ORIGINAL PAPER

RELATIONS BETWEEN THE FRAMES OF A CURVE AND ITS PEDAL IN E^3

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Abstract. *The main goal of this paper using the Bishop frame and the Frenet-Serret frame vectors of the given space curve with the constant support function, some characteristic properties of the pedal curve of its are investigated. Finally, we present one example to confirm our claim.*
Keywords: *Frenet-Serret Frame, Bishop Frame (Relatively Parallel Adapted Frame), Natural Curvatures, Support Function, Pedal Curve.*

1. INTRODUCTION

It is well known that one of the most important subjects of the differential geometry is the theory curves. The primary goal in the geometric theory of curves is to measure their shapes in ways that do not take in to account how they are parametrized or how Euclidean space is coordinatized. However, it is generally hard to measure anything without coordinatizing space and parametrizing the curve. The theory of space curves dates back to Clairaut in 1731. He considered them as the intersection of two surfaces given by equations. Clairaut showed that space curves have two curvatures, but they did not corresponds exactly to the curvature and torsion. The subject was later taken up by Euler who was the first to work with parametrized curves and use arc-length as a parameter. Lancret in 1806 introduced the concepts of the unit tangent, the principal normal and the bi-normal and with those curvature and torsion as we now understand them. It is possible that Monge had some inklings of what torsion was, but he never presented an explicit formula. Cauchy in 1826 considerably modernized the subject and formulated some of the relations that later became part of the Serret and Frenet equations. In differential geometry the curves and surfaces in 3-dimensional Euclidean space E^3 is usually studied employing the vector formalism. The differential geometry of curves [6, 10, 29] traditionally begins with a vector $X(s)$ that describes the curve parametrically as a function of s that is at least twice differentiable. In the case of a differentiable curve, at each point a triad of mutually orthogonal unit vectors (called tangent, normal and binormal) is constructed and the rates of change of these vectors along the curve define the curvature and torsion of the curve. These two functions characterize the curve completely except for its position and orientation in space. Then, the following questions immediately come to our mind:

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1. Can the Frenet frame be always established along a space curve? For example, let M be given by the coordinate neighbourhood (I,α). If $\alpha'(s_0)\neq 0$ and $\alpha''(s_0)=0$ is it possible to establish a Frenet frame at the point $\alpha(s_0)\in M$.

2. If the Frenet frame can not be established, we will characterize how the curve?

This is an important issue. This problem was overcome through the work of Bishop. In 1975, Bishop introduced a beautiful frame the relatively parallel adapted frame or the Bishop frame, which could the desired means to ride along a space curve with the class $C^k, k\geq 2,$ and $\alpha'(s_0)\neq 0$ for all $s\in I,$ [1]. Subsequently, a large number researcher has used the Bishop frame and produced papers in Euclidean, Minkowskian and dual spaces [2, 3, 11-14, 18, 19, 23, 31]. Because, a Bishop frame also has the advantage of being globally defined even if a curve has points of zero curvature. Naturally, it also finds applications in problems which make use of frames along curves, such as in rotation minimizing frames in rigid body dynamics [7], computer graphics and visualization [14], robotics [31], and also in mathematical biology in the study of DNA [5].

On the other hand, the notion of the pedal surface(or curve) of a given surface(or curve) in with respect to a chosen origin is well- known in literature ([8-10], [16], [20-22], [24-28], and [29]). Georgiou, Hasanis and Koutroufiotis [8] have studied the differential geometry of the pedal surface of a surface with respect to a chosen origin and they investigated the applications in geometrical optics. Recently, the pedal surface with respect to a point in the interior of a closed, convex and smooth surface M in E^3 has studied by Kuruoğlu and the author has given some new characteristic properties of the pedal surface of M., [20]. Furthermore, the strip curve of the pedal of a given surface with the constant support function in the Minkowskian space has studied by Sarıoğlugil and Kuruoğlu, [27]. Besides, the plane pedal curves have studied by Sarıoğlugil, Kuruoğlu and Çalışkan, [24].

The paper organized as follows. In section 2*;* we briefly considered basic concepts of curves in Euclidean space E^3. Furthermore the Bishop frame is presented and the relationships between the Frenet-Serret frame and the Bishop frame were given. Then, the pedal curve of the given space curve is introduced, and some results related to the pedal curve of its, which are obtained by Sarıoğlugil and Kuruoğlu [25] were stated. In section 3*;* the geometry of the pedal curve of the space curve with the constant support function in E^3 is expressed depend on both the Frenet-Serret frame and the Bishop frame of the space curv2. At the end of this section, we present one example to confirm our claim.

2. PRELIMINARIES

To begin with, we recall the fundamentals of the differential geometry of curves in 3-dimensional Euclidean space E^3 with the inner product $g=\langle\ ,\ \rangle$.

The Euclidean metric is given by

$$g = ds^2 = dx^2 + dy^2 + dz^2 \tag{1}$$

where $\{x,y,z\}$ is a local coordinate system in E^3. On the other hand, the wedge (cross or vectorial) product of the vectors $X=\sum_{i=1}^{3} x_i e_i$ and $Y=\sum_{j=1}^{3} y_j e_j$ is defined by

$$X \Lambda Y = \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \tag{2}$$

where $\{e_1, e_2, e_3\}$ is the standard basis of E^3, [10].

Let M be a regular curve of the class C^3 in E^3 denoted by

$$\begin{aligned} \alpha : I \subset \mathbb{R} \to E^3 \\ t \to \alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t)) \end{aligned} \tag{3}$$

where $\alpha_i \in C^3(I, \mathbb{R})$. In this case, the vector

$$\dot{\alpha}(t)\Big|_{\alpha(t_0)} = \left(\frac{d\alpha_1}{dt}, \frac{d\alpha_2}{dt}, \frac{d\alpha_3}{dt} \right) \neq 0 \tag{4}$$

is called the velocity vector of M at the point $\alpha(t_0)$ and the space spanned by the vector $\dot{\alpha}(t)$ is a line. This space is called the tangent line of M at $\alpha(t_0)$.

