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NEW SECOND DERIVATIVE FREE ITERATIVE METHOD FOR SOLVING NONLINEAR EQUATIONS USING HALLEY'S METHOD

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Abstract. *In this paper, we suggest and analyze a new third order convergence iterative method free from second derivative and use for solving of nonlinear equation* $f(x) = 0$. *This method based on Halley's method using predictor and corrector technique. The convergence analysis of our method is discussed. This technique can be used to suggest a wide class of new iterative methods for solving a system of nonlinear equations and gives better result.*

Keywords: *non linear equations, convergence analysis, iterative methods, second derivatives, Halley's method.*

1. INTRODUCTION

Numerical analysis is the area of mathematics and computer sciences that creates, analyzes and implements algorithms for solving numerically the problems of continuous mathematics. Such problems originate generally from real – world applications of algebra, geometry and calculus and they involve variables which vary continuously. These problems occur throughout the natural sciences, social sciences, engineering, medicine and business. In recent years, many researchers have developed several iterative methods for solving nonlinear equations. In this paper, we are going to develop second derivative free method to find approximations of the root α of $f(x) = 0$. Numerical method for finding approximate solutions of using several different techniques including Taylor's series, quadrature formulae, homotopy and decomposition techniques [1-29] and the references therein. The classical Newton' s method (NM) [12] for solving nonlinear equations is written as:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}$$

which is a well known basic method [28]. The Newton's method (1) was modified by Steffensen's who replaced the first derivative $f'(x)$ in Newton's method by forward difference approximation [12].

$$f'(x) = \frac{f(x_n + f(x_n)) - f(x_n)}{f(x_n)} \tag{2}$$

and obtained the famous Steffensen's method (SM) [6, 7, 10]

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$$x_{n+1} = x_n - \frac{[f(x_n)]^2}{f(x_n+f(x_n))-f(x_n)} \tag{3}$$

Newton's method and Steffensen's method are of second order convergence.

For a given x_0,to compute approximate solution x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - \frac{2f(x_n)\ f'(x_n)}{2f'^2(x_n) - f(x_n)f''(x_n)} \tag{4}$$

which is known as Halley's method of cubic order of convergence [18, 25].

We use Predictor – corrector methods, to discuss the application of the explicit and implicit multistep methods for the solution of the initial value problems. We use explicit (predictor) method for predicting a value and then use the implicit (corrector) method iteratively until the convergence is obtained [6].

2. METHODS

2.1. ITERATIVE METHOD

Numerical analysis is the area of mathematics and computer sciences, which arise in various fields of pure and applied sciences can be formulated in terms of nonlinear equations. Recently, N. Ahmad and V.P.Singh have obtained some new three step iterative methods for solving non linear equations using Steffensen's method and Halley – method (NTSM-2) [22].

NTSM – 2: for a given initial choice x_0, find the approximate solution x_{n+1} by the iterative Schemes

$$a_n = x_n - \frac{[f(x_n)]^2}{f(x_n+f(x_n))-f(x_n)}$$

$$b_n = a_n - \frac{2f(a_n)\ f'(a_n)}{2f'^2(a_n) - f(a_n)f''(a_n)}$$

$$x_{n+1} = x_n - \frac{2\ f(x_n)}{[f'(b_n)+f'(x_n)]}$$

for n=0, 1, 2, 3, …

NTSM-2 has seventh order convergence.

A significant part in developing our new iterative methods free from second derivative with respect to b. To be more precise, we now approximate $f''(a_n)$, to reduce the number of evaluations per iteration by a combination. Toward this end, an estimation of the function $P_2(t)$ is taken into consideration as follows:

$$P_2(t) = a + b\,(t - x_n) + c\,(t - x_n)^2 + d(t - x_n)^3 \tag{5}$$

Also consider that this approximation polynomial satisfies the interpolation conditions $(x_n) = P_2(x_n)$, $f(a_n) = P_2(a_n)$, $f'(x_n) = P_2'(x_n)$, and $f'(a_n) = P_2'(a_n)$ by substituting the known values in $P_2(t)$ we have a system of three linear equations with three unknowns. By solving this system and simplifying we have [15].

