ORIGINAL PAPER

FIXED POINTS OF RANDOM *F*-CONTRACTIONS IN COMPLETE METRIC SPACES

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Abstract. *In this paper, in complete metric spaces, we introduce random F-contraction and random F-weak contraction benefiting from the Wardowski and Van Dung's work. Later, we prove some random fixed point theorems in complete metric spaces. We also give some examples to support our results.*

Keywords: *Random F-contraction, random F-weak contraction, random operator, metric space.*

1. INTRODUCTION

The Banach contraction principle is the simplest and important result in fixed point theory [1]. This principle has many applications and was extended by several authors [2-6]. Authors have obtained fixed point theorems for various operators using this principle. The random operator is one of them. Random operator theory is needed for the study of various classes of random equations. Important contributions to the study of the mathematical aspects of such random equations have been presented in [7, 8] among others. The problem of fixed points for random mappings was initiated by the Prague school of probability research. Random fixed point theorems for contraction mappings were proved by Hanš [9,10], Hanš and Špaček [11] and Mukherjee [12, 13].

In this paper, we prove some random fixed point theorems in complete metric spaces. Firstly, we introduce random F-contraction and random F-weak contraction in complete metric spaces. Later, we prove some random fixed point theorems in complete metric spaces. We also state some examples to support our results.

2. PRELIMINARIES

Definition 2.1. [3] Let $(\Omega,\sum)$ be a measurable space with $\sum$-a sigma algebra of subsets of and M be a nonempty subset of a metric space X=(X,d). Let 2^M be the family of nonempty subsets of M and C(M) the family of all nonempty closed subsets of M. A mapping $G\colon \Omega \to 2^M$ is called measurable if for each open subset U of M, $G^{-1}(U)\in\sum$, where $G^{-1}(U)=\{\omega\in\Omega : G(\omega)\cap U\neq\emptyset\}$.

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Definition 2.2. [3] A mapping $\xi: \Omega \to M$ is called a measurable selector of a measurable mapping $G: \Omega \to 2^M$ if ξ is measurable and $\xi(\omega) \in G(\omega)$ for each $\omega \in \Omega$. The mapping $T: \Omega \times M \to X$ is said to be a random operator if and only if for each fixed $x \in M$, the mapping $T(.,x): \Omega \to X$ is measurable.

Definition 2.3. [3] A random operator $T: \Omega \times M \to X$ is said to be continuous random operator if for each fixed $x \in M$ and $\omega \in \Omega$, the mapping $T(\omega,.): M \to X$ is continuous.

Definition 2.4. [3] A measurable mapping $\xi: \Omega \to M$ is a random fixed point of a random operator $T: \Omega \times M \to X$ if and only if $T(\omega, \xi(\omega)) = \xi(\omega)$ for each $\omega \in \Omega$.

Definition 2.5. [4] Let $\mathcal{F}$ be the family of all functions $F: (0, +\infty) \to R$ such that
(F1) F is strictly increasing, that is, for all $\alpha, \beta \in (0, +\infty)$ if $\alpha < \beta$ then $F(\alpha) < F(\beta)$,
(F2) For each sequence $\{\alpha_n\}$ of positive numbers, the following holds:
$\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$,
(F3) There exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Example 2.6. [4] The following functions $F: (0, +\infty) \to R$ are the elements of $\mathcal{F}$:
(1) $F\alpha = \ln \alpha$,
(2) $F\alpha = \ln \alpha + \alpha$
(3) $F\alpha = -\frac{1}{\sqrt{\alpha}}$
(4) $F\alpha = \ln \alpha^2 + \alpha$.

Definition 2.7. [4] Let (X, d) be a metric space. A map $T: X \to X$ is said to be an F-contraction on (X, d) if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$,

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)).$$

Theorem 2.8. [4] Let (X, d) be a complete metric space and $T: X \to X$ be an F-contraction. Then we have
(1) T has a unique fixed point $x^* \in X$.
(2) $\forall x \in X$, the sequence $\{T^n x\}$ is convergent to x^*.

Remark 2.9. [4] Let T be an F-contraction. Then $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ such that $Tx \neq Ty$. Also, T is a continuous map.

Definition 2.10. [5] Let (X, d) be a metric space. A map $T: X \to X$ is said to be an F-contraction on (X, d) if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$,

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(max\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}).$$

3. MAIN RESULTS

In this paper, we prove random fixed point theorems for random F-contraction and random F-weak contraction in complete metric spaces.