On the other hand, if $\|\dot{\alpha}(t)\| = 1, \forall t \in I$, the parameter t is called the arc-length parameter of M and denoted by s. In this case, the curve M is called the unit speed curve in E^3, $[6]$. The tangent vector $\dot{\alpha}(s)$ is called the unit tangent vector of the curve M and denoted by T. For $\|T(s)\| = 1$ we get $\langle \dot{\alpha}(s), \ddot{\alpha}(s) \rangle = 0$ where

$$\ddot{\alpha}(s_0) = \left(\frac{d^2\alpha_1}{ds^2}, \frac{d^2\alpha_2}{ds^2}, \frac{d^2\alpha_3}{ds^2} \right)\Big|_{\alpha(s_0)}. \tag{5}$$

Then, the vector $\ddot{\alpha}(s_0)$ is called the acceleration vector of M at the point $\alpha(s_0)$. Since $\ddot{\alpha}(s)$ is perpendicular to the unit tangent vector $T(s) = \dot{\alpha}(s)$, the unit vector $\frac{1}{\|\ddot{\alpha}(s)\|} \ddot{\alpha}(s)$ is called the principal normal vector of M and denoted by $N(s)$. From here, we can say that the vector $T(s) \Lambda N(s)$ is a unit vector which is perpendicular to both the tangent vector $T(s)$ and the principal normal vector $N(s)$. Then, the vector $T(s) \Lambda N(s)$ is called the binormal vector of M and denoted by $B(s)$. Thus, the vector system

$$\left\{ T(s) = \dot{\alpha}(s), N(s) = \frac{1}{\|\ddot{\alpha}(s)\|} \ddot{\alpha}(s), B(s) = T(s) \Lambda N(s) \right\} \tag{6}$$

is the Frenet-Serret frame of the curve M at $\alpha(s)$, (Fig.1).

For the arbitrary parameter $t \in I$, the Frenet-Serret frame of M at the point $\alpha(t)$ is

$$\left\{T(t)=\frac{1}{\|\dot{\alpha}(t)\|}\dot{\alpha}(t), N(t)=B(t)\Lambda T(t), B(t)=\frac{1}{\|\dot{\alpha}(t)\Lambda\ddot{\alpha}(t)\|}\left(\dot{\alpha}(t)\Lambda\ddot{\alpha}(t)\right)\right\}. \tag{7}$$

At each point of the curve, the planes spanned by $\{T,N\},\{T,B\}$ and $\{N,B\}$ are known as the oscilating plane, the rectifying plane, and the normal plane, respectively (Fig. 1).

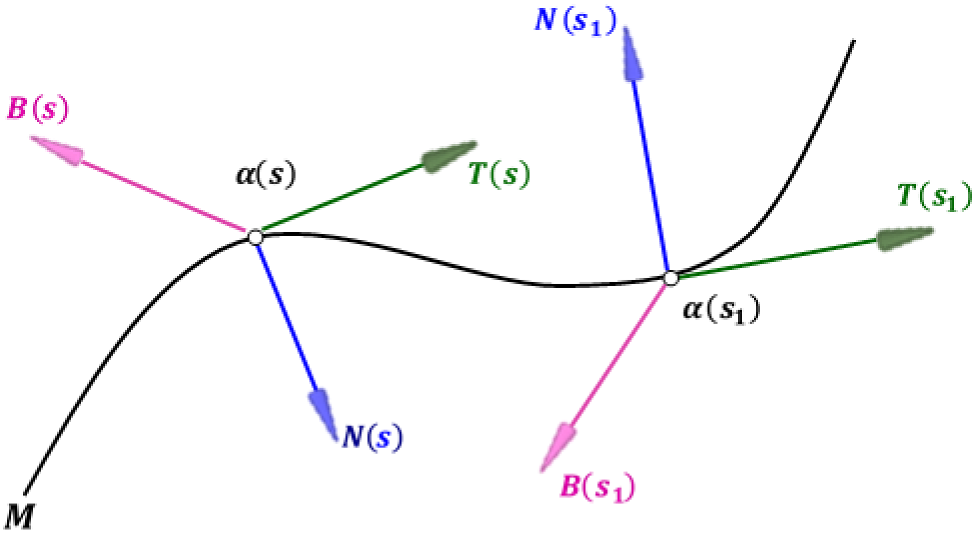

Figure 1. The Frenet-Serret frame for the curve M with non-vanishing curvature.

On the other hand, the geometry of a curve is essentially characterized by two scalar functions, curvature κ and torsion τ , which represent the rate of change of the tangent vector and the osculating plane, respectively. Then, the curvature of M is

$$\kappa(s)=\|\dot{T}(s)\|=\|\ddot{\alpha}(s)\|. \tag{8}$$

In other words, the curvature of M is the magnitude of the acceleration vector of M Thus, from (6) and (8) we obtain

$$\dot{T}(s)=\kappa(s)N(s). \tag{9}$$

From (6) we may write

$$\begin{cases} B(s)=T(s)\Lambda N(s) \\ \langle B(s),B(s)\rangle=1 \end{cases}. \tag{10}$$

Differentiating (10) with respect to the parameter s and using (9) we get

$$\begin{cases} \dot{B}(s) = T(s)\Lambda\dot{N}(s) \\ \left\langle \dot{B}(s), B(s)\right\rangle = 0 \end{cases}. \tag{11}$$

From here, we can say that the vector $\dot{B}(s)$ perpendicular to both the tangent vector $T(s)$ and the binormal vector $B(s)$. Then the vector $\dot{B}(s)$ must be parallel to $N(s)$. Therefore

$$\dot{B}(s) = -\tau(s)N(s) \tag{12}$$

From (6) we may write

$$N(s) = B(s)\Lambda T(s) \tag{13}$$

Differentiating (12) with respect to the parameter s we get

$$\dot{N}(s) = \dot{B}(s)\Lambda T(s) + B(s)\Lambda\dot{T}(s). \tag{14}$$

Substituting (9) and (12) into (14) and rearranging the last equation we obtain

$$\dot{N}(s) = -\kappa(s)T(s) + \tau(s)B(s). \tag{15}$$

Combining (9), (12) and (15) we have

$$\begin{cases} \dot{T}(s) = \kappa(s)N(s) \\ \dot{N}(s) = \kappa(s)T(s) \; +\kappa(s)B(s) \\ \dot{B}(s) = -\tau(s)N(s) \end{cases}. \tag{16}$$

These equations are called Frenet- Serret equations of M $[8]$.

