ORIGINAL PAPER

DEVELOPED MOTION OF ROBOT END-EFFECTOR OF SPACELIKE RULED SURFACES (THE FIRST CASE)

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Abstract. *The trajectory of a robot end effector is described by a ruled surface and a spin angle about the ruling of the ruled surface. In this paper, we analyzed the problem of describing trajectory of a robot end-effector by a spacelike ruled surface with spacelike ruling. We obtained the developed frame* $\{t_1, r_1, k_1\}$ *by rotating the generator frame* $\{r, t, k\}$ *at an Darboux angle* $\theta = \theta(s)$ *in the plane* $\{r, k\}$*, which is on the striction curve* β *of the spacelike ruled surface* X *. Afterword, natural frame, tool frame and surface frame which is necessary for the movements of robot are defined derivative formulas of the frames are founded by calculating the Darboux vectors. Tool frame* $\{O, A, N\}$ *are constituted by means of this developed frame. Thus, robot end effector motion is defined for the spacelike ruled surface* φ *generated by the orientation vector* $t_1 = O$*. Also, by using Lancret curvature of the surface and rotation angle (Darboux angle) in the developed frame the robot end-effector motion is developed. Therefore, differential properties and movements an different surfaces in Minkowski space is analyzed by getting the relations for curvature functions which are characterized a spacelike ruled surface with spacelike directix. Finally, to be able to get a member of trajectory surface family which has the same trajectory curve is shown with the examples in every different choice of the Darboux angle which is used to described the developed frame.*

AMS: *53C40, 53C50.*

Keywords: *Curvature theory, Darboux angle, Developed frame, Robot end-effector, Trajectory curve.*

1. INTRODUCTION

The motion of a robot end effector is referred to as the robot trajectory. The trajectory of a robot end effector is described by a ruled surface and a spin angle about the ruling of the ruled surface. A robot trajeectory consist of; (i) a sequence of positions, velocities and accelerations of a point fixed in the end effector, and (ii) a sequence of orientations, angular velocities and angular accelerations of the end effector. The point fixed in the end effector will be referred to as the Tool Center Point and denoted as the TCP.

Ruled surfaces were first investigated by G. Monge who established the partial differential equation satisfied by all ruled surfaces. Ruled surfaces have been widely applied in designing cars, ships, manufacturing (e.g. CAD/CAM) of products and many other areas such as motion analysis and simulation of rigid body, as well as model-based object recognition system. However, ruled surfaces are stil widely used in many areas in modern

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surface modelling systems. Ruled surfaces in Minkowski 3-space have been studied in a lot of fields. More information about timelike ruled surfaces in Minkowski 3-space may be also found in Turgut and Hacısalihoğlu’s papers in [1-2] and Öğrenmiş et al. [3].

Curvature theory investigates the intrinsic geometric properties of the trajectory of points, lines, and planes embedded in a moving rigid body. Curvature theory is also concerned with the velocity and acceleration distribution of a moving rigid body in constrained motion. The curvature theory is using to determine the differential properties of the motion of a robot end effector. The differential properties of the robot end effector motion are then related to the time dependent properties of the motion which are essential in the robot trajectory planning. The differential properties of the ruled surface generate the linear and angular motion properties of the robot end effector for robot path planning [4]. Also, the curvature theory of line trajectories seeks to characterize the shape of the trajectory ruled surface and relates it to the motion of body carrying the linet hat generates it [5]. Ryuh and Pennock [6] applied the curvature theory of a ruled surface to study the instantaneous motion properties of a robotic device. The differential properties of motion of the end effector were determined from the curvature theory. Also, they proposed a method of robot trajectory planning based on the curvature theory of a ruled surface incorporated with the geometric modeling technique in [4]. In this method, it is shown that how a ruled surface may be generated using the geometric modeling technique of a curve. Ryuh, Lee and Moon in [7] studied a precision control method of a robot path generation based on the dual curvature theory a ruled surface. In [8], the authors are developed a new adjustment method for improving machining accuracy of tool path in five-axis flank milling of ruled surfaces. They proposed a feedrate adjustment rule that automatically controls the tool motion at feedrate-sensitive corners based on a bisection method. Also they are conducted on different ruled surfaces to verify the effectiveness of the proposed method. Kim et al. [9] developed a real-time trajectory generation method and control approach for a five-axis NC machine. They describe the spatial trajectory of the tool of the five axis machine by a ruled surface, and the differential motion parameters of the tool were obtained from the curvature theory of the ruled surface. Also, they were used the Fergusen geometric modeling technique to present the tool trajectory as a ruled surface. Also, in [10-12] the authors have studied manipulators.

The motion of robot end-effector is a research topic of various studies in Minkowski 3-space. Ekici et al. studied the differential properties of robot end-effectors motion using the curvature theory of timelike ruled surfaces with timelike ruling in [13]. In [14], Turhan and Ayyıldız used the curvature theory of ruled surfaces with lightlike ruling in Minkowski 3-space. They also derived the relation between these functions and the curvature functions of the central normal surface whose ruling spacelike.

