ORIGINAL PAPER

A NEW APPROACH TO INEXTENSIBLE FLOWS OF CURVES WITH BLASCHKE FRAME

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Abstract. *Inextensible flows of curves plays an important role in practical applications. We construct a new method for inextensible flows of dual curves in dual space* $\mathbb{D}^3$. *In this paper, we study inextensible flows of dual curves according to Blaschke frame in dual space* $\mathbb{D}^3$. *The concepts with the inextensible flows are analyzed by using Blaschke frame.*

Keywords: *Dual space curve, Inextensible flows, Blaschke frame.*

1. INTRODUCTION

Dual numbers were investigated by W.K. Clifford [1] and rediscovered by Study [2]. Then he used dual numbers and dual vectors in his research on the geometry of lines and kinematics, and defined the mapping which is called by his name (E. Study's mapping): the set of oriented straight lines in the Euclidean 3-space $\mathbb{E}^3$ is one-to-one correspondence with the dual points on the surface of a dual unit sphere $\hat{S}^2$ in the dual space $\mathbb{D}^3$ of triples of dual numbers. It can be seen that the dual number is a powerful mathematical tool for spatial mechanism design. Hence, dual numbers are analysis have important applications to differential line geometry and kinematics [3-9].

In differential geometry, the special curves are an important subject of curve theory [10-16]. A large number of aspects of applied physics and engineering utilized gaseous and smooth flows. Gaseous flows are extremely essential in spacecraft, automobiles, and aeroplanes. The research of smooth flow is extremely important meant for the uses of naviero, just like the style of boats and various tasks in municipal design including the design of the harbor and the safety of seaside. Physically, inextensible curve and surface flows are characterized by the absence of any strain energy induced from the motion. Kwon etc [17] investigated inextensible flows of curves and developable surfaces in $\mathbb{R}^3$. Then, they derived the corresponding equations for the inextensible flow of a developable surface and showed that it suffices to its evolution in terms of two inextensible curve flows [17]. Additionally, there are many works related with inextensible flows [18-24].

Inextensible flows of curves plays an important role in practical applications. We construct a new method for inextensible flows of dual curves in dual space $\mathbb{D}^3$. In this paper, we study inextensible flows of dual curves according to Blaschke frame in dual space $\mathbb{D}^3$. The concepts with the inextensible flows are analyzed by using Blaschke frame.

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2. MATERIALS AND METHODS

We start our discussion by reviewing some of the basic concepts of dual numbers [7]. Dual numbers are the set of all pairs of real numbers written as

$$\varpi = \varpi + \varepsilon \varpi^{*}, \varpi, \varpi^{*} \in \mathbb{R},$$

where the dual unit ε satisfies the relationships

$$\varepsilon \neq 0, \varepsilon 1 = 1\varepsilon, \varepsilon^{2} = 0.$$

The set of $\mathbb{D} = \{\varpi = \varpi + \varepsilon\varpi^{*} \mid \varpi, \varpi^{*} \in \mathbb{R}\}$ of dual number is a commutative ring according to the operations

$$(\varpi + \varepsilon\varpi^{*}) + (n + \varepsilon n^{*}) = (\varpi + n) + \varepsilon(\varpi^{*} + n^{*}),$$

$$(\varpi + \varepsilon\varpi^{*})(n + \varepsilon n^{*}) = \varpi n + \varepsilon(\varpi n^{*} + n\varpi^{*}).$$

The dual number $\varpi = \varpi + \varepsilon\varpi^{*}$ divided by the dual number $\hat{n} = n + \varepsilon n^{*}$ that provided $n \neq 0$ can be defined as

$$\frac{\varpi}{\hat{n}} = \frac{\varpi}{n} + \varepsilon\left(\frac{\varpi^{*} n - \varpi n^{*}}{n^{2}}\right).$$

