ORIGINAL PAPER

ABOUT SOME POWER SERIES INEQUALITIES AND TRACE INEQUALITIES

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Abstract. *The aim of this paper is to present: several new inequalities for power series with real coefficients by using a Young-type inequality for sequences of complex numbers, a matrix analogue for the Hilbert-Schmidt norm and a trace inequality for positive operators in Hilbert spaces, starting from a refinement of the classical Kittaneh-Manasrah inequality. Then several consequences as applications will be presented for special functions such as polylogarithm and for elementary functions.*

Keywords: *Young-type inequality, Hilbert-Schmidt norm, Power series, Trace inequalities*

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1. INTRODUCTION

The famous Young's inequality, also known as the weighted arithmetic mean-geometric mean for two numbers, state that:

$$a^{\nu} b^{1-\nu} < \nu\, a + (1-\nu) b,$$

where a and b are distinct positive numbers and $\nu \in (0,1)$, see [36]. This inequality is used in the classical Holder's inequality, given below, which is a very important tool in real and complex analysis,

$$\frac{1}{p} x^p + \frac{1}{q} y^q \geq xy,\ \ p > 1,\ \ \frac{1}{p} + \frac{1}{q} = 1,$$

which takes place for any positive numbers.

These inequalities have many applications in various fields and there exist a lot of interesting generalizations of this well-known inequality and its reverse, see for example [1, 9-11, 14, 18, 22, 23] and references therein.

As in [1], we will consider $A_{\nu}(a,b) = \nu\, a + (1-\nu) b$ and $G_{\nu}(a,b) = a^{\nu} b^{1-\nu}$. In [1] are presented new results which extend many generalizations of Young's inequality given before.

The below result from [1], is a generalization of the left-hand side of a refinement of the inequality of Young proved in 2010 and 2011 by Kittaneh and Manasrah in [22] and [23] and it is a very important tool in the demonstrations of our next theorems.

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Theorem 1. Let λ, ν and τ be real numbers with $\lambda \geq 1$ and $0 < \nu < \tau < 1$. Then

$$\left(\frac{\nu}{\tau}\right)^{\lambda} < \frac{A_{\nu}(a,b)^{\lambda} - G_{\nu}(a,b)^{\lambda}}{A_{\tau}(a,b)^{\lambda} - G_{\tau}(a,b)^{\lambda}} < \left(\frac{1-\nu}{1-\tau}\right)^{\lambda}$$

for all positive and distinct real numbers a and b.

Moreover, both bounds are sharp.

The technique to find other inequalities for functions using power series was given by Mortici in [28] and in Ibrahim, Dragomir and Darus in [20] and using this method we can extend some of the known inequalities, which can have applications in many fields.

As in [20], we will consider an analytic function defined by the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with real coefficients and convergent on the unit disk $D(0,R), R > 0$. Let $f_A(z)$ be a new power series defined by $\sum_{n=0}^{\infty} |a_n| z^n$, where $a_n = |a_n| sgn(a_n)$ and $sgn(x)$ is the real signum function as in [20]. The power series $f_A(z)$ has the same radius of convergence as the original power series $f(z)$.

It is necessary to recall the following inequalities which have been obtained by Ibrahim, Dragomir and Darus in [20], Theorem 1, Theorem 2 and Theorem 3 for power series (see Theorem A, Theorem B and Theorem C).

Theorem A. Let $f(z) = \sum_{n=0}^{\infty} p_n z^n$ and $g(z) = \sum_{n=0}^{\infty} q_n z^n$ be two power series with real coefficients and convergent on the open disk $D(0,R), R > 0$. If $x, y \in \mathbb{C},\ x, y \neq 0,\ p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ so that $xy, |x|^p, |y|^p,\ |x|^q, |y|^q\ \in D(0,R)$ then

$$\frac{1}{p} g_A(|x|^p) f_A(|y|^p) + \frac{1}{q} f_A(|x|^q) g_A(|y|^q) \geq |f(xy)g(xy)|$$

and

$$\frac{1}{p} g_A(|x|^p) f_A(|y|^q) + \frac{1}{q} f_A(|x|^q) g_A(|y|^p) \geq |f(x|y|^{q-1}) g(x|y|^{p-1})|.$$