In this paper, we address the path planning problem using the curvature theory of a ruled surface. The objects consist of point in the coordinate plane. We can locate such coordinates by rotating these objects in a specific direction. This allows the calculation of the robot's next motion. So, any errors and miscalculations that may arise in trajectory planning can be prevented. Each robot has a unique coordinate system. However, the appropriate choice of coordinates for the robot motion allows us to define the work area of the robot more efficiently. Therefore, we obtained the developed frame $\{t_1, r_1, k_1\}$ by rotating the generator frame $\{r, t, k\}$ at an angle $\theta = \theta\,(s)$ in the plane $\{r, k\}$ to provide a practical work area. It is useful in animation motion planning, and tool path planning in CAD/CAM. Thus, this study represents robot path as a ruled surface generated at the Tool Center Point and by the unit vector $(t_1 = O)$ of the tool frame. The vector $t_1 = O$ is depending on the Darboux angle function $\theta = \theta\,(s)$. New direction vectors are achieved by changing the angle function. The robot trajectory changes depending on the angle function. Therefore, we obtained trajectory ruled surface family with a common trajectory curve in the rotation trihedron. Any other generated trajectory corresponds to a member of this trajectory ruled surface family. The given calculations (i.e, positional variation of the TCP, linear velocity, angular velocity) are valid for all members of the trajectory ruled surface family. Therefore, we defined the desired trajectory of the robot end effector motion and give the differential properties of robot endeffectors motion using the curvature theory. Also, the motion of robot end effector is illustrated with examples by two members of the spacelike trajectory ruled surface family.

2. PRELIMINARIES

Let us consider Minkowski 3-space $IR_1^3 = [IR_1^3, (-,+,+)]$ and let the Lorentzian inner product of $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3) \in IR_1^3$ be

$$\langle X, Y \rangle = -x_1 y_1 + x_2 y_2 + x_3 y_3 .$$

The norm of $X \in IR_1^3$ is denoted by $\|X\|$ and defined as $\|X\| = \sqrt{|\langle X, X \rangle|}$. A vector $X = (x_1, x_2, x_3) \in IR_1^3$ called a spacelike, timelike and null (lightlike) vector if $\langle X, X \rangle > 0$ or X=0 , $\langle X, X \rangle < 0$ and $\langle X, X \rangle = 0$ for $X \neq 0$, respectively. A timelike vector is to be positive (resp.negative) if and only if $x_3 > 0$ (resp. $x_3 < 0$) , [15].

The vector product of vectors $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ in IR_1^3 is defined by

$$X \times Y = (x_2 y_3 - x_3 y_2, x_1 y_3 - x_3 y_1, x_2 y_1 - x_1 y_2) .$$

For X and Y spacelike vectors in IR_1^3, if the inequality $|\langle X, Y \rangle| > \|X\| \|Y\|$ is satisfied, there is a unique real number a such that,

$$\langle X, Y \rangle = \|X\| \, \|Y\| \cosh \alpha .$$

If the inequality $|\langle X, Y \rangle| \leq \|X\| \|Y\|$ is satisfied, there is a unique real number a such that

$$\langle X, Y \rangle = \|X\| \, \|Y\| \cos \alpha .$$

Let X be a spacelike vector and Y be a positive timelike vector in IR_1^3. Then there is a unique nonnegative real number α such that $\langle X, Y \rangle = \|X\| \, \|Y\| \sinh \alpha$.

For X and Y be timelike vectors in IR_1^3.Then there is a unique real number α such that $\langle X, Y \rangle = \|X\| \, \|Y\| \cosh \alpha$ [16].

Theorem 2.1. Let X, Y $\in IR_1^3$. We have
i) *If X and Y are the spacelike vectors,* $X \times Y$ *is a timelike vector*
ii) *If X and Y are the timelike vectors,* $X \times Y$ *is a spacelike vector*
iii) *If X is the spacelike vector and Y is the timelike vector,* $X \times Y$ *is a spacelike vector*
where $\times$ is Lorentzian cross product [1-2].

A smooth regular curve $\alpha : I \subset IR \to IR_1^3$ is said to be timelike, spacelike or lightlike curve if the velocity vector α' is a timelike, spacelike or lightlike vector, respectively [15]. In fact, a timelike curve corresponds to the path of on observer moving at less than the speed of light. Null curves correspond to moving at the speed of light and spacelike curves to moving faster than light.

Let $\alpha = \alpha(s)$ be a unit speed curve in IR_1^3; by $\kappa(s)$ and $\tau(s)$ we denote the natural curvature and torsion of $\alpha(s)$, respectively. Consider the Frenet frame $\{e_1, e_2, e_3\}$ associated with curve $\alpha = \alpha(s)$ such that $e_1 = e_1(s)$, $e_2 = e_2(s)$ and $e_3 = e_3(s)$ are the unit tangent, the principal normal and the binormal vector fields, respectively. If $\alpha = \alpha(s)$ is a spacelike curve, then the structural equations (or Frenet formulas) of this frame are given as

$$e_1'(s) = \kappa(s)e_2(s) \quad , \quad e_2'(s) = \varepsilon\kappa(s)e_1(s) + \tau(s)e_3(s), \quad e_3'(s) = \tau(s)e_2(s),$$

where $\varepsilon = \begin{cases} -1, e_3 \ \textit{is timelike}, \\ 1, e_3 \ \textit{is spacelike.} \end{cases}$

