ORIGINAL PAPER

NUMERICAL TECHNIQUE FOR THE SOLUTION OF THIRD ORDER BOUNDARY VALUE PROBLEMS IN PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. *In the present research article, we described a numerical method for the solution of third order two-point boundary value problems in partial differential equations. We outlined the derivation and the order of the method. In the numerical section we considered model problems and solved those problems by the propose method to illustrate the performance in term of the efficiency and accuracy.*

Keywords: *boundary value problems, finite difference method, partial differential equations, third order differential equation, two point BVPs.*

AMS Subject Classification 2010: *65M06, 65M12.*

1. INTRODUCTION

Let us consider third order partial differential equation and corresponding boundary value problem in the following form

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + f\left(t, x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right), \qquad a < x < b \text{ and } 0 \le t < T \qquad (1)$$

and initial-boundary conditions are

$$u(x,0) = \alpha\,,\ u(a,t) = \alpha_1, u_x(a,t) = \beta_1 \text{ and } u_x(b,t) = \beta_2 \qquad (2)$$

where $u_x(a,t) = \left(\frac{\partial u}{\partial x}\right)_{(a,t)}$, $u_x(b,t) = \left(\frac{\partial u}{\partial x}\right)_{(b,t)}$ and $\alpha, \alpha_1, \beta_1$,β_2 are either real finite constants or real continuous function of x and t i.e. $\alpha(x), \alpha_1(t), \beta_1(t), \beta_2(t)$. Let us assume that $f(t, x, u, u_x, u_{xx})$ is continuous function in $[a, b]$and $t \ge 0$.

The applications of different orders of partial differential equations can be found in the mathematical modeling of physical problems in and around us. In particular, third order partial differential equations are used to describe the mathematical models of the heat and the wave equations in a many subfields of engineering and natural sciences. In most cases it is impossible to solve these modeled problems under realistic conditions analytically. However, we have enriched source of studies on both analytically and numerically on the heat and the wave equations available in the literature, but not on class of third order partial differential equation.

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In recent years, for the numerical solution ofclasses of the third order of partial differential equations, different class numerical techniques have been reported in the literature. Some reported literary work area collection of criteria for deriving stability conditions of difference schemes in Mengzhao [1], a one-step two-parameter method in Djidjeli [2], Adomian decomposition method in Wazwaz [3], method of lines for different class of third order partial differential equations in Mechee [4] and reference therein.

In this article we have developed numerical technique: a finite difference method for the numerical solution of (1) which is second order accurate in space and first or second order accurate in time. The existence and uniqueness of the solution for the problem (1) are assumed Agarwal [5]. To the best of our knowledge, no method similar to the proposed method for the numerical solution has been discussed in the literature to date.

We have presented our work in this article as follows. In the next section we will discuss thefinite difference method and in Section 3 we will derive our propose method. In Section 4, we have discussed stability analysis and the applications of the proposed method to the model problems andnumerical results have been produced to show the efficiency in Section 5. Discussion and conclusion on the performance of the method are presented in Section 6.

2. DEVELOPMENTOF THE FINITE DIFFERENCE METHOD

We substitute rectangular domain $R = [a, b] \times [0, T[$ by a discrete set of mesh points and we wish to determine the numerical solution of the problem (1) at these discrete mesh points. Thus space interval $[a, b]$ will be partitioned $a = x_0 < x_1 <..< x_N = b$, into N subintervals each of length h so that $b = a + Nh$ and the time variable will be discretized $0 = t_0 < t_1 < t_2 <..$ in steps of length k. Let ∂R the boundary of the region R is parallel to coordinate axes. Thus region R and its boundary ∂R consisting of ordinate $x = a$, $x = b$ and the axis $t = 0$ covered by discrete mesh points with coordinates $(a + ih, jk)$ where $i = 0,1, .., N$ and $j = 0,1, ..$ as shown below in the Fig. 1.

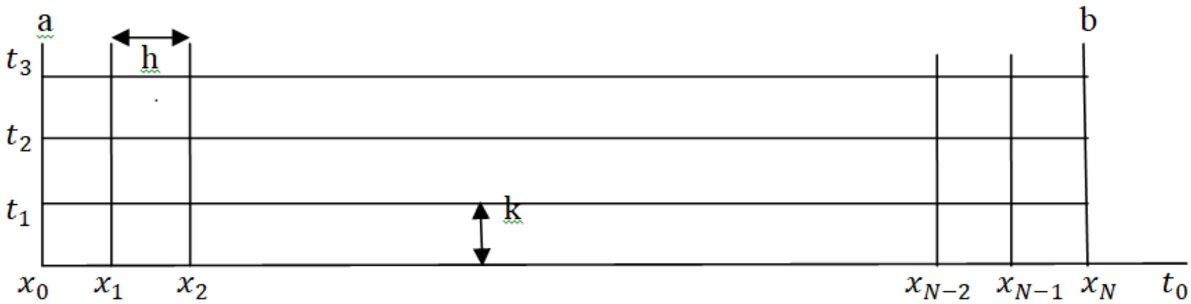

Figure 1.Region R and its boundary ∂R, consisting of ordinate x=a , x=b and the axis t=0, covered by discrete mesh points with coordinates (a+ih,jk) where i=0,1,..,N and j=0,1,….