For the arbitrary parameter t we have

$$\begin{cases} \kappa(t) = \dfrac{\left\|\dot{\alpha}(t)\Lambda\ddot{\alpha}(t)\right\|}{\left\|\dot{\alpha}(t)\right\|^3} \\ \\ \tau(t) = \dfrac{\det\left(\dot{\alpha}(t),\ddot{\alpha}(t),\dddot{\alpha}(t)\right)}{\left\|\dot{\alpha}(t)\Lambda\ddot{\alpha}(t)\right\|^2} \end{cases}. \tag{17}$$

So far, we have been interested in bi-regular curves, ($\dot{\alpha}(s)\neq 0$ and $\ddot{\alpha}(s)\neq 0$), in E^3. But what If the acceleration vector(so the curvature) vanishes at some points? The Frenet frame the before and after the zero-curvature set can be entirely different, so there may be no way to define a unique continuous Frenet frame over the whole curve, and any heuristics one might use the mend the situation is arbitrary. This difficulty led to introduce an alternative framing based on parallel transport rather than local curve derivatives [12]. The parallel transport frame(also known as the Bishop frame) is based on the observation that, while $T(s)$ for a given curve model is unique, we may choose any convenient arbitrary basis $\{N_1(s), N_2(s)\}$ for the remainder of the frame, so long as it is in the normal plane perpendicular to $T(s)$ at each point. If the derivates of the vectors $N_1(s)$ and $N_2(s)$ depend only on $T(s)$ and not each other we can make the vectors $N_1(s)$ and $N_2(s)$ vary smoothly throughout the path regardless of the curvature, Then we may choose the alternative frame equations (Fig. 2),

$$\begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} T(s) \\ N_1(s) \\ N_2(s) \end{bmatrix}. \tag{18}$$

where $v(s)$ is the speed of curve M, [1].

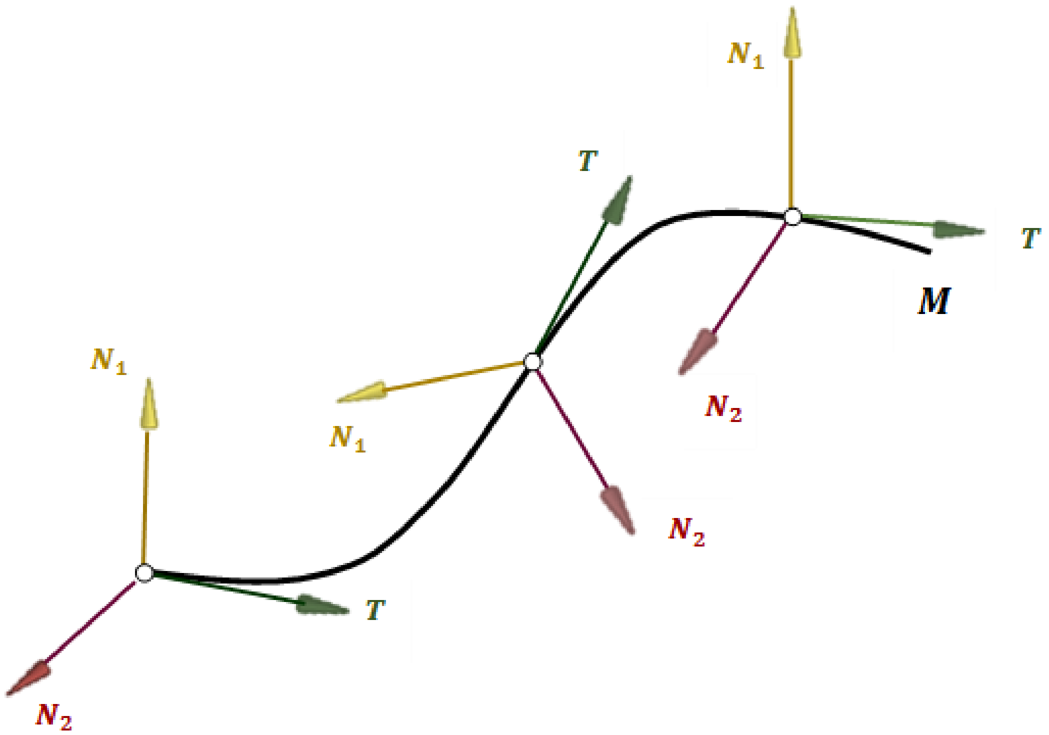

Figure 2. The parallel transport (or Bishop) frame for the curve M with vanishing curvature.