If $\alpha = \alpha(s)$ is a timelike curve, then above equations are given as [15]:

$$e_1'(s) = \kappa(s)e_2(s) \quad , \quad e_2'(s) = \kappa(s)e_1(s) - \tau(s)e_3(s), \quad e_3'(s) = \tau(s)e_2(s),$$

A surface M in IR_1^3 is called a spacelike surface if the induceded metric on the surface is a positive defination metric. The normal vector on the spacelike surface is a timelike vector [15]. A spacelike ruled surface in IR_1^3 is obtained by a spacelike straight line moving along spacelike curve [15]. The spacelike ruled surface M is given parameterization

$$\psi : I \times IR \to IR_1^3 \ , \ \psi(s,v) = \alpha(s) + vX(s) \ \text{ in } \ IR_1^3 .$$

3. FRAMES OF REFERENCE

A timelike ruled surface which indicates the tool path has a parametric representation,

$$X(s,v) = \alpha(s) + v\overline{R}(s) \tag{3.1}$$

where $\alpha(s)$ a spacelike curve is the specified path of the TCP, v is a real-valued parameter, and $\overline{R}(s)$ spacelike straight line is the vector generating the timelike ruled surface (called the ruling).

The striction curve of spacelike ruled surface X is

$$\beta(s)=\alpha(s)-\mu(s)\overline{R}(s) \tag{3.2}$$

where the parameter

$$\mu(s)=\left\langle \alpha^{'}(s),\overline{R}^{'}(s)\right\rangle \tag{3.3}$$

indicates the distance from the striction point of the spacelike ruled surface to the TCP.

For the generator trihedron, there are two cases. The generator trihedron is defined by three mutually orthogonal unit vectors, namely;

i) the spacelike generator vector $r=(1/R)\overline{R}(s)$, the spacelike central normal vector $t=\overline{R}^{'}$, and the timelike central tangent vector $k=r\times t$, where R is $\left\|\overline{R}(s)\right\|$.

ii) the spacelike generator vector $r=(1/R)\overline{R}(s)$, the timelike central normal vector $t=\overline{R}^{'}$, and the spacelike central tangent vector $k=t\times r$, where R is $\left\|\overline{R}(s)\right\|$.

Now let's make the necessary calculations for the first of these cases. Likewise it can be done in the other.

The first order positional variation of the striction line of the spacelike ruled surface may be expressed in the generator trihedron as

$$\beta^{'}=\Gamma r+\Delta k \tag{3.4}$$

where

$$\begin{aligned}\Gamma&=\frac{1}{R}\left\langle \alpha^{'},\overline{R}\right\rangle-\mu^{'}R\\ \Delta&=\frac{1}{R}\left\langle \alpha^{'},\overline{R}^{'}\times\overline{R}\right\rangle\end{aligned} \tag{3.5}$$

is referred to as the curvature functions of the spacelike ruled surface.

First order angular variation of the generator trihedron may be expressed in the matrix form as

$$\frac{d}{ds}\begin{bmatrix} r\\ t\\ k\end{bmatrix}=\frac{1}{R}\begin{bmatrix} 0 & 1 & 0\\ -1 & 0 & \gamma\\ 0 & \gamma & 0\end{bmatrix}\begin{bmatrix} r\\ t\\ k\end{bmatrix} \tag{3.6}$$

where

$$\gamma=\left\langle \overline{R}^{''},\overline{R}^{'}\times\overline{R}\right\rangle$$

is referred to as the geodesic curvature of the curve drawn by the ruling vectors $\overline{R}(s)$ of the spacelike ruled surface.

For the Darboux vector of generator trihedron of spacelike ruled surface, we can write

$$U_r = t \times t' = \frac{1}{R}(-\gamma r + k)$$

(3.7)

Also, the Lancret curvature of spacelike ruled surface X is

$$\lambda = \|t'\| = \sqrt{\left|\frac{1-\gamma^2}{R^2}\right|} \tag{3.8}$$

4. DEVELOPED TRIHEDRON

Let us rotate the generator trihedron $\{r,t,k\}$ on the striction curve of the spacelike ruled surface X at the central point at an Darboux Lorentzian angle $\theta = \theta(s)$, $\theta \neq$ fixed, in the plane $\{r,k\}$. So, it can be written in matrix form as

$$\begin{bmatrix} r_1 \\ k_1 \end{bmatrix} = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} r \\ k \end{bmatrix} \tag{4.1}$$

In fact, by using the above rotation equation and we have relations

$$\begin{aligned} r_1 &= \cosh\theta r - \sinh\theta k, \\ t_1 &= t, \\ k_1 &= -\sinh\theta r + \cosh\theta k. \end{aligned}$$

The orthonormal system $\{t_1, r_1, k_1 = U_r\}$ is called the developed trihedron of the spacelike ruled surface X. Here, k_1 is Darboux vector of generator trihedron of the spacelike ruled surface *X*.