Let us denote the numerical approximation of $u(x, t)$ at mesh point (x_i, t_j) by $u_{i,j}$. Also we denote the theoretical value of the forcing function $f\left(t, x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right)$ at mesh point (x_i, t_j) by $f_{i,j}$. Thus, using these finite difference, the problem $(1 - 2)$ reduced to the following discrete problem at node (x_i, t_j),

$$\left(\frac{\partial u}{\partial t}\right)_{i,j} - f_{i,j} = \left(\frac{\partial^3 u}{\partial x^3}\right)_{i,j} \tag{3}$$

andinitial-boundary conditions are

$$u_{i,0} = \alpha/\alpha_i\ ,\ u_{0,j} = \alpha_1/\alpha_{1,j},\ u_{x0,j} = \beta_1/\beta_{1,j} \text{ and } u_{xN,j} = \beta_2\ /\ \beta_{2,j} \tag{4}$$

Let us define following approximations:

$$\overline{u}_{xi,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2h} \tag{5}$$

$$\overline{u}_{xxi,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} \tag{6}$$

and

$$\overline{f}_{i,j} = f\left(x_i, t_j, u_{i,j}, \overline{u}_{xi,j}, \overline{u}_{xxi,j}\right) \tag{7}$$

Hence, following the idea in Pandey [6-8], we propose following difference method for the numerical solution of the (3)

$$\frac{u_{i,j+1} - u_{i,j}}{k} - \overline{f}_{i,j}$$

$$= \frac{1}{150h^3}\begin{cases} 50\left(8u_{i-1,j} - 9u_{i,j} + u_{i+2,j} + 6hu'_{i-1,j}\right), & i = 1 \\ 6\left(u_{i-1,j} + 18u_{i,j} - 33u_{i+1,j} + 14u_{i+2,j} + 6hu'_{i-2,j}\right), & i = 2 \\ 75\left(-u_{i-2,j} + 2u_{i-1,j} - 2u_{i+1,j} + u_{i+2,j}\right), & 3 \le i \le N-2 \\ 50\left(-u_{i-2,j} + 9u_{i,j} - 8u_{i+1,j} + 6hu'_{i+1,j}\right), & i = N-1 \\ 150\left(u_{i-3,j} - 6u_{i-2,j} + 15u_{i-1,j} - 10u_{i,j} + 6hu'_{i,j}\right), & i = N \end{cases} \tag{8}$$

If we replace $\left(\frac{\partial u}{\partial t}\right)_{i,j}$ by first order difference approximation $\frac{u_{i,j}-u_{i,j-1}}{k}$ in (3) then the method (8) has order $O(k + h^2)$. However, in computations we replace $u_{i,j}$ by the mean of the values $u_{i,j+1}$ and $u_{i,j-1}$.

3. DERIVATIONOF THE FINITE DIFFERENCE METHOD

In this section we discuss the derivation of the proposed finite difference method (8). Let us consider the following linear combination of the solution and derivative of the solution of the problem (1), $u\left(x_i + h, t_j\right), u\left(x_i + 2h, t_j\right), u\left(x_i - h, t_j\right)$, $u'\left(x_i - h, t_j\right)$ and $u(x_i, t_j)$, i.e.

$$h^3\left(\frac{\partial^3 u}{\partial x^3}\right)_{(x_i,t_j)} = \tag{9}$$

$$a_1 u(x_i - h, t_j) + a_2 u(x_i, t_j) + a_3 u(x_i + h, t_j) + a_4 u(x_i + 2h, t_j) + a_5 h u'(x_i - h, t_j)$$

where$a_i, i = 1,2,\cdots,5$are constant. To determine the constants, expand each term in (9) in a Taylor series about point(x_i, t_j). In so obtained Taylor series compare the coefficients of $\mathrm{h^p, p = 01,2,\cdots,4}$. Thus we obtained a system of linear equations in a_i. Solving the above obtained system of equation, we have

$$(a_1, a_2, a_3, a_4, a_5) = \frac{1}{3}(8, -9, 0, 1, 6) \tag{10}$$

Using (10) in (9), we have

$$\left(\frac{\partial^3 u}{\partial x^3}\right)_{(x_i, t_j)} = \frac{1}{3h^3}\Big(8u(x_i - h, t_j) - 9u(x_i, t_j) + u(x_i + 2h, t_j) + 6hu'(x_i - h, t_j)\Big)$$