For any $\varpi = \varpi + \varepsilon\varpi^{*}, \hat{n} = n + \varepsilon n^{*} \in \mathbb{D}^{3}$, the scalar product and the vector product of ϖ and $\hat{n}$ are defined as, respectively [6],

$$< \varpi, \hat{n} > = < \varpi, n > + \varepsilon(< \varpi, n^{*} > + < \varpi^{*}, n >),$$

$$\varpi \wedge \hat{n} = (\varpi_{2}\hat{n}_{3} - \varpi_{3}\hat{n}_{2}, \varpi_{3}\hat{n}_{1} - \varpi_{1}\hat{n}_{3}, \varpi_{1}\hat{n}_{2} - \varpi_{2}\hat{n}_{1}),$$

where $\varpi = \varpi + \varepsilon\varpi^{*}, \hat{n} = n + \varepsilon n^{*} \in \mathbb{D}^{3}, 1 \leq i \leq 3$. If $x \neq 0$, the norm $\|\varpi\|$ of $\varpi = \varpi + \varepsilon\varpi^{*}$ is defined as

$$\|\varpi\| = \sqrt{< \varpi, \varpi >} = \|\varpi\| + \varepsilon \frac{< \varpi, \varpi^{*} >}{\|\varpi\|}.$$

The set

$$\hat{S}^{2} = \{\varpi = \varpi + \varepsilon\varpi^{*}, \|\varpi\| = (1,0); \varpi, \varpi^{*} \in \mathbb{R}^{3}\},$$

is called the dual unit sphere with the center $\hat{O}$ in $\mathbb{D}^{3}$.

The set of all oriented lines in the Euclidean 3-space $\mathbb{E}^3$ is in one-to-one correspondence with the set of points of dual unit sphere in the dual lines in the Euclidean 3-space $\mathbb{E}^3$ is in one-to-one correspondence with the set of points of dual unit sphere in the dual 3-space $\mathbb{D}^3$.The representation of directed lines in $\mathbb{E}^3$ by dual unit vectors brings about several advantages and from now on we do not distinguish between oriented lines and their representing dual unit vectors.If every $x_i(s)$ and $x_i^*(s)$, $1 \leq i \leq 3$, real valued functions are differentiable, the dual space curve

$$\varpi \ : \ I \subset R \to D^3$$

$$t \ \to \ \varpi(s) = \left(\varpi_1(s) + \varepsilon\varpi_1^*(s), \varpi_2(s) + \varepsilon\varpi_2^*(s), \varpi_3(s) + \varepsilon\varpi_3^*(s)\right),$$

in $\mathbb{D}^3$ is differentiable.

Let $\{\hat{\mathbf{T}}, \hat{\mathbf{N}}, \hat{\mathbf{B}}\}$ be the Blaschke frame of the differentiable dual curve in the dual space $\mathbb{D}^3$ [5] . Then, the Blaschke frame equations are

$$\begin{aligned} \hat{\mathbf{T}}' &= \kappa\hat{\mathbf{N}}, \\ \hat{\mathbf{N}}' &= -\kappa\hat{\mathbf{T}} + (\tau + \varepsilon)\hat{\mathbf{B}}, \\ \hat{\mathbf{B}}' &= (-\tau - \varepsilon)\hat{\mathbf{N}}. \end{aligned}$$

3. FLOWS WITH BLASCHKE FRAME

Assume that $\hat{\chi}(u,t)$ is a one parameter family of smooth dual curves in dual space $\mathbb{D}^3$.

Then, the flow of $\hat{\chi}$ can be represented as

$$\begin{aligned} \frac{\partial \chi}{\partial t} &= f\mathbf{T} + g\mathbf{N} + h\mathbf{B}, \\ \frac{\partial \chi^*}{\partial t} &= f\mathbf{T}^* + f^*\mathbf{T} + g\mathbf{N}^* + g^*\mathbf{N} + h\mathbf{B}^* + h^*\mathbf{B}. \end{aligned}$$

Definition 3.1. The flow $\frac{\partial \chi}{\partial t} + \varepsilon \frac{\partial \chi^*}{\partial t}$ in $\mathbb{D}^3$ are said to be inextensible if

$$\frac{\partial}{\partial t}\left|\frac{\partial(\chi + \varepsilon\chi^*)}{\partial u}\right| = 0.$$

Lemma 3.2. Let $\frac{\partial \chi}{\partial t}+\varepsilon\frac{\partial \chi^*}{\partial t}$ be a smooth flow of the dual curve $\hat{\chi}$. The flow is inextensible if and only if

$$\frac{\partial v}{\partial t}=\frac{\partial f}{\partial u}-gv\kappa. \qquad (1)$$