Definition 3.1. Let (X,d) be a complete metric space and M be a nonempty separable closed subset of X. A map $T(\omega,.):\Omega\times M\to M$ is said to be a random F-contraction, if there exist $F\in\mathcal{F}$ and $\tau(\omega)>0$ such that for all $x,y\in M$ and $\omega\in\Omega$,

$$d\Big(T\big(\omega,x(\omega)\big),T\big(\omega,y(\omega)\big)\Big)>0$$

$$\Rightarrow \tau(\omega)+F(d(T(\omega,x(\omega)),T(\omega,y(\omega))))\leq F(d(x(\omega),y(\omega))). \tag{3.1}$$

Remark 3.2. Let T be an random F-contraction. Then, from (F1) and 3.1 we get $d(T(\omega,x(\omega)),T(\omega,y(\omega)))<d(x(\omega),y(\omega))$ for all $x,y\in M$ and $\omega\in\Omega$ such that $T\big(\omega,x(\omega)\big)\neq T\big(\omega,y(\omega)\big)$. Thus every random F-contraction T is a contractive operator. Also, T is a continuous map.

Theorem 3.3. Let (X,d) be a complete metric space and M be a nonempty separable closed subset of X and $T:\Omega\times M\to M$ be a random F-contraction operator. Then T has a unique random fixed point $x^*(\omega)\in\Omega\times M$ and for every $x(\omega)\in\Omega\times M$, the sequence $\{T^n x(\omega)\}$ is convergent to $x^*(\omega)$.

Proof. Choose $x_0(\omega)\in\Omega\times M$ for each $\omega\in\Omega$ and define a sequence $\{x_n(\omega)\}$ by

$$x_1(\omega)=T(\omega,x_0(\omega)),\ \ x_2(\omega)=T(\omega,x_1(\omega))=T^2(\omega,x_0(\omega))$$

$$,\ldots,$$

$$x_{n+1}(\omega)=T(\omega,x_n(\omega))=T^{n+1}(\omega,x_0(\omega)), \tag{3.2}$$

for all $n\in N$. If there exists $n\in N$ such that $d(x_n(\omega),T\big(\omega,x_n(\omega)\big))=0$, the proof is complete. So, we assume that

$$0<d(x_n(\omega),T\big(\omega,x_n(\omega)\big))=d(T(\omega,x_{n-1}(\omega)),T(\omega,x_n(\omega))), \tag{3.3}$$

for all $n\in N$ and $\omega\in\Omega$. For any $n\in N$ we have

$$\tau(\omega)+F(d(T(\omega,x_{n-1}(\omega)),T(\omega,x_n(\omega))))\leq F(d(x_{n-1}(\omega),x_n(\omega))),$$

i.e.,

$$F(d(T(\omega,x_{n-1}(\omega)),T(\omega,x_n(\omega))))\leq F(d(x_{n-1}(\omega),x_n(\omega)))-\tau(\omega).$$

Repeating this process, we get

$$\begin{aligned}
&F(d(T(\omega,x_{n-1}(\omega)),T(\omega,x_n(\omega))))\leq F(d(x_{n-1}(\omega),x_n(\omega)))-\tau(\omega)\\
&=F(d(T(\omega,x_{n-2}(\omega)),T(\omega,x_{n-1}(\omega))))-\tau(\omega)\\
&\leq F(d(x_{n-2}(\omega),x_{n-1}(\omega)))-2\tau(\omega)\\
&=F(d(T(\omega,x_{n-3}(\omega)),T(\omega,x_{n-2}(\omega))))-2\tau(\omega)\\
&\leq F(d(x_{n-3}(\omega),x_{n-2}(\omega)))-3\tau(\omega)\\
&\ldots\\
&\leq F(d(x_0(\omega),x_1(\omega)))-n\tau(\omega).
\end{aligned} \tag{3.4}$$

Taking the limit as $n \to \infty$ in 3.4, we obtain

$$\lim_{n\to\infty} F(d(T(\omega, x_{n-1}(\omega)), T(\omega, x_n(\omega))) = -\infty, \tag{3.5}$$

then from (F2) $\lim_{n\to\infty} d(T\big(\omega, x_{n-1}(\omega)\big), T\big(\omega, x_n(\omega)\big)) = 0$, i.e.,

$$\lim_{n\to\infty} d(x_n(\omega), x_{n+1}(\omega)) = 0. \tag{3.6}$$

and from (F3) there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} (d(x_n(\omega), x_{n+1}(\omega)))^k F(d(x_n(\omega), x_{n+1}(\omega))) = 0. \tag{3.7}$$