Thus we write above equation in defined discrete notations as given below,

$$u_{xxx\, i,j} = \frac{1}{3h^3}\left(8u_{i-1,j} - 9u_{i,j} + u_{i+2,j} + 6hu'_{i-1,j}\right), \quad i = 1 \tag{11}$$

Replace $\left(\frac{\partial u}{\partial t}\right)_{i,j}$ by finite difference approximations

$$u_{ti,j} = \frac{1}{k}\left(u_{i,j+1} - u_{i,j}\right) \tag{12}$$

Using (11) and (12) in (3) , we have

$$\frac{1}{k}\left(u_{i,j+1} - u_{i,j}\right) - f_{i,j} = \frac{1}{3h^3}\left(8u_{i-1,j} - 9u_{i,j} + u_{i+2,j} + 6hu'_{i-1,j}\right), \quad i = 1 \tag{13}$$

Following the similar method , we can derive equations for different value of i in (8).

Also if we replace $\left(\frac{\partial u}{\partial t}\right)_{i,j}$ by finite difference approximations, $u_{ti,j} = \frac{1}{k}\left(u_{i,j} - u_{i,j-1}\right)$ then we will obtain

$$\frac{1}{k}\left(u_{i,j} - u_{i,j-1}\right) - f_{i,j} = \frac{1}{3h^3}\left(8u_{i-1,j} - 9u_{i,j} + u_{i+2,j} + 6hu'_{i-1,j}\right), \quad i = 1 \tag{14}$$

The method (14) in terms of the function values at the $(j+1)^{th}$ and j^{th} level may be written as:

$$\frac{1}{k}\left(u_{i,j+1} - u_{i,j}\right) - f_{i,j+1} = \frac{1}{3h^3}\left(8u_{i-1,j+1} - 9u_{i,j+1} + u_{i+2,j+1} + 6hu'_{i-1,j+1}\right), \qquad i = 1 \tag{15}$$

Averaging method (13) and (15), we get a method of improved order i.e. $O(k^2 + h^2)$.

Thus we have proposed /developed the Crank-Nicolson type numerical technique for solution of third order initial-boundary value problems in partial differential equation.

4.THE STABILITY ANALYSIS OF THE METHOD

In this section we analysis method (5) for the purpose of its convergence. So we will consider following linear initial-boundary value problem

$$\frac{\partial u}{\partial t}=\frac{\partial^3 u}{\partial x^3}+f(t,x,),\qquad a<x<b \text{ and } 0\le t<T \qquad (16)$$

and initial-boundary conditions are

$$u(x,0)=\alpha\ ,\ u(a,t)=\alpha_1, u_x(a,t)=\beta_1 \text{ and } u_x(b,t)=\beta_2 \qquad (17)$$

We will solve the problem (16) by proposed method (8) and we write in the matrix form. Let $\boldsymbol{U}$ be the approximate solution of the system of equations (8), thus we have

$$\boldsymbol{A}\cdot\boldsymbol{U}^{k+1}=(-\boldsymbol{A}+\boldsymbol{2I})\cdot\boldsymbol{U}^{k}+\boldsymbol{RH} \qquad (18)$$

Above equation (18) is the matrix form of our proposed finite difference equation (8) relating the solution values along the $(k+1)^{th}$ and k^{th} time levels. In equation (18), **A** , **U, I** and **RH** are respectively the coefficient matrices, column vector of the solution, identity matrix and the column vector of known boundary values and forcing function values of (16) and these are,

$$\boldsymbol{A}=[R_1,R_2,R_3,\cdots,R_{N-2},R_{N-1},R_N]^T_{1\times N}$$

where

$R_1=\left[1+\frac{3}{2}\lambda,0,-\frac{1}{6}\lambda,0,\cdots,0\right]_{1\times N}$,$R_2=\left[-\frac{1}{50}\lambda,1-\frac{9}{25}\lambda,\frac{33}{50}\lambda,\frac{-7}{25}\lambda,0,\cdots,0\right]_{1\times N}$,

$R_3=\left[\frac{1}{4}\lambda,-\frac{1}{2}\lambda,1,\frac{1}{2}\lambda,-\frac{1}{4}\lambda,0,\cdots,0\right]_{1\times N}$,$R_{N-2}=\left[0,\cdots,0,\frac{1}{4}\lambda,-\frac{1}{2}\lambda,1,\frac{1}{2}\lambda,-\frac{1}{4}\lambda\right]_{1\times N}$,

$R_{N-1}=\left[0,\cdots,0,\frac{1}{6}\lambda,1-\frac{3}{2}\lambda,\frac{4}{3}\lambda\right]_{1\times N}$, $R_N=\left[0,\cdots,0,-\frac{1}{2}\lambda,3\lambda,-\frac{15}{2}\lambda,1+\ 5\lambda\right]_{1\times N}$.