Proof. As $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial u}$ are commutative and $v^2=<\frac{\partial \hat{\chi}}{\partial u},\frac{\partial \hat{\chi}}{\partial u}>$, we have

$$\frac{\partial v}{\partial t}=<v(\mathbf{T}+\varepsilon\mathbf{T}^*),(\frac{\partial f}{\partial u}-gv\kappa)\mathbf{T}+(fv\kappa+\frac{\partial g}{\partial u}-\tau vh)\mathbf{N}+(gv\tau+\frac{\partial h}{\partial u})\mathbf{B}$$

$$+\varepsilon((\frac{\partial f^*}{\partial u}-\kappa vg^*)\mathbf{T}+(\frac{\partial f}{\partial u}-gv\kappa)\mathbf{T}^*+(f^*\kappa v+\frac{\partial g^*}{\partial u}-\tau vh^*-vh)\mathbf{N}$$

$$+(f\kappa v+\frac{\partial g}{\partial u}-\tau vh)\mathbf{N}^*+(vg+\tau vg^*+\frac{\partial h^*}{\partial u})\mathbf{B}+(gv\tau+\frac{\partial h}{\partial u})\mathbf{B}^*)>.$$

which completes the proof.

Theorem 3.3. Let $\frac{\partial \hat{\chi}}{\partial t}$ be a smooth flow of the dual curve $\hat{\chi}$. The flow is inextensible if and only if

$$\frac{\partial f}{\partial u}=g\kappa,$$

$$\frac{\partial f^*}{\partial u}=g^*\kappa.$$

Proof. Now let $\frac{\partial \chi}{\partial t}+\varepsilon\frac{\partial \chi^*}{\partial t}$ be extensible. From Lemma 3.2, we have

$$\frac{\partial}{\partial t}s(u,t)=\int_0^u\frac{\partial v}{\partial t}du=\int_0^u(\frac{\partial(f+\varepsilon f^*)}{\partial u}-(g+\varepsilon g^*)v\kappa)du=0,$$

Finally, we express the desired result.

We now restrict ourselves to arc length parametrized curves.

Lemma 3.4.

$$\frac{\partial \mathbf{T}}{\partial t}=(f\kappa+\frac{\partial g}{\partial s}-\tau h)\mathbf{N}+(g\tau+\frac{\partial h}{\partial s})\mathbf{B},$$

$$\frac{\partial \mathbf{N}}{\partial t}=(-f\kappa-\frac{\partial g}{\partial s}+\tau h)\mathbf{T}+\psi\mathbf{B},$$

$$\frac{\partial \mathbf{B}}{\partial t}=-(g\tau+\frac{\partial h}{\partial s})\mathbf{T}-\psi\mathbf{N},$$

$$\frac{\partial \mathbf{T}^{*}}{\partial t}=(f^{*}\kappa+\frac{\partial g^{*}}{\partial s}-\tau h^{*}-h)\mathbf{N}$$

$$+(f\kappa+\frac{\partial g}{\partial s}-\tau h)\mathbf{N}^{*}+(g+\tau g^{*}+\frac{\partial h^{*}}{\partial s})\mathbf{B}+(g\tau+\frac{\partial h}{\partial s})\mathbf{B}^{*},$$

$$\frac{\partial \mathbf{N}^{*}}{\partial t}=(-f^{*}\kappa-\frac{\partial g^{*}}{\partial s}+\tau h^{*}+h)\mathbf{T}-(f\kappa+\frac{\partial g}{\partial s}-\tau h)\mathbf{T}^{*}+\psi\mathbf{B}^{*}+\psi^{*}\mathbf{B},$$

$$\frac{\partial \mathbf{B}^{*}}{\partial t}=-(g+\tau g^{*}+\frac{\partial h^{*}}{\partial s})\mathbf{T}-(g\tau+\frac{\partial h}{\partial s})\mathbf{T}^{*}-\psi\mathbf{N}^{*}-\psi^{*}\mathbf{N},$$

where $\hat{\psi}=\psi+\varepsilon\psi^{*}=\left\langle\frac{\partial \hat{\mathbf{N}}}{\partial t},\hat{\mathbf{B}}\right\rangle$.

