];[\nu_2+1]\end{array};\frac{4y_1^2}{81},\frac{4y_2^2}{81}\right],$$

(3.13)

where $\nu'_s=\nu_1+\nu_2;\mathrm{Re}(\rho+\sigma)>0;\mathrm{Re}(\rho+\nu'_s)>0$ and $\mathrm{Re}(\nu_j)>-1$,(j=1,2).

Corollary 3.14. If we assign n = 3 in (2.4), then we have the following integral (which is given in terms of Srivastava triple hypergeometric series):

$$\int_0^1 x^{\rho-1}(1-x)^{2(\rho+\sigma)-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\rho+\sigma-1}\prod_{j=1}^{3} w_{\nu_j,c_j}^{b_j}\left(xy_j\left(1-\frac{x}{3}\right)^2\right)dx=\left(\frac{2}{3}\right)^{(\rho+\nu_s'')}$$

$$\times\prod_{j=1}^{3}\frac{\left(\frac{y_j}{2}\right)^{\nu_j}\Gamma(\rho+\sigma)\Gamma(\rho+\nu_s'')}{\Gamma\left(\nu_j+\frac{1+b_j}{2}\right)\Gamma(2\rho+\sigma+\nu_s'')}$$

$$\times F^{(3)}\left[\begin{array}{c}\frac{(\rho+\nu_s'')}{2},\frac{(\rho+\nu_s''+1)}{2}::-;-;-;-;-;-;-;-;\\ \frac{2\rho+\sigma+\nu_s''}{2},\frac{2\rho+\sigma+\nu_s''+1}{2}::-;-;-;\left[\nu_1+\frac{1+b_1}{2}\right],\left[\nu_1+\frac{1+b_1}{2}\right];\left[\nu_3+\frac{1+b_3}{2}\right]\end{array};-\frac{4c_1y_1^2}{81},-\frac{4c_2y_2^2}{81},-\frac{4c_3y_3^2}{81}\right], \tag{3.14}$$

where $\nu_s''=\nu_1+\nu_2;\mathrm{Re}(\rho+\sigma)>0;\mathrm{Re}(\rho+\nu_s')>0$ and $\mathrm{Re}(\nu_j)>-\frac{1+b_j}{2}$, (j=1,2,3) and $F^{(3)}[x,y,z]$ is the Srivastava triple hypergeometric series [13, p.69, eq.(39)].

Corollary 3.15. Making $b_j= c_j= 1$; (j = 1,2,3) in (3.14), the following integral holds true:

$$\int_0^1 x^{\rho-1}(1-x)^{2(\rho+\sigma)-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\rho+\sigma-1}\prod_{j=1}^{3} w_{\nu_j,c_j}^{b_j}\left(xy_j\left(1-\frac{x}{3}\right)^2\right)dx=\left(\frac{2}{3}\right)^{(\rho+\nu_s'')}$$

$$\times\prod_{j=1}^{3}\frac{\left(\frac{y_j}{2}\right)^{\nu_j}\Gamma(\rho+\sigma)\Gamma(\rho+\nu_s'')}{\Gamma(\nu_j+1)\Gamma(2\rho+\sigma+\nu_s'')}$$

$$\times F^{(3)}\left[\begin{array}{c}\frac{(\rho+\nu_s'')}{2},\frac{(\rho+\nu_s''+1)}{2}::-;-;-;-;-;-;-;-;\\ \frac{2\rho+\sigma+\nu_s''}{2},\frac{2\rho+\sigma+\nu_s''+1}{2}::-;-;-;[\nu_1+1],[\nu_1+1];[\nu_n+1]\end{array};-\frac{4y_1^2}{81},-\frac{4y_2^2}{81},-\frac{4y_3^2}{81}\right], \tag{3.15}$$

where $\nu_s'=\nu_1+\nu_2;\mathrm{Re}(\rho+\sigma)>0;\mathrm{Re}(\rho+\nu_s')>0$ and $\mathrm{Re}(\nu_j)>-1$, (j=1,2,3).

Corollary 3.16. Making $b_j= 1$ and $c_j= -1$; (j = 1,2,3) in (3.14), we have

$$\int_0^1 x^{\rho-1}(1-x)^{2(\rho+\sigma)-1}\left(1-\frac{x}{3}\right)^{2\rho-1}\left(1-\frac{x}{4}\right)^{\rho+\sigma-1}\prod_{j=1}^{3} w_{\nu_j,c_j}^{b_j}\left(xy_j\left(1-\frac{x}{3}\right)^2\right)dx=\left(\frac{2}{3}\right)^{2(\rho+\nu_s'')}$$

$$\times\prod_{j=1}^{3}\frac{\left(\frac{y_j}{2}\right)^{\nu_j}\Gamma(\rho+\sigma)\Gamma(\rho+\nu_s'')}{\Gamma\left(\nu_j+\frac{1+b_j}{2}\right)\Gamma(2\rho+\sigma+\nu_s'')}$$

$$\times F^{(3)}\left[\begin{array}{l} \frac{(\rho+\nu_s'')}{2},\frac{(\rho+\nu_s''+1)}{2}::-;.-;-;.-;-;-; \\ \frac{2\rho+\sigma+\nu_s''}{2},\frac{2\rho+\sigma+\nu_s''+1}{2}::-;-;-;[\nu_1+1],[\nu_1+1];[\nu_n+1] \end{array};\frac{4y_1^2}{81},\frac{4y_2^2}{81},\frac{4y_3^2}{81}\right], \tag{3.16}$$

where $\nu_s' = \nu_1 + \nu_2; \mathrm{Re}(\rho+\sigma) > 0; \mathrm{Re}(\rho+\nu_s') > 0$ and $\mathrm{Re}(\nu_j) > -1$, (j=1,2,3).

4. CONCLUDING REMARKS

The present paper is mainly motivated by the work of Khan and Ghayasuddin [10]. Here we have investigated some new integral formulae involving n generalized Bessel functions $w_{\nu_j,c_j}^{b_j}(z)$; (j = 1,2,2…,n). Also, we have derived some integrals involving a product of Bessel functions $J_{\nu_j}(z)$ and modified Bessel functions $I_{\nu_j}(z)$ (j = 1,2,…, n) as special cases of our main results. Since for b = 2 and c = 1, $w_{\nu_j,c_j}^{b_j}(z)$ reduces to $\frac{2j_{\nu_j}}{\sqrt{\pi}}$, while for b = 2 and c = −1, $w_{\nu_j,c_j}^{b_j}(z)$ reduces to $\frac{2i_{\nu_j}}{\sqrt{\pi}}$ see [6], therefore, by using these relations in our main results we can establish some more interesting integrals.

Furthermore, the generalized Bessel function $w_{\nu,c}^{b}(z)$ has also relations with sine and cosine functions see [6]. So, our main results can establish numerous other interesting integral formulae.

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