Eqns. (3.7) and (3.9) may be written as

$$\sinh\theta + \gamma\cosh\theta = 0 \tag{4.2}$$

By using Eqn. (4.2) into Eqn. (3.8) we get the relations

$$\begin{aligned} \frac{1}{R} &= \lambda\cosh\theta \\ \frac{\gamma}{R} &= -\lambda\sinh\theta \end{aligned} \tag{4.3}$$

The first-order angular variation of developed trihedron $\{t_1, r_1, k_1\}$ may be expressed in the matrix form as

$$\frac{d}{ds}\begin{bmatrix} t_1 \\ r_1 \\ k_1 \end{bmatrix} = \begin{bmatrix} 0 & -\lambda & 0 \\ \theta^{'} & 0 & \lambda \\ 0 & \theta^{'} & 0 \end{bmatrix}\begin{bmatrix} t_1 \\ r_1 \\ k_1 \end{bmatrix} \tag{4.4}$$

where $\theta^{'}$ is the curvature and λ is the Lancret curvature of the spacelike ruled surface X.

Also; t_1, r_1 and k_1 be the spacelike generator vector, the spacelike central normal vector and the timelike central tangent vector, respectively.

Each vector of developed trihedron in end effector defines its own ruled surface while the robot moves. Let us take the following spacelike ruled surface (robot trajectory ruled surface) as

$$\varphi(s,v) = \alpha(s) + vt_1(s) \tag{4.5}$$

where $\alpha(s)$ spacelike curve is the specified path of the TCP (called the directrix of the timelike ruled surface X and φ), v is a real valued parameter, and $t_1(s)$ is the spacelike vector generating the spacelike ruled surface φ (called the ruling or direction). Also, this surface is trajectory surface of robot.

If you take a surface formed by a k_1 timelike central tangent vector, such a surface is not defined in IR_1^3. So, is not talk about the robot trajectory ruled surface.

If you take a surface formed by a r_1 spacelike central normal vector, such a surface is to be central normal surface, will examine in section 5.

The striction curve of spacelike ruled surface φ is

$$\beta_{t_1}(s) = \alpha(s) - \mu_{t_1}(s)t_1(s) \tag{4.6}$$

where the parameter

$$\mu_{t_1}(s) = -\frac{\Gamma\cosh\theta - \Delta\sinh\theta + \mu^{'}R\cosh\theta}{\lambda} \tag{4.7}$$

Also, Γ and Δ are referred to as the curvature functions of the spacelike ruled surface (Eqn. (3.1)). Differentiating Eqn.(4.6) gives first order positional variation of the striction point of the spacelike ruled surface φ. By using Eqns. (4.4) and (4.7) we can write Eqn.(4.6) with respect to developed trihedron as

$$\beta^{'}_{t_1}(s) = \Gamma_{t_1}k_1 + \Delta_{t_1}t_1 \tag{4.8}$$

where

$$\begin{aligned} \Gamma_{t_1} &= \Delta\cosh\theta - \Gamma\sinh\theta - \mu' R\sinh\theta - \mu_{t_1}^{'} \\ \Delta_{t_1} &= \mu - \mu_{t_1}^{'} \end{aligned} \tag{4.9}$$

How that, Γ and Δ are curvature functions which characterize the spacelike ruled surface here Γ_{t_1} and Δ_{t_1} are curvature functions which characterize the robot trajectory spacelike ruled surface.

The positional variation of the striction line may be considered as the linear velocity. As in the case of the developed frame (4.4) may also be written as

$$U_{t_1} = r_1 \times r_1^{'} = \theta^{'} t_1 + \lambda k_1 \tag{4.10}$$

which is Darboux vector of the developed frame. In a planar curve, the first term will drop out and the developed frame will rotate around the k_1 vector with an angular velocity. This formulation is useful for studying the rotational motions of rigid body attached to the developed frame moving along a curve.

5. CENTRAL NORMAL SURFACE

As the developed trihedron moves along the striction curve β_{t_1}, the central normal vector generates another spacelike ruled surface which is called the spacelike central normal surface. The spacelike central normal surface is defined as

$$\varphi_{r_1}(s,v) = \beta_{t_1}(s) + v r_1(s) \tag{5.1}$$

The striction curve of spacelike central normal surface is

$$\beta_{r_1}(s) = \beta_{t_1}(s) - \mu_{r_1}(s) r_1(s) \tag{5.2}$$

Differentiating (5.2) and by using Eqn. (4.8) into Eqn. (4.4) gives

$$\mu_{r_1}(s) = \frac{\lambda \Delta_{t_1} - \theta^{'} \Gamma_{t_1}}{\lambda^2 - \theta^{'2}} \tag{5.3}$$

The natural trihedron is defined by the following three orthonormal vectors; the timelike central normal vector r_1, spacelike principal normal vector r_2, and spacelike binormal vector r_3, as shown in figure 1. Also, the natural trihedron is used to study the angular and positional variation of the normal vector.

For the natural trihedron $\{r_1, r_2, r_3\}$, there are two cases:

i) r_2 timelike vector, r_1 and r_3 spacelike generator vector:

These three vectors are defined, respectively, as

$$r_1 = -\frac{t_1^{'}}{\lambda}$$
$$r_2 = \frac{1}{\kappa} r_1^{'} \tag{5.4}$$
$$r_3 = r_2 \times r_1$$

where $\kappa = \left\| r_1^{'} \right\|$ is the curvature of the spacelike ruled surface φ. Also, here is $r_1 \times r_2 = -r_3$, $r_2 \times r_3 = -r_1$ and $r_3 \times r_1 = r_2$.

ii) r_3 timelike vector, r_1 and r_2 spacelike generator vector:
These three vectors are defined, respectively, as

$$r_1 = -\frac{t_1^{'}}{\lambda}$$
$$r_2 = \frac{1}{\kappa} r_1^{'}$$
$$r_3 = r_1 \times r_2$$

where $\kappa = \left\| r_1^{'} \right\|$ is the curvature of the spacelike ruled surface φ. Also, here is $r_1 \times r_2 = r_3$, $r_2 \times r_3 = -r_1$ and $r_3 \times r_1 = -r_2$.