It follows from 3.4

$$d(x_n(\omega), x_{n+1}(\omega))^k (F(d(x_n(\omega), x_{n+1}(\omega)) - F(d(x_0(\omega), x_1(\omega)))$$

$$\leq -d(x_n(\omega), x_{n+1}(\omega))^k n\tau(\omega) \leq 0 \tag{3.8}$$

for all $n \in N$. By using 3.6, 3.7 and taking the limit $n \to \infty$ in 3.8, we get

$$\lim_{n\to\infty} (n(d(x_n(\omega), x_{n+1}(\omega)))^k) = 0, \tag{3.9}$$

then there exists $n_1 \in N$ such that $n(d(x_n(\omega), x_{n+1}(\omega)))^k \leq 1$ for all $n \geq n_1$, that is,

$$d(x_n(\omega), x_{n+1}(\omega))) \leq \frac{1}{n^{\frac{1}{k}}}. \tag{3.10}$$

For all $m > n \geq n_1$, by using 3.10 and the triangle inequality, we get

$$d(x_m(\omega), x_n(\omega)) \leq d(x_m(\omega), x_{m-1}(\omega)) + \ldots + d(x_{n+1}(\omega), x_n(\omega))$$

$$< \sum_{i=n}^{\infty} d(x_{i+1}(\omega), x_i(\omega)) \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \tag{3.11}$$

Since the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ is convergent, taking the limit as $n \to \infty$ in 3.11, we get $\lim_{n,m\to\infty} d(x_m(\omega), x_n(\omega)) = 0$. This proves $\{x_n(\omega)\}$ is a Cauchy sequence in $\Omega \times M$. By completeness of (X, d), $\{x_n(\omega)\}$ converges to some point $x(\omega) \in \Omega \times M$. Finally the continuity of T yields

$$d\Big(T\big(\omega, x(\omega)\big), x(\omega)\Big) = \lim_{n\to\infty} d\Big(T\big(\omega, x_n(\omega)\big), x_n(\omega)\Big) = \lim_{n\to\infty} d\big(x_{n+1}(\omega), x_n(\omega)\big)$$

$$= d(x^*(\omega), x^*(\omega)) = 0.$$

Now, let us to show that T has at most one fixed point. Indeed, if $x,y \in M$ be two distinct fixed points of T, that is, $T(\omega, x(\omega)) = x(\omega) \neq y(\omega) = T(\omega, y(\omega))$. Therefore,

$$d(T(\omega, x(\omega)), T(\omega, y(\omega)) = d(x(\omega), y(\omega)) > 0,$$

then we get

$$\begin{aligned} &F(d(x(\omega), y(\omega))) \\ &= F(d(T(\omega, x(\omega)), T(\omega, y(\omega)))) \\ &< \tau(\omega) + F(d(T(\omega, x(\omega)), T(\omega, y(\omega)))) \\ &\leq F(d(x(\omega), y(\omega))), \end{aligned}$$

which is a contradiction. Therefore, the random fixed point is unique.

Example 3.4 Let $F: R_+ \to R$ be given by the formula $F(a) = lna$. It is clear that F satisfies (F1)-(F3)-(F3) for any $k \in (0,1)$. Let $\Omega = [0,1]$ and $\sum$ be the sigma algebra of Lebesgue's measurable subset of $[0,1]$. Take $X = M = R$ with $d(x(\omega), y(\omega)) = |x(\omega) - y(\omega)|$ for all $x, y \in R$ and $\tau(\omega) = ln\,4$. Define random mapping $T: \Omega \times X \to X$ as $T(\omega, x(\omega)) = \frac{w-x}{4}$. Then a measurable mapping $x^* \in M$ defined as $x^*(\omega) = \frac{w}{5}$ for all $\omega \in \Omega$, serve as a unique random fixed point of T.