On the other hand, the relationship between the Frenet- Serret frame and the Bishop frame is

$$\begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} T(s) \\ N_1(s) \\ N_2(s) \end{bmatrix}. \tag{19}$$

Then, the relationships between the frame parameters are

$$\begin{cases} k_1(s) = \kappa(s)\cos\theta \\ k_2(s) = \kappa(s)\sin\theta \end{cases} \tag{20}$$

$$\theta(s) = arctg\left(\frac{k_2(s)}{k_1(s)}\right) \tag{21}$$

$$\kappa^2(s) = k_1^{\ 2}(s) + k_2^{\ 2}(s) \tag{22}$$

and

$$\tau(s) = \frac{d\theta}{ds} = \frac{k_1(s)\dot{k}_2(s) - \dot{k}_1(s)k_2(s)}{k_1^{\ 2}(s) + k_2^{\ 2}(s)}. \tag{23}$$

From (18) it can be easily seen that

$$\begin{cases} k_1(s)\sin\theta + k_2(s)\cos\theta = \kappa \\ \\ k_1(s)\sin\theta - k_2(s)\cos\theta = 0 \end{cases} \tag{24}$$

and

$$\dot{k}_1(s)\sin\theta - \dot{k}_2(s)\cos\theta = -\kappa\tau \tag{25}$$

Let M be a regular and unit speed curve and let O be an arbitrary point in E^3, $(O \notin M)$. We choose O as the origin. Then the pedal curve of M is defined by

$$\overline{X} = -hN \tag{26}$$

where h and N are the support function and the principal normal vector of M, respectively, and denoted by $\overline{M}$, $[25]$. Then, the support function of M is defined by

$$h = -\langle X, N\rangle \tag{27}$$

where X is the position vector of M with respect to O.

Geometrically, we can construct the pedal curve M, as follows:

We draw the tangent line L at $P \in M$ and from O we get the normal to the tangent line L.. The normal meets to tangent line L at the point $\overline{P}$. The locus of all $\overline{P}$. is called the pedal curve of M and denoted by $\overline{M}$ (Fig. 3). It will be taken as the support function of the curve M is constant along this study.

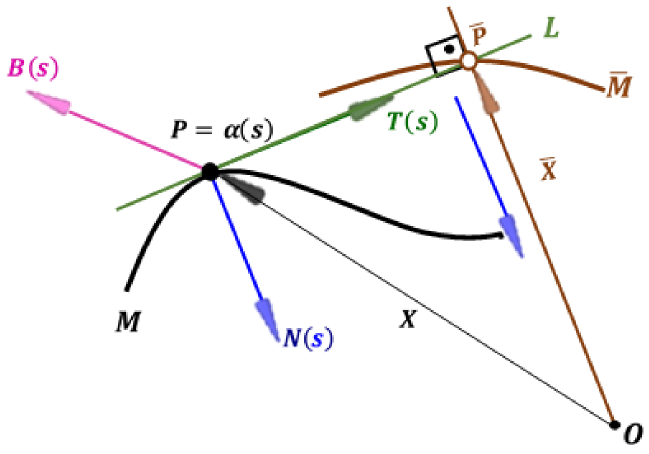

Figure 3. The pedal curve of M .

Some characteristic properties of the pedal curves of a curve with the constant support in E^3 were studied by A. Sarioğlugil and N. Kuruoğlu [25], and the following results have been obtained

$$\begin{cases} \overline{T}=\dfrac{1}{\sqrt{\kappa^2+\tau^2}}\left(\kappa T-\tau B\right) \\ \overline{N}=\dfrac{-1}{\sqrt{\kappa^2\eta\left(\kappa^2+\tau^2\right)}}\left[\kappa^2\tau\left(\dfrac{\tau}{\kappa}\right)^{\bullet}T-\left(\kappa^2+\tau^2\right)^2N+\kappa^3\left(\dfrac{\tau}{\kappa}\right)^{\bullet}B\right] \\ \overline{B}=\dfrac{1}{\eta}\left[\tau\left(\kappa^2+\tau^2\right)T+\kappa^2\left(\dfrac{\tau}{\kappa}\right)^{\bullet}N+\kappa\left(\kappa^2+\tau^2\right)B\right] \end{cases} \tag{28}$$

and

$$\begin{cases} \overline{\kappa}=\dfrac{\eta}{h\left(\kappa^2+\tau^2\right)^{\frac{3}{2}}} \\ \overline{\tau}=\dfrac{\left(\kappa^2+\tau^2\right)\left(\ddot{\kappa}\tau-\kappa\ddot{\tau}\right)+\dfrac{3}{2}\kappa^2\left(\dfrac{\tau}{\kappa}\right)^{\bullet}\left(\kappa^2+\tau^2\right)^{\bullet}}{h\eta^2} \end{cases} \tag{29}$$

where

$$\eta=\sqrt{\left(\kappa^2+\tau^2\right)^2+\kappa^4\left[\left(\dfrac{\tau}{\kappa}\right)^{\bullet}\right]^2}.$$

3. MAIN RESULTS

In this section, considering (24) and (25) we will investigate the relationships the relatively parallel-adapted frames (or Bishop Frames) of the curve M and its pedal curve. Then Iit will be assumed that the support function of the curve M is constant throughout in this study. Since h is a constant function, the all derivatives of h are zero. Then we may give the following the definition.

We will call the pedal curve of the curve M with the constant function if the support function of M is constant along the curve. Substituting (17) into (24) we may write the pedal curve depend on the vectors of the relatively parallel adapted frames (or Bishop Frames) of the curve M such that

$$\overline{X} = -h\left(\cos\theta\ N_1 + \sin\theta\ N_2\right) \tag{30}$$

where θ is the angle between the vectors N and N_1.
Differentiating (30) with respect to the arc parameter s of M we obtain

$$\dot{\overline{X}} = h\left[\left(k_1\cos\theta + k_2\sin\theta\right)T + \dot{\theta}\sin\theta N_1 - \dot{\theta}\cos\theta N_2\right]. \tag{31}$$

Substituting (20), (21) and (25) into (31) we get

$$\dot{\overline{X}} = h\left[\kappa T + \frac{\tau}{\kappa}\left(k_2 N_1 - k_1 N_2\right)\right]. \tag{32}$$

The vector $\dot{\overline{X}}$ is a velocity vector of $\overline{M}$ and the norm of $\dot{\overline{X}}$ is

$$\left\|\dot{\overline{X}}\right\| = h\left[\kappa^2 + \tau^2\right] \neq 1. \tag{33}$$