$\boldsymbol{U}=[u_1,u_2,\cdots,u_N]^T$,and

$$\boldsymbol{RH}=\begin{pmatrix}\frac{k}{2}\left(f_{1,j+1}+f_{1,j}\right)+\frac{\lambda}{3}\left(4\left(u_{0,j+1}+u_{0,j}\right)+3h\left(u'_{0,j+1}+u'_{0,j}\right)\right)\\ \frac{k}{2}\left(f_{2,j+1}+f_{2,j}\right)+\frac{3h\lambda}{25}\left(u'_{0,j+1}+u'_{0,j}\right)\\ \frac{k}{2}\left(f_{3,j+1}+f_{3,j}\right)\\ \vdots\\ \frac{k}{2}\left(f_{N-1,j+1}+f_{N-1,j}\right)+h\lambda\left(u'_{N,j+1}+u'_{N,j}\right)\\ \frac{k}{2}\left(f_{N,j+1}+f_{N,j}\right)+3h\lambda\left(u'_{N,j+1}+u'_{N,j}\right)\end{pmatrix}_{1\times N}\text{, where}\lambda=\frac{k}{h^3}$$

Table 1. Matrix A in table form.

$1+\frac{3}{2}\lambda$	0	$-\frac{1}{6}\lambda$	0			0
$-\frac{1}{50}\lambda$	$1-\frac{9}{25}\lambda$	$\frac{33}{50}\lambda$	$\frac{-7}{25}\lambda$	0		
$\frac{1}{4}\lambda$	$-\frac{1}{2}\lambda$	1	$\frac{1}{2}\lambda$	$-\frac{1}{4}\lambda$	0	
	$\ddots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$	
		$\frac{1}{4}\lambda$	$-\frac{1}{2}\lambda$	1	$\frac{1}{2}\lambda$	$-\frac{1}{4}\lambda$
			0	$\frac{1}{6}\lambda$	$1-\frac{3}{2}\lambda$	$\frac{4}{3}\lambda$
0			$-\frac{1}{2}\lambda$	3λ	$-\frac{15}{2}\lambda$	$1+5\lambda$

We consider the only homogenous part of the difference scheme (18) which enable us to establish the stability of considered problem (16). Thus we write (18) in the following form,

$$\boldsymbol{U}^{k+1} = (-\boldsymbol{I} + \boldsymbol{2A^{-1}}) \cdot \boldsymbol{U}^{k} \qquad (19)$$

Let $\boldsymbol{u}^{k+1}$ and $\boldsymbol{u}^{k}$ be the exact solution of the proposed difference method at respectively $(k+1)^{th}$ and k^{th} time level. So we have errors in these approximate and exact solutionat different time level in absence of round off errors. Let these errors are at $(k+1)^{th}$ and k^{th} time level are,

$$\boldsymbol{\varepsilon}^{k+1} = \boldsymbol{U}^{k+1} - \boldsymbol{u}^{k+1} \text{ and } \quad \boldsymbol{\varepsilon}^{k} = \boldsymbol{U}^{k} - \boldsymbol{u}^{k}\ .$$

Thus we have error equation using above errors at $(k+1)^{th}$ and k^{th} time level in the following form,

$$\boldsymbol{\varepsilon}^{k+1} = (-\boldsymbol{I} + \boldsymbol{2A^{-1}}) \cdot \boldsymbol{\varepsilon}^{k} \qquad (20)$$

It is easy to prove that $\|-\boldsymbol{I} + \boldsymbol{2A^{-1}}\|$=0 but it is amplification matrix $(-\boldsymbol{I} + \boldsymbol{2A^{-1}})$ in (19) and (20). The eigen values of matrix $(-\boldsymbol{I} + \boldsymbol{2A^{-1}})$ will not exceed 0 (Jain [9], Varga [10]). Hence our proposed method is stable and stability of the proposed method is absolute and independent of the mesh ratio parameter λ.