Definition 3.5 Let (X, d) be a complete metric space and M be a nonempty separable closed subset of X and $T: \Omega \times M \to M$ is said to be a random F-weak contraction, if there exist $F \in \mathcal{F}$ and $\tau(\omega) > 0$ such that for all $x, y \in M$ and $\omega \in \Omega$,

$$\begin{aligned} &d(T(\omega, x(\omega)), T(\omega, y(\omega))) > 0 \Rightarrow \tau(\omega) + F(d(T(\omega, x(\omega)), T(\omega, y(\omega)))) \\ &\leq F(max\{ d(x(\omega), y(\omega)), d(x(\omega), T(\omega, x(\omega))), d(y(\omega), T(\omega, y(\omega))), \\ &\frac{d(x(\omega), T(\omega, y(\omega))) + d(y(\omega), T(\omega, x(\omega)))}{2}\}). \end{aligned} \tag{3.12}$$

Remark 3.6 Every random F-contraction is a random F-weak contraction. But the converse is not true.

Example 3.7 Let $F: R_+ \to R$ be given by the formula $F(a) = lna$. It is clear that F satisfies (F1)-(F3)-(F3) for any $k \in (0,1)$. Let $\Omega = [0,1]$ and $\sum$ be the sigma algebra of Lebesgue's measurable subset of $[0,1]$. Take $X = [0,1]$ with $d(x(\omega), y(\omega)) = |x(\omega) - y(\omega)|$ and M= X for all $x, y \in M$. Let $T: \Omega \times [0,1] \to [0,1]$ be given by

$$T(\omega, x(\omega)) = \begin{cases} \frac{\omega - x}{4} & if \quad x(\omega) \in [0,1) \\ \frac{1}{4} & if \quad x(\omega) = 1. \end{cases}$$

Since T is not continuous, T is not random F-contraction by Remark 2.9. For $x \in [0,1)$ and $y = 1$, we have

$$d(T(\omega, x(\omega)), T(\omega, y(\omega))) = d(\frac{\omega - x}{4}, \frac{1}{4}) = \left|\frac{\omega - x}{4} - \frac{1}{4}\right| > 0,$$

and

$$max\{\, d(x(\omega),1), d(x(\omega),T(\omega,x(\omega))), d(1,T1), \frac{d(x,T1)+d(1,Tx)}{2}\} \geq d(1,T1) = \frac{3}{4}.$$

Therefore, by choosing $F\alpha = ln\,\alpha$, $\alpha \in (0,+\infty)$ and $\tau(\omega) = ln\frac{3}{2}$, we see that T is random F-weak contraction.

Theorem 3.8 Let (X,d) be a complete metric space and M be a nonempty separable closed subset of X and $T{:}\,\Omega \times M \to M$ be a random F-weak contraction operator. If T or F is continuous, then T has a unique random fixed point $x(\omega) \in \Omega \times M$ and for every $x(\omega) \in \Omega \times M$, the sequence $\{T^n x(\omega)\}$ is convergent to $x^*(\omega)$.

Proof. Let $x(\omega) \in \Omega \times M$ be arbitrary and fixed. We define $x_{n+1}(\omega) = T(\omega, x_n(\omega))$ for each $\omega \in \Omega$ and for all $n \in N \cup \{0\}$, where $x_0(\omega) = x(\omega)$. If there exists $n_0 \in N \cup \{0\}$ such that $x_{n_0+1}(\omega) = x_{n_0}(\omega)$, then $Tx_{n_0}(\omega) = x_{n_0}(\omega)$. This proves that $x_{n_0}(\omega)$ is a random fixed point of T. Now we suppose that $x_{n+1}(\omega) \neq Tx_n(\omega)$ for all $n \in N \cup \{0\}$. Then $d(x_{n+1}(\omega), x_n(\omega)) > 0$ for all $n \in N \cup \{0\}$. It follows from Definition 3.5 that for each $n \in N$:

$$F\left(d\left(x_{n+1}(\omega), x_n(\omega)\right)\right) = F\left(d\left(T\left(\omega, x_n(\omega)\right), T\left(\omega, x_{n-1}(\omega)\right)\right)\right)$$

$$\leq F(max\{\, d(x_n(\omega), x_{n-1}(\omega)), d(x_n(\omega), T(\omega, x_n(\omega))), d(x_{n-1}(\omega), T(\omega, x_{n-1}(\omega))$$

$$, \frac{d(x_n(\omega), T(\omega, x_{n-1}(\omega))) + d(x_{n-1}(\omega), T(\omega, x_n(\omega)))}{2}\}) - \tau(\omega)$$