Now let's make the necessary calculations for the first of these cases. Likewise it can be done in the other.

Let η be the angle between the spacelike vectors k_1 and r_3, see Fig. 1. Here, the developed trihedron and the natural trihedron have the timelike central normal vector in common. Then, we have

$$\begin{aligned} k_1 &= \cosh \eta r_2 + \sinh \eta r_3 \\ t_1 &= \sinh \eta r_2 + \cosh \eta r_3 \end{aligned} \tag{5.5}$$

Substituting Eqn. (4.4) into Eqn. (5.5) and using $r_1^{'} = \kappa r_2$ it follows that

$$\cosh \eta = \frac{\theta^{'}}{\kappa} \quad , \quad \sinh \eta = -\frac{\lambda}{\kappa}. \tag{5.6}$$

From Eqn. (5.6), adding the result and rearranging gives the curvature

$$\kappa = \sqrt{\theta^{'2} - \lambda^2} \tag{5.7}$$

The Darboux vector of developed trihedron may be obtained in the natural trihedron as follows. Substituting Eqn. (5.6) into Eqn. (5.5) gives

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \frac{1}{\kappa}\begin{bmatrix} 0 & \kappa & 0 \\ \lambda & 0 & \theta' \\ \theta' & 0 & \lambda \end{bmatrix}\begin{bmatrix} t_1 \\ r_1 \\ k_1 \end{bmatrix} \tag{5.8}$$

Hence, the Darboux vector of the developed trihedron may be written as

$$U_{t_1} = \kappa r_3 \tag{5.9}$$

which shows that the binormal vector plays the role of the opposite direction of rotation for developed trihedron.

Differentiating Eqn. (5.2) and substituting Eqns.(4.8) and (5.8) into the result, we obtain

$$\beta'_{r_1} = \Gamma_{r_1} r_1 + \Delta_{r_1} r_3 \tag{5.10}$$

where

$$\begin{aligned} \Gamma_{r_1} &= -\mu'_{r_1} \\ \Delta_{r_1} &= \frac{\Delta_{t_1}\theta' - \lambda\Gamma_{t_1}}{\kappa} \end{aligned} \tag{5.11}$$

The first-order angular variation of natural trihedron may be expressed in the matrix form as

$$\frac{d}{ds}\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \tag{5.12}$$

where κ and $\tau = \langle r'_2, r_3 \rangle$ are the curvature and torsion of the spacelike ruled surface φ, respectively.

To find simpler expressions for the curvature and torsion, we substituting Eqn. (5.6) into Eqn. (5.7) which gives

$$\kappa = \theta' \operatorname{sech}\eta \tag{5.13}$$

Differentiating eqn.(5.4) and by using eqns (4.4) and (5.8) we have

$$\tau = -\eta' \tag{5.14}$$

As in the case of the natural trihedron, eqn. (5.16) may also be written as

$$U_{r_2} = \kappa r_3 - \tau r_1 \tag{5.15}$$

which is the Darboux vector of the natural trihedron.

Hence, observe that both the Darboux vector of the natural trihedron and the Darboux vector of the developed trihedron describe the angular motion of the spacelike ruled surface and the spacelike central normal surface.

6. RELATIONSHIP BETWEEN THE FRAMES

Path of a robot may be represented by a tool center point and tool frame of end-effector. For tool frame, there are two cases. The tool frame is represented by three mutually perpendicular unit vector $[O,A,N]$,

i) where O is the spacelike orientation vector, A is the timelike approach vector and N is the spacelike normal vector, shown in Fig. 1.

ii) where O is the spacelike orientation vector, A is the spacelike approach vector and N is the timelike normal vector, shown in Fig. 1.