Proof. Using definition of $\hat{\chi}$, we get

$$\frac{\partial \hat{\mathbf{T}}}{\partial t}=\frac{\partial}{\partial s}(f\mathbf{T}+g\mathbf{N}+h\mathbf{B}+\varepsilon(f\mathbf{T}^{*}+f^{*}\mathbf{T}+g\mathbf{N}^{*}+g^{*}\mathbf{N}+h\mathbf{B}^{*}+h^{*}\mathbf{B}).$$

From the Blaschke equations, we have

$$\frac{\partial \hat{\mathbf{T}}}{\partial t}=(\frac{\partial f}{\partial s}-g\kappa)\mathbf{T}+(f\kappa+\frac{\partial g}{\partial s}-\tau h)\mathbf{N}+(g\tau+\frac{\partial h}{\partial u})\mathbf{B}$$

$$+\varepsilon((\frac{\partial f^{*}}{\partial s}-\kappa g^{*})\mathbf{T}+(\frac{\partial f}{\partial u}-g\kappa)\mathbf{T}^{*}+(f^{*}\kappa+\frac{\partial g^{*}}{\partial u}-\tau h^{*}-h)\mathbf{N}$$

$$+(f\kappa+\frac{\partial g}{\partial s}-\tau h)\mathbf{N}^{*}+(g+\tau g^{*}+\frac{\partial h^{*}}{\partial s})\mathbf{B}+(g\tau+\frac{\partial h}{\partial s})\mathbf{B}^{*}).$$

Using Lemma 3.2, we get

$$\frac{\partial \hat{\mathbf{T}}}{\partial t} = (f\kappa + \frac{\partial g}{\partial s} - \tau h)\mathbf{N} + (g\tau + \frac{\partial h}{\partial u})\mathbf{B}$$

$$+\varepsilon((f^{*}\kappa + \frac{\partial g^{*}}{\partial u} - \tau h^{*} - h)\mathbf{N} + (f\kappa + \frac{\partial g}{\partial s} - \tau h)\mathbf{N}^{*}$$

$$+(g + \tau g^{*} + \frac{\partial h^{*}}{\partial s})\mathbf{B} + (g\tau + \frac{\partial h}{\partial s})\mathbf{B}^{*}).$$

Using $\left\langle \frac{\partial \mathbf{N}}{\partial t} + \varepsilon \frac{\partial \mathbf{N}^{*}}{\partial t}, (\mathbf{N} + \varepsilon \mathbf{N}^{*}) \right\rangle = \left\langle \frac{\partial \mathbf{B}}{\partial t} + \varepsilon \frac{\partial \mathbf{B}^{*}}{\partial t}, \hat{\mathbf{B}} \right\rangle = 0,$ we obtain

$$\frac{\partial \mathbf{N}}{\partial t} = (-f\kappa - \frac{\partial g}{\partial s} + \tau h)\mathbf{T} + \psi \mathbf{B},$$

$$\frac{\partial \mathbf{N}^{*}}{\partial t} = (-f^{*}\kappa - \frac{\partial g^{*}}{\partial s} + \tau h^{*} + h)\mathbf{T}$$

$$-(f\kappa + \frac{\partial g}{\partial s} - \tau h)\mathbf{T}^{*} + \psi \mathbf{B}^{*} + \psi^{*}\mathbf{B},$$

$$\frac{\partial \mathbf{B}}{\partial t} = -(g\tau + \frac{\partial h}{\partial s})\mathbf{T} - \psi \mathbf{N},$$

$$\frac{\partial \mathbf{B}^{*}}{\partial t} = -(g + \tau g^{*} + \frac{\partial h^{*}}{\partial s})\mathbf{T}$$

$$-(g\tau + \frac{\partial h}{\partial s})\mathbf{T}^{*} - \psi \mathbf{N}^{*} - \psi^{*}\mathbf{N},$$

where $\psi = \left\langle \frac{\partial \mathbf{N}}{\partial t}, \mathbf{B} \right\rangle, \psi^{*} = \left\langle \frac{\partial \mathbf{N}}{\partial t}, \mathbf{B}^{*} \right\rangle + \left\langle \frac{\partial \mathbf{N}^{*}}{\partial t}, \mathbf{B} \right\rangle$, which completes the proof.