Theorem B. Let $f(z$ and $g(z)$ be as in Theorem A. Then one has the inequalities:

$$\frac{1}{p} g_A(|x|^p) f_A(|y|^q) + \frac{1}{q} f_A(|x|^p) g_A(|y|^q) \geq |f(|x|^{p-1}|y|^{q-1}) g(xy)|$$

and

$$\frac{1}{p} f_A(|x|^p) g_A(|y|^2) + \frac{1}{q} g_A(|x|^2) g_A(|y|^q) \geq \left| f(xy) g\left(|x|^{\frac{2}{q}} |y|^{\frac{2}{p}}\right)\right|.$$

Theorem C. Let $f(z$ and $g(z)$ be as in Theorem A. Then one has the inequalities:

$$\frac{1}{p} g_A(|x|^2) f_A(|y|^q) + \frac{1}{q} f_A(|x|^p) g_A(|y|^2) \geq |f(|x|^{p-1}|y|^{q-1}) g(|x|^{\frac{2}{p}} |y|^{\frac{2}{q}})|$$

and

$$\frac{1}{p}g_A(|x|^2)f_A(|y|^p)+\frac{1}{q}f_A(|x|^2)g_A(|y|^q)\geq\left|f\left(|x|^{\frac{2}{q}}y\right)g\left(|x|^{\frac{2}{p}}y\right)\right|.$$

As in [12], let H be a Hilbert space and $B_1(H)$ the trace class operators in $B(H)$. We define the trace of a trace class operator $A\in B_1(H)$ to be $tr(A)=\sum_{i\in I}<Ae_i,e_i>$, where $\{e_i\}_{i\in I}$ is an orthonormal basis of H.

If H is finit dimensional then we can see that this coincides with usual definition of the trace. It is known, see [12] and the references therein, that previous series converges absolutely and it is independent of the choice of $\{e_i\}_{i\in I}$.

A trace inequality via Kittaneh-Manasrah result was proven by Dragomir in [12], Theorem 1 (see Theorem D).

Theorem D. Let A, B be two positive operators and $P,Q\in B_1(H)$ with $P,Q>0$. Then for any $\nu\in[0,1]$ we have

$$r\left(\frac{tr(PA)}{tr(P)}-2\frac{tr\left(PA^{\frac{1}{2}}\right)}{tr(P)}\frac{tr\left(QB^{\frac{1}{2}}\right)}{tr(Q)}+\frac{tr(QB)}{tr(Q)}\right)\leq$$

$$\leq(1-\nu)\frac{tr(PA)}{tr(P)}+\nu\frac{tr(QB)}{tr(Q)}-\frac{tr(PA^{1-\nu})}{tr(P)}\frac{tr(QB^{\nu})}{tr(Q)}\leq$$

$$\leq R\left(\frac{tr(PA)}{tr(P)}-2\frac{tr\left(PA^{\frac{1}{2}}\right)}{tr(P)}\frac{tr\left(QB^{\frac{1}{2}}\right)}{tr(Q)}+\frac{tr(QB)}{tr(Q)}\right),$$

where $r=min\{1-\nu,\nu\}$ and $R=max\{1-\nu,\nu\}$.

We recall the definition of the Frobenius norm, known as Hilbert-Schmidt norm. For any $A=(a_{ij})$ in M_m, where M_m is the set of $m\times m$ square matrix we have:

$$||A||_2=|tr(AA^*)|=\left(\sum_{i,j=1}^{m}|a_{ij}|^2\right)^{\frac{1}{2}}.$$

The following inequality from [1] improves the Frobenius norm version of Young's inequality given by Kosaki [24], Bhatia and Parthasarathy [5] and [18, 22] (see below Theorem E).

Theorem E. Let ν,τ be real numbers with $0<\nu<\tau<1$. If $A,B,X\in M_m$ with A and B positive semidefinite, then we have

$$\left(\frac{\nu}{\tau}\right)^2<\frac{||\nu AX+(1-\nu)XB||_2^2-\left||A^{\nu}XB^{1-\nu}\right||_2^2}{||\tau AX+(1-\tau)XB||_2^2-||A^{\tau}XB^{1-\tau}||_2^2}<\left(\frac{1-\nu}{1-\tau}\right)^2,$$

provided the fracvtion is defined.