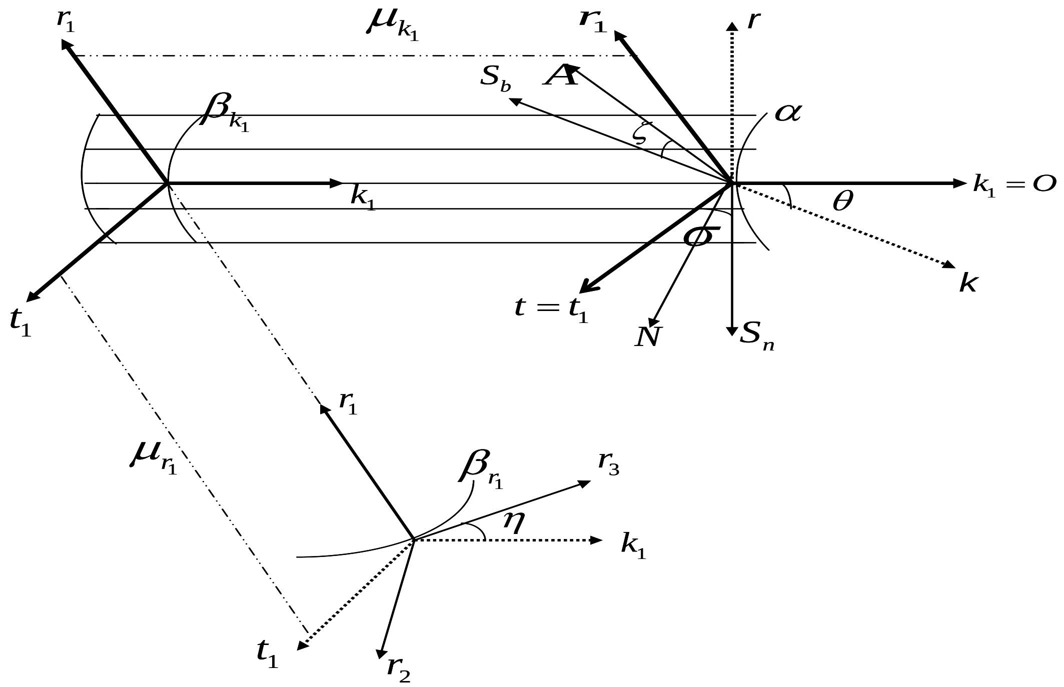

Figure 1. The relationship between frames.

Each vector of tool frame in end-effector defines its own ruled surface while the robot moves. Let $O = t_1$ and the vector O are called directix and ruling, respectively. Then, the surface frame, $[O, S_n, S_b]$, of spacelike ruled surface φ may be determined as follows:

$$S_n = \frac{\varphi_s \times \varphi_v}{\left\|\varphi_s \times \varphi_v\right\|}\Big|_{v=0} \tag{6.1}$$

which is the unit timelike normal of spacelike ruled surface φ in TCP.

Substituting Eqns. (4.4), (4.8) and Eqn.(4.6) into Eqn.(6.1) we obtain

$$S_n = \frac{\Gamma_{t_1} r_1 - \lambda \mu_{t_1} k_1}{\sqrt{\Gamma_{t_1}{}^2 - \left(\lambda \mu_{t_1}\right)^2}} \tag{6.2}$$

where Γ_{t_1} and μ_{t_1} are as defined by Eqs.(4.9), (4.7), (5.3) and (3.3), respectively.

$$S_b = S_n \times O \tag{6.3}$$

is unit spacelike binormal vector of the spacelike ruled surface.

Substituting Eqn.(6.2) into Eqn.(6.3) gives

$$S_b = \frac{\Gamma_{t_1} k_1 - \lambda \mu_{t_1} r_1}{\sqrt{\Gamma_{t_1}{}^2 - \left(\lambda \mu_{t_1}\right)^2}} \tag{6.4}$$

The orientation of the surface frame relative to the tool frame and the developed trihedron is shown in figure 1. Let ζ be the angle between timelike vector S_n and the spacelike approach vector A . Then, we have

$$\left\langle S_n, A \right\rangle = \sinh \zeta, \quad A \times O = -N \tag{6.5}$$

We may express the results in matrix form as

$$\begin{bmatrix} O \\ A \\ N \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sinh \zeta & \cosh \zeta \\ 0 & \cosh \zeta & \sinh \zeta \end{bmatrix} \begin{bmatrix} O \\ S_n \\ S_b \end{bmatrix} \tag{6.6}$$

Let the angle between the vectors S_b and t_1 be σ . Then, we have

$$\begin{aligned} S_n &= \sinh \sigma r_1 + \cosh \sigma k_1 \\ S_b &= \cosh \sigma r_1 + \sinh \sigma k_1 \end{aligned} \tag{6.7}$$

From Eqns.(6.6) and (6.7) we can write

$$\begin{bmatrix} O \\ A \\ N \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh \Sigma & \sinh \Sigma \\ 0 & \sinh \Sigma & \cosh \Sigma \end{bmatrix} \begin{bmatrix} t_1 \\ r_1 \\ k_1 \end{bmatrix} \tag{6.8}$$

where $\Sigma = \zeta + \sigma$ describes the orientation of the end-effector.

7. DIFFERENTIAL MOTION OF THE TOOL FRAME

The spacelike curve generated by TCP from eqn. (3.2) is

$$\alpha(s)=\beta(s)+\mu\overline{R}(s) \tag{7.1}$$

Differentiating eqn. (7.1) with respect to the arc length, using eqns.(3.6) and (3.4), the first-order positional variation of the TCP, expressed in the generator trihedron is

$$\alpha^{'}(s)=\left(\Gamma+\mu^{'}R\right)r+\mu t+\Delta k \tag{7.2}$$

Substituting eqn. (4.1) into eqn.(7.2), gives

$$\alpha^{'}=\left(-\left(\Gamma+\mu^{'}R\right)\sinh\theta+\Delta\cosh\theta\right)k_1+\left(\left(\Gamma+\mu^{'}R\right)\cosh\theta-\Delta\sinh\theta\right)r_1+\mu t_1 \tag{7.3}$$