$$= F(max\{\, d(x_n(\omega), x_{n-1}(\omega)), d(x_n(\omega), x_{n+1}(\omega)), d\left(x_{n-1}(\omega), x_n(\omega)\right)$$

$$, \frac{d(x_{n-1}(\omega),\ x_{n+1}(\omega))}{2}\}) - \tau(\omega)$$

$$\leq F(max\{\, d(x_n(\omega), x_{n-1}(\omega)), d(x_n(\omega), x_{n+1}(\omega))$$

$$, \frac{d(x_{n-1}(\omega),\ x_n(\omega)) + d(x_n(\omega),\ x_{n+1}(\omega))}{2}\}) - \tau(\omega)$$

$$= F(max\{\, d(x_n(\omega), x_{n-1}(\omega)), d(x_n(\omega), x_{n+1}(\omega))\}) - \tau(\omega)\,. \tag{3.13}$$

If there exists $n \in N$ such that

$$max\{\, d(x_n(\omega), x_{n-1}(\omega)), d(x_n(\omega), x_{n+1}(\omega))\} = d(x_n(\omega), x_{n+1}(\omega))$$

then 3.13 becomes

$$F(d(x_{n+1}(\omega), x_n(\omega))) \leq F(d(x_{n+1}(\omega), x_n(\omega))) - \tau(\omega) < F(d(x_{n+1}(\omega), x_n(\omega))).$$

It is a contradiction. Therefore,

$$max\{ d(x_n(\omega), x_{n-1}(\omega)), d(x_n(\omega), x_{n+1}(\omega))\} = d(x_n(\omega), x_{n-1}(\omega))$$

for all $n \in N$. Thus, from 3.13, we have

$$F(d(x_{n+1}(\omega), x_n(\omega))) \leq F(d(x_n(\omega), x_{n-1}(\omega))) - \tau(\omega)$$

for all $n \in N$. It implies that

$$F(d(x_{n+1}(\omega), x_n(\omega))) \leq F(d(x_1(\omega), x_0(\omega))) - n\tau(\omega) \tag{3.14}$$

for all $n \in N$. Taking the limit as $n \to \infty$ in 3.14, we get

$$\lim_{n\to\infty} F(d(x_{n+1}(\omega), x_n(\omega))) = -\infty$$

that together with (F2) gives

$$\lim_{n\to\infty} d(x_{n+1}(\omega), x_n(\omega)) = 0 \tag{3.15}$$

and from (F3), there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty}((d(x_{n+1}(\omega), x_n(\omega)))^k F(d(x_{n+1}(\omega), x_n(\omega))) = 0. \tag{3.16}$$

It follows from 3.14 that

$$(d(x_{n+1}(\omega), x_n(\omega)))^k (F(d(x_{n+1}(\omega), x_n(\omega)))) - F(d(x_1(\omega), x_0(\omega))))$$

$$\leq -(d(x_{n+1}(\omega), x_n(\omega)))^k n\tau(\omega) \leq 0 \tag{3.17}$$

for all $n \in N$. By using 3.15, 3.16 and taking the limit as $n \to \infty$ in 3.17, we get

$$\lim_{n\to\infty}(n(d(x_{n+1}(\omega), x_n(\omega)))^k) = 0. \tag{3.18}$$

Then there exists $n_1 \in N$ such that $n(d(x_{n+1}(\omega), x_n(\omega)))^k \leq 1$ for all $n \geq n_1$, that is,

$$d(x_{n+1}(\omega), x_n(\omega)) \leq \frac{1}{n^{\frac{1}{k}}} \tag{3.19}$$

for all $n \geq n_1$. For all $m > n \geq n_1$, by using 3.19 and the tringular inequality, we get

$$d(x_m(\omega), x_n(\omega)) \leq \sum_{i=n}^{m-1} \frac{1}{i^{\frac{1}{k}}} d(x_{i+1}(\omega), x_i(\omega))$$

$$\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \tag{3.20}$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{k}}}$ is convergent, taking the limit as $n \to \infty$ in 3.20, we get $\lim_{n,m\to\infty} d(x_m(\omega), x_n(\omega)) = 0$. This proves that $\{x_n(\omega)\}$ is Cauchy sequence in $\Omega \times M$. By completeness of (X,d), there exists $x^*(\omega)$ is a point such that $x_n(\omega) \to x^*(\omega)$ and $x^*(\omega)$ is a fixed point of T by two following cases.