$$P_2(x_n) = \text{a} = (x_n),$$

$$P_2{}'(x_n) = \text{b} = f'(x_n),$$

$$P_2{}''(x_n) = 2\text{c} = {}''(x_n),$$

$$\text{c} = \frac{f'(a_n) - f'(x_n)}{a_n - x_n} - \frac{f''(a_n)}{2},$$

$$\text{d} = \frac{f''(a_n)}{3\,(a_n - x_n)} - \frac{(f'(a_n) - f'(x_n))}{3(a_n - x_n)^2},$$

$$f'(a_n) = 2\left(\frac{f(a_n) - f(x_n)}{a_n - x_n}\right) - f'(x_n) = P_1(x_n, a_n),$$

$$f''(a_n) = \frac{2}{a_n - x_n}\left[\left(\frac{f(a_n) - f(x_n)}{a_n - x_n}\right) - f'(x_n)\right] = P_2(x_n, a_n). \tag{6}$$

Now using NTSM – 2 and equation (6) to suggest the following new iterative method for solving nonlinear equation. It is established that the following new method has convergence order three which will denoted by Najmuddin Vimal method (NVM) then NTSM – 2 can be written in the form:

Theorem 2.1 (NVM) For a given x_0 compute approximate solution x_{n+1} by the iterative schemes

$$a_n = x_n - \frac{[f(x_n)]^2}{f(x_n + f(x_n)) - f(x_n)},$$

$$b_n = a_n - \frac{2f(a_n)\,f'(a_n)}{2f'^2(a_n) - f(a_n)P_2(x_n, a_n)},$$

$$x_{n+1} = x_n - \frac{2\,f(x_n)}{[f'(b_n) + f'(x_n)]},$$

where $$P_2(x_n, a_n) = \frac{2}{a_n - x_n}\left[\left(\frac{f(a_n) - f(x_n)}{a_n - x_n}\right) - f'(x_n)\right]$$

for n=0, 1, 2, 3, …

It is known as NVM method.

3. RESULTS AND DISCUSSION

Let us now discuss the convergence analysis of the above method NVM.

Theorem 2.2. Let $\alpha \epsilon I$ be a simple zero of sufficiently differential function $f : I \subseteq R \rightarrow R$ for an open interval I, if x_0 is sufficiently close to α then the second derivative free iterative method defined by Theorem 2.1 is third order convergence.

Proof. Let α be a simple zero of f. Then by expanding $f(x_n)$ about α we have

$$f(x_n)=e_nc_1+e_n^2c_2+e_n^3c_3+\ldots\ldots.. \quad (7)$$

$$f'(x_n)=c_1+2c_2e_n+3c_3e_n^2+4c_4e_n^3+\ldots \quad (8)$$

where $c_k=\frac{1}{k!}f^{(k)}(\alpha)$, k=1, 2, 3, … and $e_n=x_n-\alpha$.

$$[f(x_n)]^2=c_1^2e_n^2+2\,c_1c_2e_n^3+c_2^2e_n^4+\ldots \quad (9)$$

$$f\big(x_n+f(x_n)\big)=c_1(1+c_1)\,e_n+(3c_1c_2+c_1^2c_2+2c_2^2)e_n^2+\ldots \quad (10)$$

Equation (7) and (10) yields,

$$f\big(x_n+f(x_n)\big)-f(x_n)=c_1^2e_n+(3c_1c_2+c_1^2c_2+2c_2^2)e_n^2+\ldots \quad (11)$$

From (9) and (11), we have

$$\frac{[f(x_n)]^2}{f\big(x_n+f(x_n)\big)-f(x_n)}=e_n-\left(\frac{c_2}{c_1}+c_2+2\frac{c_2^2}{c_1^2}\right)e_n^2+\ldots \quad (12)$$

From (12), we have

$$a_n=\alpha+\left(\frac{c_2}{c_1}+c_2+2\frac{c_2^2}{c_1^2}\right)e_n^2+\ldots \quad (13)$$

Let us set,A = $a_n-\alpha$. Then the equation (13) can be re – written in the form

$$A=\left(\frac{c_2}{c_1}+c_2+2\frac{c_2^2}{c_1^2}\right)e_n^2+\ldots \quad (14)$$