Also, substituting eqn. (6.11) into eqn. (7.3), gives

$$\begin{aligned}\alpha^{'}=\mu O&+\left[\cosh\Sigma\left(\left(\Gamma+\mu^{'}R\right)\cosh\theta-\Delta\sinh\theta\right)+\sinh\Sigma\left(-\left(\Gamma+\mu^{'}R\right)\sinh\theta+\Delta\cosh\theta\right)\right]A\\&+\left[\sinh\Sigma\left(\left(\Gamma+\mu^{'}R\right)\cosh\theta-\Delta\sinh\theta\right)+\cosh\Sigma\left(-\left(\Gamma+\mu^{'}R\right)\sinh\theta+\Delta\cosh\theta\right)\right]N\end{aligned} \tag{7.4}$$

Differentiating eqn. (6.9) and substituting eqn.(4.4) into the result the first order angular variation of the tool frame gives

$$\frac{d}{ds}\begin{bmatrix}O\\A\\N\end{bmatrix}=\begin{bmatrix}0&\theta^{'}&0\\\lambda\cosh\Sigma&\sinh\Sigma\left(\theta^{'}+\Sigma^{'}\right)&\cosh\Sigma\left(\theta^{'}+\Sigma^{'}\right)\\\lambda\sinh\Sigma&\cosh\Sigma\left(\theta^{'}+\Sigma^{'}\right)&\sinh\Sigma\left(\theta^{'}+\Sigma^{'}\right)\end{bmatrix}\begin{bmatrix}t_1\\r_1\\k_1\end{bmatrix} \tag{7.5}$$

The first-order angular variation of tool frame $\{O,A,N\}$ may be expressed in the matrix form as

$$\frac{d}{ds}\begin{bmatrix}O\\A\\N\end{bmatrix}=\begin{bmatrix}0&-\lambda\cosh\Sigma&\lambda\sinh\Sigma\\\lambda\cosh\Sigma&0&\theta^{'}+\Sigma^{'}\\\lambda\sinh\Sigma&\theta^{'}+\Sigma^{'}&0\end{bmatrix}\begin{bmatrix}O\\A\\N\end{bmatrix}$$

(7.6)

As in the case of the tool frame, may also be written as

$$U_A=\left(\theta^{'}+\Sigma^{'}\right)O-\lambda\sinh\Sigma A+\lambda\cosh\Sigma N \tag{7.7}$$

which is the Darboux vector of the tool frame. Here, we obtain by using eqn.(6.8)

$$U_A = \left(\theta' + \Sigma'\right)t_1 + \lambda k_1 \tag{7.8}$$

Also, substituting eqn. (4.10) into eqn. (7.8), gives

$$U_A = U_{t_1} + \Sigma' t_1 \tag{7.9}$$

Hence, angular variation of tool frame that according to developed frame is the rotation around the r_1 vector.

8. EXAMPLES

Example 1. Consider the spacelike ruled surface

$$X\left(s,v\right) = \left(\cosh s(1-v), s+2v, \sinh s(-1+v)\right)$$

where $\alpha(s) = \left(\cosh s, s, -\sinh s\right)$ (spacelike) is the base curve $\overline{R}(s) = \left(-\cosh s, 2, \sinh s\right)$ (spacelike) is the genarator. $-\pi \le s \le 0 \quad and \quad -1 \le v \le 1$, (Fig. 2), the generator trihedron is

$$r = \frac{1}{\sqrt{3}}\left(-\cosh s, 2, \sinh s\right),$$
$$t = \left(-\sinh s, 0, \cosh s\right),$$
$$k = \frac{1}{\sqrt{3}}\left(-2\cosh s, -1, 2\sinh s\right).$$

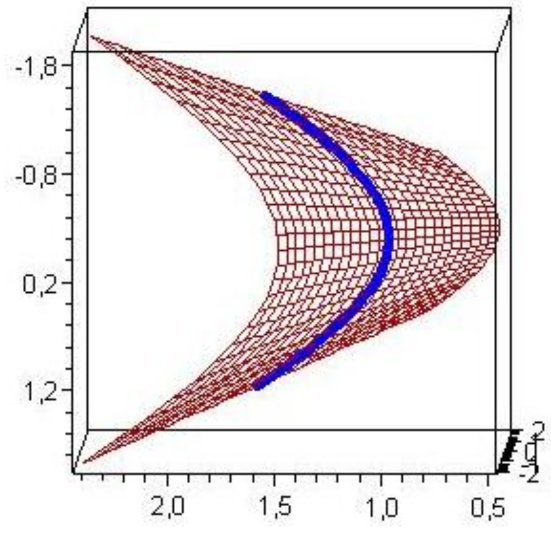

Figure 2. Spacelike ruled surface with generator vector $\overline{R}$.

A straight forward computation shows that

$$\mu\left(s\right) = 1, \qquad \Gamma\left(s\right) = \frac{2}{\sqrt{3}}, \Delta\left(s\right) = \frac{1}{\sqrt{3}} \text{ and } \gamma = \left\langle \overline{R}'', \overline{R}' \times \overline{R} \right\rangle = 2.$$

Also, the Darboux vector of generator trihedron $U_r = \left(0,1,0\right)$.