Case 1. T is continuous. Then, we have

$$d(x^*(\omega), T(\omega, x^*(\omega))) = \lim_{n\to\infty} d(x_n(\omega), T(\omega, x_n(\omega))) = \lim_{n\to\infty} d(x_n(\omega), x_{n+1}(\omega)) = 0.$$

This proves that $x^*(\omega)$ is a fixed point of T.

Case 2 F is continuous. In this case, we consider two following subcases.

Subcase 2.1 For each $n \in N$, there exists $i_n \in N$ such that $x_{i_{n+1}}(\omega) = T(\omega, x^*(\omega))$ and $i_n > i_{n-1}$ where $i_0 = 1$. Then we have

$$x^*(\omega) = \lim_{n\to\infty} x_{i_{n+1}}(\omega) = \lim_{n\to\infty} T(\omega, x^*(\omega)) = T(\omega, x^*(\omega)).$$

This proves that $x^*(\omega)$ is a random fixed point of T.

Subcase 2.2 There exists $n_0 \in N$ such that $x_{n+1}(\omega) \neq Tx^*(\omega)$ for each $\omega \in \Omega$ and for all $n \geq n_0$. That is $d(T(\omega, x_n(\omega)), T(\omega, x^*(\omega))) > 0$ for all $n \geq n_0$. It follows from 3.12 and (F1)

$$\begin{aligned}\tau(\omega) + F(d(x_{n+1}(\omega), T(\omega, x^*(\omega)))) &= \tau(\omega) + F(d(T(\omega, x_n(\omega)), T(\omega, x^*(\omega))))\\ &\leq F(max\{ d(x_n(\omega), x^*(\omega)), d(x_n(\omega), T^*(\omega, x_n^*(\omega))), d(x^*(\omega), T(\omega, x^*(\omega))),\\ &\qquad \frac{d(x_n(\omega),T(\omega,x^*(\omega)))+d(x^*(\omega),T(\omega,x_n(\omega)))}{2}\})\\ &\leq F(max\{ d(x_n(\omega), x^*(\omega)), d(x_n(\omega), T^*(\omega, x_n^*(\omega))), d(x^*(\omega), T(\omega, x^*(\omega))),\\ &\qquad \frac{d(x_n(\omega),T(\omega,x^*(\omega)))+d(x^*(\omega),T(\omega,x^*(\omega)))+d(x^*(\omega),x_{n+1}(\omega))}{2}\}).\end{aligned} \tag{3.21}$$

If $d(x^*(\omega), T(\omega, x^*(\omega))) > 0$ then by the fact

$$\lim_{n\to\infty} d(x_n(\omega), x^*(\omega)) = \lim_{n\to\infty} d(x^*(\omega), x_{n+1}(\omega)) = 0,$$

there exists $n_1 \in N$ such that for all $n \geq n_1$, we have

$$\begin{aligned}&max\{ d(x_n(\omega), x^*(\omega)), d(x_n(\omega), x_{n+1}(\omega)), d(x^*(\omega), T(\omega, x^*(\omega))),\\ &\quad \frac{d(x_n(\omega),T(\omega,x^*(\omega)))+d(x^*(\omega),T(\omega,x^*(\omega)))+d(x^*(\omega),x_{n+1}(\omega))}{2}\})\\ &= d(x^*(\omega), T(\omega, x^*(\omega)))\end{aligned}$$

then from 3.21, we get

$$\tau(\omega) + F(d(x_{n+1}(\omega), T(\omega, x^*(\omega)))) \leq F(d(x^*(\omega), T(\omega, x^*(\omega)))), \tag{3.22}$$

for all $n \geq max\{n_0, n_1\}$. Since F is continuous, taking the limit as $n \to \infty$ in (3.22), we obtain

$$\tau(\omega)+F(d(x^*(\omega),T(\omega,x^*(\omega))))\le F(d(x^*(\omega),T(\omega,x^*(\omega)))).$$

It is contradiction. Therefore, $d(x^*(\omega),T(\omega,x^*(\omega)))=0$, that is, $x^*(\omega)$ is a random fixed point of T.