Now, expanding $f(a_n)$, $f\,'(a_n)$ about α and using (13), we have

$$f(a_n)=Ac_1+A^2c_2+A^3c_3+\ldots \quad (15)$$

$$f\,'(a_n)=c_1+A2c_2+A^23c_3+A^34c_4+\ldots \quad (16)$$

$$P_2(x_n,a_n)=2(c_1c_2+c_2)+\left(4c_3-\frac{c_2^2}{c_1}+2c_3c_1^2+6c_1c_3+c_1c_2^2+2c_2^2\right)e_n+\ldots \quad (17)$$

$$2f(a_n)f'(a_n)\backslash=2Ac_1^2+6c_1c_2A^2+4c_2^2A^3+\ldots \quad (18)$$

$$2[f'(a_n)]^2=2c_1^2+8c_1c_2A+8c_2A^2+\ldots \quad (19)$$

$$f(a_n)P_2(x_n,a_n)=2\left(\frac{c_2}{c_1}+c_2+2\frac{c_2^2}{c_1^2}\right)(c_1^2c_2+c_1c_2)e_n^2+\backslash\left(\frac{c_2}{c_1}+c_2+2\frac{c_2^2}{c_1^2}\right)(4c_1c_3-c_2^2+2c_3c_1^3$$
$$+6c_1^2c_3+c_1^2c_2^2+2c_1c_2^2\,)\,e_n^3+\ldots \quad (20)$$

$$2[f'(a_n)]^2f(a_n)P_2(x_n,a_n)=2c_1^2-\left[4c_1c_2^2+c_2^2+2c_2^3+c_1^2c_2^2+\frac{2c_2^3}{c_1}-8c_1c_2\right)e_n^2+\ldots \quad (21)$$

$$2f(a_n)f'(a_n) = 2c_1^2\backslash(\frac{c_2}{c_1} + c_2 + 2\frac{c_2^2}{c_1^2})e_n^2 + \ 6c_1c_2(\frac{c_2}{c_1} + c_2 + 2\frac{c_2^2}{c_1^2})^2 e_n^4 + \dots \quad (22)$$

$$\frac{2f(a_n)f'(a_n)}{2[f'(a_n)]^2 - f(a_n)P_2(x_n,a_n)} = \mathrm{A} + \left(\frac{c_2}{c_1} + c_2 + 2\frac{c_2^2}{c_1^2}\right)\left[\frac{c_2^2}{c_1^2} + \frac{5c_2^2}{c_1} + \frac{8c_2^3}{c_1^3} - \frac{4c_2}{c_1} + c_2^2\right]e_n^4 \quad (23)$$

$$b_n = \ \alpha + (\frac{c_2}{c_1} + c_2 + 2\frac{c_2^2}{c_1^2})\left[\frac{4c_2}{c_1} - \frac{c_2^2}{c_1^2} - \ \frac{5c_2^2}{c_1} - \frac{8c_2^3}{c_1^3} - c_2^2\right]e_n^4 \quad (24)$$

Also expanding $f'(b_n)$ about α and using (24), we have

$$f'(b_n) = c_1 + 2c_2\ (\frac{c_2}{c_1} + c_2 + 2\frac{c_2^2}{c_1^2})\left[\frac{4c_2}{c_1} - \frac{c_2^2}{c_1^2} - \ \frac{5c_2^2}{c_1} - \frac{8c_2^3}{c_1^3} - c_2^2\right]e_n^4 \dots \quad (25)$$

by substituting (7), (8) and (25) in VNM and after some simple calculation, we obtain

$$x_{n+1} = \quad \alpha + \frac{c_2^2}{c_1^2}e_n^3 + \ O(\,e_n^4\,), \quad (26)$$

$$e_{n+1} = \quad \frac{c_2^2}{c_1^2}e_n^3 + \ O(\,e_n^4\,). \quad (27)$$

This shows that Theorem 2.2 is third order convergence.

3.1. NUMERICAL EXAMPLES

In this section, we present some numerical example by employing Najmuddin Vimal method (NVM) to solve some nonlinear equations and compare it with Newton's method (NM)[12], Steffensen's method (SM)[12], Dehghan method (DM)[16], Super-Halley method (SHM)[10] and Improvement of Super-Halley method (ISHM)[8]. Test functions and their roots in Table 1 and number of iterations in Table 2.