From here, it can be said that the pedal curve $\overline{M}$ is not unit speed .
Differentiating (32) with respect to the arc parameter s of M and rearranging the obtained equation we get

$$\ddot{\overline{X}} = h\begin{bmatrix}\left(\dot{\kappa} - \tau\left(k_1\sin\theta - k_2\cos\theta\right)\right)T + \left(\dot{\tau}\sin\theta + \tau^2\cos\theta + \kappa k_1\right)N_1 \\ +\left(-\dot{\tau}\cos\theta + \tau^2\sin\theta + \kappa k_2\right)N_2\end{bmatrix}. \tag{34}$$

Substituting (20) and (24) into (34) we have

$$\ddot{\overline{X}} = h\left[\dot{\kappa}T + \frac{1}{\kappa}\left[\left(\dot{\tau}\frac{k_2}{\kappa} + k_1\left(\kappa^2 + \tau^2\right)\right)N_1 + \left(-\dot{\tau}\frac{k_1}{\kappa} + k_2\left(\kappa^2 + \tau^2\right)\right)N_2\right]\right]. \tag{35}$$

Similarly, differentiating (35) with respect to the arc parameter s of M and rearranging the obtained equation we get

$$\overline{\dddot{X}}=h\begin{bmatrix}\left(\ddot{\kappa}-\frac{\tau^2}{\kappa}\left(\kappa^2+k_1^{\,2}\right)-\kappa^2\right)T+\left(\left(\ddot{\tau}-\tau^3\right)\frac{k_2}{\kappa}+\left(3\tau\dot{\tau}+2\kappa\dot{\kappa}+\kappa^2\frac{\dot{k}_1}{k_1}\right)\frac{k_1}{\kappa}\right)N_1\\+\left(-\left(\ddot{\tau}-\tau^3\right)\frac{k_1}{\kappa}+\left(3\tau\dot{\tau}+2\kappa\dot{\kappa}+\kappa^2\frac{\dot{k}_2}{k_2}\right)\frac{k_2}{\kappa}\right)N_2\end{bmatrix}. \tag{36}$$

Theorem 1: Let M be a regular curve wth the constant function h in E^3. The Frenet-Serret vectors of the pedal curve $\overline{M}$ are

$$\begin{cases}\overline{T}=\dfrac{1}{\sqrt{\kappa^2+\tau^2}}\left[\kappa T+\dfrac{\tau}{\kappa}\left(k_2N_1-k_1N_2\right)\right]\\[2ex]\overline{N}=\dfrac{1}{\eta\sqrt{\kappa^2+\tau^2}}\begin{bmatrix}-\left(\dfrac{\tau}{\kappa}\right)^{\cdot}\kappa^2\tau^2T+\left(\left(\dfrac{\tau}{\kappa}\right)^{\cdot}\kappa^2k_2+\dfrac{k_1}{\kappa}\left(\kappa^2+\tau^2\right)^2\right)N_1\\+\left(-\left(\dfrac{\tau}{\kappa}\right)^{\cdot}\kappa^2k_1+\dfrac{k_2}{\kappa}\left(\kappa^2+\tau^2\right)^2\right)N_2\end{bmatrix}\\[2ex]\overline{B}=\dfrac{1}{\eta}\begin{bmatrix}\tau\left(\kappa^2+\tau^2\right)T+\left(\left(\dfrac{\tau}{\kappa}\right)^{\cdot}\kappa k_1-k_2\left(\kappa^2+\tau^2\right)\right)N_1\\+\left(\left(\dfrac{\tau}{\kappa}\right)^{\cdot}\kappa k_2+k_1\left(\kappa^2+\tau^2\right)\right)N_2\end{bmatrix}\end{cases} \tag{37}$$

where the vector systems $\{T(s),N_1(s),N_2(s)\}$ and (κ,τ) are the Bishop frame and the Frenet-Serret frame aparatus of M, respectively, and

$$\eta=\sqrt{\tau^2\left[\tau^4+3\kappa^2\left(\kappa^2+\tau^2\right)+2\kappa^3\left(\frac{\tau}{\kappa}\right)^{\cdot}\right]+\kappa^4\left[\kappa^2+\left(\left(\frac{\tau}{\kappa}\right)^{\cdot}\right)^2\right]}.$$

Proof. From (7) we may write

$$\begin{cases} \overline{T} = \dfrac{1}{\left\| \dot{\overline{X}} \right\|} \dot{\overline{X}} \\ \overline{N} = \overline{B} \wedge \overline{T} \\ \overline{B} = \dfrac{1}{\left\| \dot{\overline{X}} \wedge \ddot{\overline{X}} \right\|} \left(\dot{\overline{X}} \wedge \ddot{\overline{X}} \right) \end{cases} . \tag{38}$$

Substituting (32) and (33) into (38) we get

$$\overline{T} = \frac{1}{\sqrt{\kappa^2 + \tau^2}} \left[\kappa T + \frac{\tau}{\kappa} \left(k_2 N_1 - k_1 N_2 \right) \right] . \tag{39}$$

Then, by (32) and (34) the vector $\dot{\overline{X}} \wedge \ddot{\overline{X}}$ is

$$\dot{\overline{X}} \wedge \ddot{\overline{X}} = h^2 \begin{bmatrix} \tau\left(\kappa^2 + \tau^2\right) T + \left(\left(\dfrac{\tau}{\kappa} \right)^{\cdot} \kappa k_1 - k_2 \left(\kappa^2 + \tau^2 \right) \right) N_1 \\ + \left(\left(\dfrac{\tau}{\kappa} \right)^{\cdot} \kappa k_2 + k_1 \left(\kappa^2 + \tau^2 \right) \right) N_2 \end{bmatrix} \tag{40}$$

and the norm of this vector is

$$\eta = \left\| \dot{\overline{X}} \wedge \ddot{\overline{X}} \right\| = \sqrt{\tau^2 \left[\tau^4 + 3\kappa^2 \left(\kappa^2 + \tau^2 \right) + 2\kappa^3 \left(\frac{\tau}{\kappa} \right)^{\cdot} \right] + \kappa^4 \left[\kappa^2 + \left(\left(\frac{\tau}{\kappa} \right)^{\cdot} \right)^2 \right]} . \tag{41}$$