The developed trihedron is defined by,

$$\begin{cases} k_1 = (-\frac{1}{\sqrt{3}}\cosh s(\sinh\theta(s)+2\cosh\theta(s)),\frac{1}{\sqrt{3}}(2\sinh\theta(s)-\cosh\theta(s)),\frac{1}{\sqrt{3}}\sinh s(\sinh\theta(s)+2\cosh\theta(s))) \\ r_1 = (-\frac{1}{\sqrt{3}}\cosh s(\cosh\theta(s)+2\sinh\theta(s)),\frac{1}{\sqrt{3}}(2\cosh\theta(s)-\sinh\theta(s)),\frac{1}{\sqrt{3}}\sinh s(\cosh\theta(s)+2\sinh\theta(s))), \\ t_1 = (-\sinh s, 0, \cosh s). \end{cases}$$

Therefore, spacelike central normal trajectory ruled surface family with a common trajectory curve is defined by

$$\varphi(s,v) = \begin{pmatrix} \cosh s(1-\frac{v}{\sqrt{3}}(\cosh\theta(s)+2\sinh\theta(s))), s+\frac{v}{\sqrt{3}}(2\cosh\theta(s)-\sinh\theta(s)), \\ \sinh s(-1+\frac{v}{\sqrt{3}}(\cosh\theta(s)+2\sinh\theta(s))) \end{pmatrix} \quad (8.1)$$

If we take $\theta(s)=\cosh s$, $\frac{-\pi}{3}\le s\le\frac{\pi}{3}$ *and* $-0.5\le v\le 0.5$ then we obtain $\varphi_1=\varphi_1(s,v)$ a member of the spacelike trajectory ruled surface family with a common trajectory curve in the developed trihedron as shown in Fig. 3.

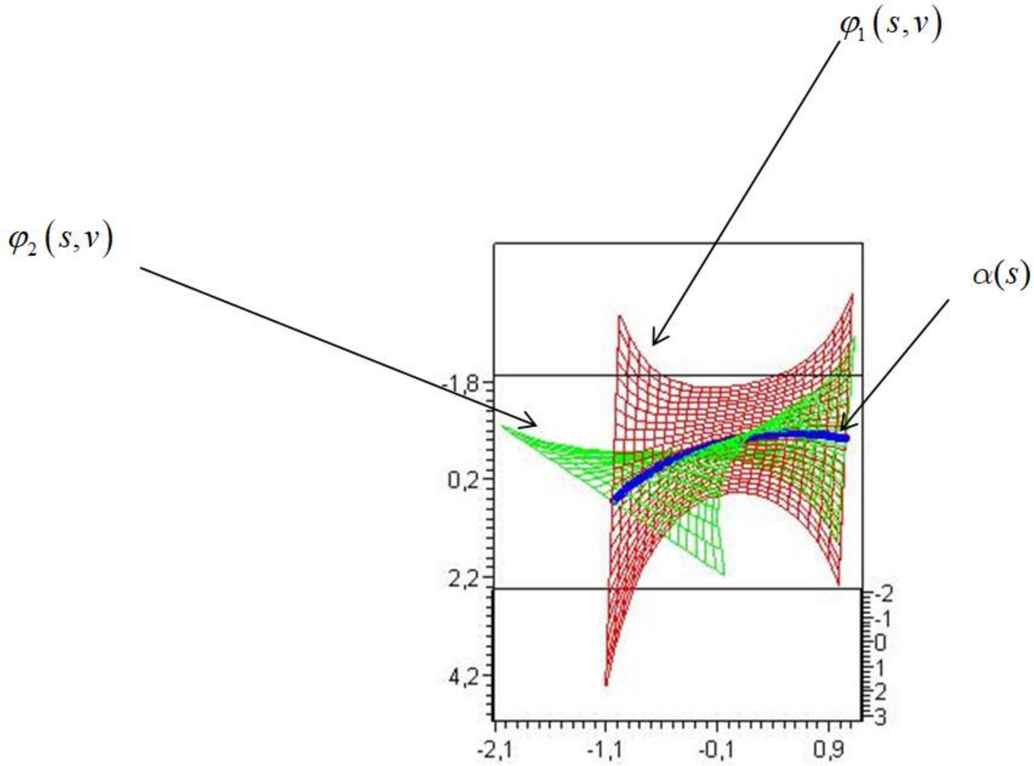

Figure 3. Trajectory spacelike ruled surface.

$$\varphi_1(s,v)=\begin{pmatrix} \cosh s(1-\frac{v}{\sqrt{3}}(\cosh(\cosh s)+2\sinh(\cosh s))), s+\frac{v}{\sqrt{3}}(2\cosh(\cosh s)-\sinh(\cosh s)), \\ \sinh s(-1+\frac{v}{\sqrt{3}}(\cosh(\cosh s)+2\sinh(\cosh s))) \end{pmatrix} \tag{8.2}$$

If we take $\theta(s)=\sinh s$, $\frac{-\pi}{3}\le s\le\frac{\pi}{3}$ *and* $-0.5\le v\le 0.5$ then we obtain $\varphi_2=\varphi_2(s,v)$ another member of the spacelike trajectory ruled surface family with a common trajectory curve in the developed trihedron as shown in Fig. 3.

$$\varphi_2(s,v)=\begin{pmatrix} \cosh s(1-\frac{v}{\sqrt{3}}(\cosh(\sinh s)+2\sinh(\sinh s))), s+\frac{v}{\sqrt{3}}(2\cosh(\sinh s)-\sinh(\sinh s)), \\ \sinh s(-1+\frac{v}{\sqrt{3}}(\cosh(\sinh s)+2\sinh(\sinh s))) \end{pmatrix} \tag{8.3}$$

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