For comparisons, we have used following methods:

Newton' s method (NM) [12]

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

Steffensen's method (SM) [12]

$$x_{n+1} = x_n - \frac{[f(x_n)]^2}{f(x_n+f(x_n)) - f(x_n)},$$

Dehghan method (DM) [16]

$$y_n = x_n - \frac{[f(x_n)]^2}{f(x_n+f(x_n)) - f(x_n)},$$

$$x_{n+1} = x_n - \frac{f(x_n)[f(x_n)+f(y_n)]}{f(x_n+f(x_n)) - f(x_n)},$$

Super-Halley method (SHM) [10]

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f''(x_n)\, f^2(x_n)}{2f^3(x_n) - 2f(x_n)\, f'(x_n)\, f''(x_n)},$$

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(x_n) + f''(x_n)\,(y_n - x_n)},$$

Improvement of Super-Halley method (ISHM) [8].

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f''(x_n)\, f^2(x_n)}{2f^3(x_n) - 2f(x_n)\, f'(x_n)\, f''(x_n)},$$

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(x_n)} - \frac{f''(x_n)\, f(y_n)}{2\, f^3(x_n)},$$

All computations are performed using MATLAB. The following examples are used for numerical testing.

Table 1. Test functions and their roots.

Functions	Roots
$f_1(x) = \cos(x) - x$	0.739085133215161
$f_2(x) = \cos(x) - \sqrt{x} + 1$	1.39058983057821
$f_3(x) = \cos(x) - xe^x$	0.517757363682458
$f_4(x) = e^x - 1.5 - tan^{-1}x$	0.767653266201279
$f_5(x) = x log_{10}x - 1.2$	2.74064609597369
$f_6(x) = x + \sin(x) - x^3$	1.31716296100603
$f_7(x) = \cos(x) - 3x + 1$	0.607101648103123
$f_8(x) = x \tan(x) + 1$	2.79838604578389
$f_9(x) = xe^x - 2$	0.852605502013726
$f_{10}(x) = \sin(x) - 1 - x^3$	-1.24905214850119

Table 2. Comparison of various iterative methods (numbers of iterations are displayed).

$f(x)$	NM	SM	DM	SHM	ISHM	VNM
$f_1, x_0 = 0.5$	4	4	3	2	3	3
$f_1, x_0 = -1$	8	5	4	4	NFTR	3
$f_2, x_0 = 1$	4	3	2	2	2	3
$f_2, x_0 = 3$	NFTR	4	3	3	NFTR	4
$f_3, x_0 = 1$	6	6	4	3	3	5
$f_3, x_0 = 5$	11	4	4	NFTR	NFTR	6
$f_4, x_0 = 1$	5	6	4	2	3	3
$f_4, x_0 = 3$	8	8	NFTR	NFTR	NFTR	6
$f_5, x_0 = 2$	4	5	3	2	3	3
$f_5, x_0 = 8$	5	6	4	3	3	4

$f_6, x_0 = 1$ $f_6, x_0 = 6$	6 8	7 NFTR	4 5	3 4	4 4	5 7
$f_7, x_0 = 3$ $f_7, x_0 = -3$	5 5	6 5	3 5	3 7	3 4	4 4
$f_8, x_0 = 2$ $f_8, x_0 = 4$	5 5	NFTR NFTR	NFTR NFTR	NFTR NFTR	4 5	4 5
$f_9, x_0 = 4$ $f_9, x_0 = 7$	4 7	6 NFTR	4 NFTR	3 NFTR	NFTR NFTR	5 7
$f_{10}, x_0 = 5$ $f_{10}, x_0 = 7$	5 7	6 19	4 12	10 11	22 26	4 5

**NFTR = Not found this root*

4. CONCLUSION

In this paper, we have suggested and analyzed new second derivative free iterative method for solving nonlinear equations. This method based on Halley method and predictor – corrector technique. Numerical tests show that the new method is comparably fast convergent than the well known existing methods and gives better result.

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