By two above cases, T has a random fixed point $x^*(\omega)$. Now, we prove that the random fixed point of T is unique. Let $x_1^*(\omega),x_2^*(\omega)$ be two random fixed points of T. Suppose to the contrary that $x_1^*(\omega)\neq x_2^*(\omega)$. Then $T(\omega,x_1^*(\omega))\neq T(\omega,x_2^*(\omega))$. It follows from 3.1 that

$$\begin{aligned}&\tau(\omega)+F(d(x_1^*(\omega),x_2^*(\omega)))=\tau(\omega)+F(d(T(\omega,x_1^*(\omega)),T(\omega,x_2^*(\omega))))\\&\le F(max\{\,d(x_1^*(\omega),x_2^*(\omega),d(x_1^*(\omega),T(\omega,x_1^*(\omega))),d(x_2^*(\omega),T(\omega,x_2^*(\omega))),\\&\quad\frac{d(x_1{}^*(\omega),T(\omega,x_2{}^*(\omega)))+d(x_2{}^*(\omega),T(\omega,x_1{}^*(\omega)))}{2}\}).\\&=F(max\{\,d\big(x_1^*(\omega),x_2^*(\omega)\big),d\big(x_1^*(\omega),x_1^*(\omega)\big),d\big(x_2^*(\omega),x_2^*(\omega)\big),\\&\quad\frac{d(x_1{}^*(\omega),x_2{}^*(\omega))+d(x_2{}^*(\omega),x_1{}^*(\omega))}{2}\}).\\&=F(d(x_1^*(\omega),x_2^*(\omega))).\end{aligned}$$

It is a contradiction. Then $d(x_1^*(\omega),x_2^*(\omega))=0$, that is $x_1^*(\omega)=x_2^*(\omega)$. This proves that the random fixed point of T is unique.

It follows from the proof of Theorem 3.8 that $\lim_{n\to\infty}T^n(\omega,x(\omega))=\lim_{n\to\infty}x_{n+1}(\omega)=x^*(\omega)$.

Example 3.9 Let F ans T be given as in Example 3.7. Then T is an random F-weak contraction. Therefore, Theorem 3.8 can be applicable to T and the unique fixed point of T is $\frac{\omega}{5}$.

Corollary 3.10 Let (X,d) be a complete metric space and M be a nonempty separable closed subset of X and $T:\Omega\times M\to M$ satisfies

$$\begin{aligned}&d(T(\omega,x(\omega)),T(\omega,y(\omega)))>0\Rightarrow\tau(\omega)+F(d(T(\omega,x(\omega)),T(\omega,y(\omega))))\\&\le ad(x(\omega),y(\omega))+bd(x(\omega),T(\omega,x(\omega)))+cd(y(\omega),T(\omega,y(\omega)))\\&+e[d(x(\omega),T(\omega,y(\omega)))+d(y(\omega),T(\omega,x(\omega)))]\end{aligned}\tag{3.23}$$

for all $x,y\in M$ and $\omega\in\Omega$ where $a,b,c\ge 0$ and $a+b+c+2e<1$.

If T or F is continuous, then T has a unique random fixed point $x^*(\omega)\in\Omega\times M$ and for every $x(\omega)\in\Omega\times M$, the sequence $\{T^nx(\omega)\}$ is convergent to $x^*(\omega)$.

Proof. For each $\omega\in\Omega$ and for all $x,y\in M$, we have

$$\begin{aligned}&ad(x(\omega),y(\omega))+bd(x(\omega),T(\omega,x(\omega)))\\&+cd(y(\omega),T(\omega,y(\omega)))+e[d(x(\omega),T(\omega,y(\omega)))+d(y(\omega),T(\omega,x(\omega)))]\\&\le(a+b+c+2e)\,max\{\,d(x(\omega),y(\omega)),d(x(\omega),T(\omega,x(\omega))),\\&\quad d(y(\omega),T(\omega,y(\omega)))\,,\frac{d(x(\omega),T(\omega,y(\omega)))+d(y(\omega),T(\omega,x(\omega)))}{2}\}\end{aligned}$$

$$\leq max\{\, d(x(\omega), y(\omega)), d(x(\omega), T(\omega, x(\omega))),$$
$$d(y(\omega), T(\omega, y(\omega)))\,, \frac{d(x(\omega),T(\omega,y(\omega)))+d(y(\omega),T(\omega,x(\omega)))}{2}\}.$$

Then, by (F1) we see that 3.12 is a consequence of 3.23. Then, the corollary is proved.

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