Substituting (40) and (41) into (38) we get

$$\overline{B} = \frac{1}{\eta} \begin{bmatrix} \tau\left(\kappa^2 + \tau^2\right) T + \left(\left(\dfrac{\tau}{\kappa} \right)^{\cdot} \kappa k_1 - k_2 \left(\kappa^2 + \tau^2 \right) \right) N_1 \\ + \left(\left(\dfrac{\tau}{\kappa} \right)^{\cdot} \kappa k_2 + k_1 \left(\kappa^2 + \tau^2 \right) \right) N_2 \end{bmatrix} \tag{42}$$

From (38). (39) and (42) we obtain

$$\overline{N}=\frac{1}{\eta\sqrt{\kappa^2+\tau^2}}\begin{bmatrix}-\left(\frac{\tau}{\kappa}\right)^{\cdot}\kappa^2\tau^2T+\left(\left(\frac{\tau}{\kappa}\right)^{\cdot}\kappa^2k_2+\frac{k_1}{\kappa}\left(\kappa^2+\tau^2\right)^2\right)N_1\\+\left(-\left(\frac{\tau}{\kappa}\right)^{\cdot}\kappa^2k_1+\frac{k_2}{\kappa}\left(\kappa^2+\tau^2\right)^2\right)N_2\end{bmatrix} \tag{43}$$

This is completed the proof,

Theorem 2: Let M be a regular curve wth the constant function h in E^3 .The curvature and the torsion of the pedal curve $\overline{M}$ are

$$\begin{cases}\overline{\kappa}=\dfrac{\eta}{\left[\kappa^2+\tau^2\right]^{\frac{3}{2}}}\\ \overline{\tau}=\dfrac{1}{h\eta^2}\begin{bmatrix}\tau\left(\kappa^2+\tau^2\right)\left(\ddot{\kappa}-\frac{\tau^2}{\kappa}\left(\kappa^2+k_1^2\right)-\kappa^2\right)-\\ \left(\ddot{\tau}-\tau^3\right)\frac{k_2}{\kappa}\left[\left(\frac{\tau}{\kappa}\right)^{\cdot}\kappa\left(k_1-k_2\right)-\left(k_1+k_2\right)\left(\kappa^2+\tau^2\right)\right]\\ +\frac{k_1}{\kappa}\left(3\tau\dot{\tau}+2\kappa\dot{\kappa}+\kappa^2\frac{\dot{k}_1}{k}\right)\left(\left(\frac{\tau}{\kappa}\right)^{\cdot}\kappa\left(k_1+k_2\right)+\left(k_1-k_2\right)\left(\kappa^2+\tau^2\right)\right)\end{bmatrix}\end{cases} \tag{44}$$

where

$$\eta=\sqrt{\tau^2\left[\tau^4+3\kappa^2\left(\kappa^2+\tau^2\right)+2\kappa^3\left(\frac{\tau}{\kappa}\right)^{\cdot}\right]+\kappa^4\left[\kappa^2+\left(\left(\frac{\tau}{\kappa}\right)^{\cdot}\right)^2\right]}.$$

Proof. From (17) we may write

$$\begin{cases}\overline{\kappa}(s)=\dfrac{\left\|\dot{X}\Lambda\ddot{X}\right\|}{\left\|\dot{X}\right\|^3}\\ \overline{\tau}(s)=\dfrac{\det\left(\dot{X},\ddot{X},\dddot{X}\right)}{\left\|\dot{X}\Lambda\ddot{X}\right\|^2}\end{cases}. \tag{45}$$

From (33), (41) and (45) we find

$$\overline{\kappa}(s)=\frac{\eta}{\left[\kappa^2+\tau^2\right]^{\frac{3}{2}}}\ . \tag{46}$$

On the other hand, by (36) and (40) we get

$$\det(\dot{\overline{X}},\ddot{\overline{X}},\dddot{\overline{X}})=h^3\left[\begin{array}{l}\tau\left(\kappa^2+\tau^2\right)\left(\ddot{\kappa}-\frac{\tau^2}{\kappa}\left(\kappa^2+k_1{}^2\right)-\kappa^2\right)-\\ \left(\ddot{\tau}-\tau^3\right)\frac{k_2}{\kappa}\left[\left(\frac{\tau}{\kappa}\right)^{\cdot}\kappa\left(k_1-k_2\right)-\left(k_1+k_2\right)\left(\kappa^2+\tau^2\right)\right]\\ +\frac{k_1}{\kappa}\left(3\tau\dot{\tau}+2\kappa\dot{\kappa}+\kappa^2\frac{\dot{k}_1}{k}\right)\left(\left(\frac{\tau}{\kappa}\right)^{\cdot}\kappa\left(k_1+k_2\right)+\left(k_1-k_2\right)\left(\kappa^2+\tau^2\right)\right)\end{array}\right] \tag{47}$$

Substituting (41) and (47) into (45) we get

$$\overline{\tau}(s)=\frac{1}{h\eta^2}\left[\begin{array}{l}\tau\left(\kappa^2+\tau^2\right)\left(\ddot{\kappa}-\frac{\tau^2}{\kappa}\left(\kappa^2+k_1{}^2\right)-\kappa^2\right)-\\ \left(\ddot{\tau}-\tau^3\right)\frac{k_2}{\kappa}\left[\left(\frac{\tau}{\kappa}\right)^{\cdot}\kappa\left(k_1-k_2\right)-\left(k_1+k_2\right)\left(\kappa^2+\tau^2\right)\right]\\ +\frac{k_1}{\kappa}\left(3\tau\dot{\tau}+2\kappa\dot{\kappa}+\kappa^2\frac{\dot{k}_1}{k}\right)\left(\left(\frac{\tau}{\kappa}\right)^{\cdot}\kappa\left(k_1+k_2\right)+\left(k_1-k_2\right)\left(\kappa^2+\tau^2\right)\right)\end{array}\right] \tag{48}$$

This is completed the proof.