5. NUMERICAL EXPERIMENTS

In this section, we have applied the proposed method (5) to solve numerically three different linear and nonlinear model problems. We have used Gauss-Seidel and Newton Raphson method respectively to solve the system of linear and nonlinear equations arises from equation (5). All computations were performed on a Windows 2007 home basic operating system in the GNU FORTRAN environment version 99 compiler (2.95 of gcc) on Intel Core i3-2330M, 2.20 Ghz PC. Let u_i^* , the numerical value calculated by formulae (8), an approximate value of the theoretical solution $u(x,t)$ at the grid point (x_i, t_M). We have used following formula in calculation of MAE, themaximum absolute error

$$MAE(\mathrm{u}) = \max_{1\leq\, i\, \leq N} |u(x_i, \mathrm{t}) - u_i^*|$$

are shown in Tables 1-2, for different value of t,N and M. The iteration is continued until either the maximum difference between two successive iterates is less than 10^{-10} or the number of iteration reached 2×10^3.

Example 1. Consider the following initial-boundary value problem

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^3 u(x,t)}{\partial x^3} + f(t,x), \;\; 0 < x < 1, t > 0$$

with the initial- boundary conditions are $\mathrm{u(x,0) = \cos{(x)}}$, $\mathrm{u(0,t) = \exp(-t)}$, $\frac{\partial \mathrm{u(0,t)}}{\partial \mathrm{x}} = 0$ and $\frac{\partial \mathrm{u(1,t)}}{\partial \mathrm{x}} = -\sin(1)\exp{(-\mathrm{t})}$. The forcing function $f(t,x)$ is calculated so that $u(x,t) = \cos(x)\, exp(-t)$ is the exact solution of the considered problem.The maximum absolute error calculated at $(k+1)^{th}$ level by the application of proposed method (8) and presented in Table 2.

Example 2. Consider the following nonlinear initial-boundary value problem

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^3 u(x,t)}{\partial x^3} + 5\frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t)\frac{\partial u(x,t)}{\partial x} + \; f(t,x), \;\; 0 < x < 1, t > 0$$

with the initial- boundary conditions are $\mathrm{u(x,0) = \cos{(1+x)}}$, $\mathrm{u(0,t) = \cos(1)}\,(1-t^2)$, $\frac{\partial \mathrm{u(0,t)}}{\partial \mathrm{x}} = -\sin(1)(1-t^2)$ and $\frac{\partial \mathrm{u(1,t)}}{\partial \mathrm{x}} = -\sin(2)\,(1-t^2)$. The forcing function $f(t,x)$ is calculated so that $u(x,t) = \cos(1+x)\,(1-t^2)$ is the exact solution of the considered problem.The maximum absolute error calculated at $(k+1)^{th}$ level by the application of proposed method (8) and presented in Table 3.

Table 2. Maximum absolute error $|u(x_i, t) - u_i^*|$ in example 1.

M+1	*N+1*	*MAE*		
		$t = 0.5 \times 10^{-4}$	$t = 1.0 \times 10^{-4}$	$t = 2.0 \times 10^{-4}$
4	4	.10915762(-5)	.21831524(-5)	.45129145(-5)
	8	.25711120(-6)	.45461775(-6)	.57900775(-6)
	16	.14077891(-6)	.12904145(-6)	*overflow*
8	4	.13895994(-5)	.21235478(-5)	.42148913(-5)
	8	.44978475(-6)	.51422239(-6)	.56358113(-6)
	16	.25998821(-6)	.21619917(-6)	.31288796(-5)

Table 3. Maximum absolute error $|u(x_i, t) - u_i^*|$ in example 2.

M+1	*N+1*	*MAE*		
		$t = 0.5 \times 10^{-4}$	$t = 1.0 \times 10^{-4}$	$t = 2.0 \times 10^{-4}$
6	4	.11612502(-5)	.23502221(-5)	.45729116(-5)
	8	.29698285(-6)	.53228035(-6)	.96683073(-6)
	16	.88366598(-7)	.20445479(-6)	.48999362(-6)
12	4	.12506572(-5)	.23204198(-5)	.45431093(-5)
	8	.29698285(-6)	.53228035(-6)	.87742382(-6)
	16	.17777357(-6)	.91800537(-7)	.84762149(-6)

We observe from numerical experimental results for considering problems that the error increases as time steps decreases. Also error decreases as space steps decrease, but error increases and time increases. Thus we conclude that our proposed method converges and order of accuracy approving the estimated order.

6. CONCLUSION

A numerical method has been developed for the numerical solution of a third order initial- boundary value problem in partial differential equation in one space dimension. A developed finite difference method is stable and convergent. The method was tested on model problems. We observed numerical results are in good agreement to the theoretical results.

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