Theorem 3.5. Let the flow $\frac{\partial \hat{\chi}}{\partial t}$ be inextensible. Then, the following system of partial differential equations holds:

$$\frac{\partial\kappa}{\partial t}=\frac{\partial}{\partial s}(f\kappa)-g\tau^2-\tau\frac{\partial h}{\partial s}+\frac{\partial^2 g}{\partial s^2}-\frac{\partial}{\partial s}(\tau h),$$

$$\frac{\partial^2 g^*}{\partial s^2}=\frac{\partial}{\partial s}(\tau h^*)+2\tau\frac{\partial h^*}{\partial s}+2g^*\tau^2+g\tau-\frac{\partial}{\partial s}(f^*\kappa).$$

Proof. From $\frac{\partial}{\partial s}\frac{\partial\hat{\mathbf{T}}}{\partial t}=\frac{\partial}{\partial t}\frac{\partial\hat{\mathbf{T}}}{\partial s}$, we get

$$\frac{\partial}{\partial s}\frac{\partial\hat{\mathbf{T}}}{\partial t}=-[(f\kappa^2+\kappa\frac{\partial g}{\partial s}-\tau\kappa h+\varepsilon(f^*\kappa^2+\frac{\partial g^*}{\partial s}\kappa-\tau\kappa h^*-h\kappa)]\mathbf{T}$$

$$-\varepsilon(f\kappa^2+\frac{\partial g}{\partial s}-\tau\kappa h]\mathbf{T}^*+[(\frac{\partial}{\partial s}(f\kappa)+\frac{\partial^2 g}{\partial s^2}-\frac{\partial}{\partial s}(\tau h)-g\tau^2-\tau\frac{\partial h}{\partial s}$$

$$+\varepsilon(\frac{\partial}{\partial s}(f^*\kappa)-g\tau-2g^*\tau^2-2\tau\frac{\partial h^*}{\partial s}+\frac{\partial^2 g^*}{\partial s^2}-\frac{\partial}{\partial s}(\tau h^*))]\mathbf{N}+\varepsilon(\frac{\partial}{\partial s}(f\kappa)$$

$$-g\tau^2-\tau\frac{\partial h}{\partial s}+\frac{\partial^2 g}{\partial s^2}-\frac{\partial}{\partial s}(\tau h)]\mathbf{N}^*+[(f\kappa\tau+\tau\frac{\partial g}{\partial s}+\frac{\partial}{\partial s}(g\tau)+\frac{\partial^2 h}{\partial s^2}-\tau^2 h$$

$$+\varepsilon(f^*\kappa\tau+\tau\frac{\partial g^*}{\partial s}-\tau^2 h^*-2h\tau+f\kappa+2\frac{\partial g}{\partial s}+\frac{\partial}{\partial s}(g^*\tau)+\frac{\partial^2 h^*}{\partial s^2})]\mathbf{B}+\varepsilon(f\kappa\tau$$

$$+\frac{\partial}{\partial s}(g\tau)+\tau\frac{\partial g}{\partial s}+\frac{\partial^2 h}{\partial s^2}-\tau^2 h]\mathbf{B}^*.$$

By using the relation frame, we have

$$\frac{\partial}{\partial t}\frac{\partial\hat{\mathbf{T}}}{\partial s}=\frac{\partial\kappa}{\partial t}\mathbf{N}+(-f\kappa^2-\kappa\frac{\partial g}{\partial s}+\tau\kappa h)\mathbf{T}+\psi\kappa\mathbf{B}$$

$$+\varepsilon((-f^*\kappa^2-\kappa\frac{\partial g^*}{\partial s}+\tau\kappa h^*+h\kappa)\mathbf{T}+\frac{\partial\kappa}{\partial t}\mathbf{N}^*).$$

This completes the proof.

Corollary 3.6.

$$\psi\kappa=-f\kappa\tau+\tau\frac{\partial g}{\partial s}+\frac{\partial}{\partial s}(g\tau)+\frac{\partial^2 h}{\partial s^2}-\tau^2 h.$$

Corollary 3.7.

$$f^{*}\kappa\tau+\tau\frac{\partial g^{*}}{\partial s}-\tau^{2}h^{*}-2h\tau+f\kappa+2\frac{\partial g}{\partial s}+\frac{\partial}{\partial s}(g^{*}\tau)+\frac{\partial^{2}h^{*}}{\partial s^{2}}=0.$$

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