Theorem 3: Let M be a regular curve wth the constant function h in E^3. The support function of the pedal curve $\overline{M}$ is

$$\overline{h}=h\kappa \tag{49}$$

where κ is the curvature of M

Proof. By (25) the support function of $\overline{M}$ we may write

$$\overline{h}=-\langle\overline{X},\overline{N}\rangle\ . \tag{50}$$

Substituting (22), (25), (27) and (37) into (50) we get

$$\overline{h}=h\kappa.$$

This is completed the proof.

Example 1. Let us consider the curve M ,which is defined by

$$X(s)=\left(\cos\left(\frac{s}{\sqrt{2}}\right),\sin\left(\frac{s}{\sqrt{2}}\right),\frac{s}{\sqrt{2}}\right). \tag{51}$$

where $\theta\in[0,2\pi]$.
Differentiating (51) with respect to the parameter s we obtain

$$\dot{X}(s)=\frac{1}{\sqrt{2}}\left(-\sin\left(\frac{s}{\sqrt{2}}\right),\cos\left(\frac{s}{\sqrt{2}}\right),1\right). \tag{52}$$

and

$$\left\|\dot{X}(s)\right\|=1. \tag{53}$$

Then, the curve M is a unit speed.
Differentiating (52) with respect to the arc parameter s of M we get

$$\ddot{X}(s)=-\frac{1}{2}\left(\cos\left(\frac{s}{\sqrt{2}}\right),\sin\left(\frac{s}{\sqrt{2}}\right),0\right). \tag{54}$$

and

$$\left\|\ddot{X}(s)\right\|=\frac{1}{2}. \tag{55}$$

Using (6) the Frenet-Serret vectors of M are

$$\begin{cases} T(s)=\dfrac{1}{\sqrt{2}}\left(-\sin\left(\dfrac{s}{\sqrt{2}}\right),\cos\left(\dfrac{s}{\sqrt{2}}\right),1\right) \\[2ex] N(s)=\left(-\cos\left(\dfrac{s}{\sqrt{2}}\right),-\sin\left(\dfrac{s}{\sqrt{2}}\right),0\right) \\[2ex] B(s)=\dfrac{1}{\sqrt{2}}\left(\sin\left(\dfrac{s}{\sqrt{2}}\right),-\cos\left(\dfrac{s}{\sqrt{2}}\right),1\right) \end{cases} \tag{56}$$

From (16) the curvature and the torsion of M we find

$$\begin{cases} \kappa = \dfrac{1}{2} \\ \\ \tau = \dfrac{1}{2} \end{cases} \tag{57}$$

Then, we can give the Bishop frame of M .
By (23) and (57) we obtain

$$\theta = \int_0^{2\pi} \frac{1}{2} ds = \pi. \tag{58}$$

Substituting (58) into (20) we get

$$\begin{cases} k_1(s) = \dfrac{1}{2}\cos\pi = -\dfrac{1}{2} \\ \\ k_2(s) = \dfrac{1}{2}\sin\pi = 0 \end{cases}. \tag{59}$$

From (19) we may write

$$\begin{cases} N_1 = (\cos\theta)N - (\sin\theta)B \\ \\ N_2 = (\sin\theta)N + (\cos\theta)B \end{cases} \tag{60}$$

Combining (56), (58) and (60) the Bishop frame is

$$\begin{cases} T(s) = \dfrac{1}{\sqrt{2}}\left(-\sin\left(\dfrac{s}{\sqrt{2}}\right), \cos\left(\dfrac{s}{\sqrt{2}}\right), 1\right) \\ \\ N_1(s) = \left(\cos\left(\dfrac{s}{\sqrt{2}}\right), \sin\left(\dfrac{s}{\sqrt{2}}\right), 0\right) \\ \\ N_2(s) = \dfrac{1}{\sqrt{2}}\left(-\sin\left(\dfrac{s}{\sqrt{2}}\right), \cos\left(\dfrac{s}{\sqrt{2}}\right), -1\right) \end{cases} \tag{61}$$

On the other hand, the support function is $h = -1.$ Then the pedal curve of M is

$$\overline{X} = \left(\cos\left(\frac{s}{\sqrt{2}}\right), \sin\left(\frac{s}{\sqrt{2}}\right), 0\right). \tag{62}$$

Substituting (57), (58) and (59) into (37) we obtain

$$\begin{cases} \overline{T} = \dfrac{\sqrt{2}}{2}\left[T + N_2\right] \\ \overline{N} = -\dfrac{4\sqrt{22}}{11} N_1 \\ \overline{B} = \dfrac{4\sqrt{11}}{11}\left[T - N_2\right] \end{cases} \tag{63}$$

Similarly, substituting (57), (58) and (59) into (44) we get

$$\begin{cases} \overline{\kappa} = \dfrac{\sqrt{22}}{8} \\ \overline{\tau} = \dfrac{24}{11} \end{cases}. \tag{64}$$

Then, by maple software program the graphs of the helix curve and its pedal may be drawn as seen Fig. 4.

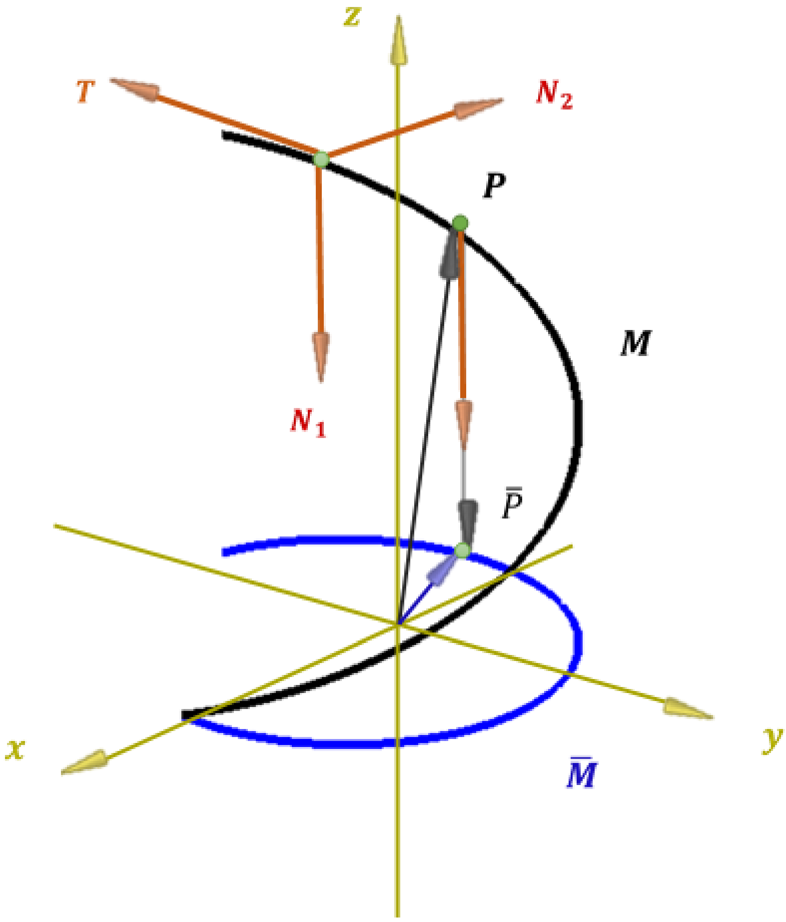

Figure 4. The helix curve and its pedal ccurve.

REFERENCES

[1] Bishop, R. L., *The American Mathematical Monthly*, **82**(3), 246, 1975 .
[2] Bükçü, B., Karacan, M.K., *Journal of Dynamical Systems and Geometric Theories*, **6**(1), 27, 2008.
[3] Bükçü, B., Karacan, M.K., *International Journal of Computational and Mathematical Sciences*, **3**(2), 63, 2009.
[4] Chen, B.Y., Dillen, F., *Bulletin of The Institute of Academia Sinica*, **33**(2), 77, 2005.
[5] Chirikjan, G.S., *Biochem. Soc. T.*, **41,** 635, 2013.
[6] Eisenhart, L.P., *A Treatise on The Differential Geometry of Curves and Surfaces*, Ginn and Company, New York, 1960.
[7] Farouki, R.T., Giannelli, C., Sampoli, M.L., Sestini, A., *Computer Aided Geometric Design*, **31**, 27, 2014.
[8] Georgiou, Chr., Hasanis, Th., Koutroufiotis, D., *The Pedal of Hypersurface Revisited*, Ioannina University, Technical Report,TR100, 1-14, 1984.
[9] Georgiou, Chr., Hasanis, Th., Koutroufiotis, D., *Geometria Dedicata*, **28**(2), 159, 1988.
[10] Gray, A., Abbena, E., Salamon, S., *Modern Differential Geometry of Curves and Surfaces with Mathematica*, Chapman & Hall/CRC, New York, 191-263, 2017.
[11] Haiming, L., Donghe, P., *Journal of Nonlinear Sciences and Applications*, **9**, 2875, 2016.
[12] Hanson, A.J., *Visualizing Quaternions*, Morgan Kaufmann, Elseiver, New York, 447, 2006.
[13] Hanson, A.J., *Quaternion Gauss Maps and Optimal Framings of Curves and Surfaces*, Indiana University, Technical Report TR518, **47**, 405-410, 1998.
[14] Hanson, A.J., Ma, H., *Parallel Transport Approach to Curve Framing*, Indiana University, Technical Report TR425, **11**, 3-7, 1995.
[15] Hasanis, Th., Koutroufiotis, D., *Arch. Mat.*, **57**, 189, 1991.
[16] Heaslip, L.W., *Polar Line Coordinates Their Application to Pedal Curves*, MSc Thesis, The University of British Columbia, 1-13, 1928.
[17] Izumiya, S., Takeuchi, J., *Turkish Journal Math.*, **28**, 153, 2004.
[18] Justh, E.W., Krishnaprasad, P.S., *Communications in Information and Systems*, **11**(1), 17, 2011..
[19] Karacan, M.K., Bükçü, B., *Journal of Vectorial Relativity*, **5**(4), 34, 2010.
[20] Kuruoğlu, N., *Pure and Applied Mathematika Sciences*, **23**(1-2), 7, 1986.
[21] Nishumura, T., *Geom. Dedicata*, **133**, 59, 2008.
[22] Nishumura, T., Kitegawa, K., *The Natural Sciences, Journal of the Faculty of Education and Human Sciences*, Yokohama National University, **10**, 39, 2008.
[23] Özyılmaz, E., *Mathematical and Computational Applications*, **16**(4), 858, 2011.
[24] Sarıoğlugil, A., Kuruoğlu, N., Çalıskan, M., *Pure and Applied Mathematika Sci.*, **40**(1-2), 23, 1994.
[25] Sarıoğlugil, A., Kuruoğlu, N., *Journal of Sciences of Ondokuz Mayıs University*, **8**(1), 1, 1997.
[26] Sarıoğlugil, A., Kuruoğlu, N., *International Journal of Applied Mathematics*, **42**(4), 357, 2003.

[27] Sarıoğlugil, A., Kuruoğlu, N., *International Journal of Applied Mathematics*, **20**(8), 357, 2007.
[28] Schellschmidt, A. S., *Results in Math.*, **52**, 369, 2008.
[29] Struik, D., *Lectures on Differential Geometry*, Dover, New York, 12-64, 1988.
[30] Ueda, K., *Mathematical Methods for Curves and Surfaces*, Vanderbilt Univ. Press, 497, 2001.
[31] Webster III, R.J., Jones, B.A., *Int. J. Robot Res.*, **29**, 1661, 2010.
[32] Yılmaz, S., Turgut, M., *Journal of Mathematical Analysis and Applications*, **371**(2), 764, 2010.