We recall an improvement of Young's inequality given in [29] in Theorem 1, which will be used below in Section 4.

Lemma 1 For $a, b \geq 1$ and $\lambda \in (0,1)$ we have

$$r\left(\sqrt{a}-\sqrt{b}\right)^2 + A(\lambda) log^2\left(\frac{a}{b}\right) \leq \lambda\, a + (1-\lambda)b - a^{\lambda}b^{1-\lambda} \leq$$

$$\leq (1-r)\left(\sqrt{a}-\sqrt{b}\right)^2 + B(\lambda) log^2\left(\frac{a}{b}\right),$$

where $r = min\{\lambda, 1-\lambda\}$, $A(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{r}{4}$ and $B(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{1-r}{4}$.

We consider the functions:

$$h(a,b) = \lambda a + (1-\lambda)b - a^{\lambda}b^{1-\lambda} - r\left(\sqrt{a}-\sqrt{b}\right)^2 - A(\lambda).\, log^2\left(\frac{a}{b}\right)$$

and

$$k(a,b) = \lambda a + (1-\lambda)b - a^{\lambda}b^{1-\lambda} - (1-r)\left(\sqrt{a}-\sqrt{b}\right)^2 - B(\lambda).\, log^2\left(\frac{a}{b}\right).$$

The below figures are graphics of this four functions for a particular value of λ.

Related to such Young-type inequalities, often appear the weighted arithmetic mean, geometric mean and harmonic mean defined by $A_{\nu}(a,b) = (1-\nu)\,a + \nu b$, $G_{\nu}(a,b) = a^{1-\nu}b^{\nu}$ and $H_{\nu}(a,b) = A_{\nu}^{-1}(a^{-1}, b^{-1}) = [(1-\nu)\,a^{-1} + \nu b^{-1}]^{-1}$, when $a, b > 0$ and $\nu \in [0,1]$.

It is necessary to recall for our goals, that for two positive definite matrices A, B, the ν -weighted arithmetic and geometric mean are defined as

$$A\nabla_{\mu}B = (1-\mu)A + \mu\, B$$

and

$$A\sharp_{\mu}B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\mu} A^{\frac{1}{2}}$$

when .$\mu \in [0,1]$. If $\mu = \frac{1}{2}$ then we write only $A\nabla B$, $A\sharp$B.

It is known that for any two square matrices A, B, A < B if B-A is positive semidefinite. Also, A < B if B-A is positive definite, see [1] and [19].

As in [8], it is also necessary to recall that for selfadjoint operators $A, B \in B(H)$ we write $A \leq B$ (or $B \geq A$) if $< Ax, x > \leq < Bx, x >$ for every vector$x \in H$. We will consider for beginning A as being a selfadjoint linear operator on a complex Hilbert space (H; < . , . >). The *Gelfand map* establishes a *- isometrically isomorphism Φ between the set C(Sp(A)) of all *continuous functions* defined on the *spectrum* of A, denoted Sp(A), and the C^* - algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows: For any f, $f \in$ C(Sp(A)) and for any α, $\beta \in \mathbb{C}$ we have

(i) Φ(αf+ βg) = α Φ(f) + β Φ (g);

(ii) Φ(fg) = Φ(f) Φ (g) and Φ (f)= Φ (f^*);

(iii) || Φ(f)|| = ||f|| := $sup_{t\in Sp(A)}$ |f(t)|;

(iv) Φ(f_0)=1_H and Φ (f_1)=A,

where $f_0(t) = 1$and $f_1(t) = t$ for t ∈Sp (A).

Using this notation, as in [8], we define f(A) := Φ (f) for all $f \in C(Sp(A))$ and we call it the *continuous functional calculus* for a selfadjoint operator A.

It is known that if A is a selfadjoint operator and f is a real valued continuous function on Sp(A), then $f(t) \geq 0$ for any $t \in$ Sp(A) implies that $f(A) \geq 0$ i.e. $f(A)$ is a *positive operator* on H.

In addition, if and f and g are real valued functions on Sp(A) then the following property holds: $f(t) \geq g(t)$ for any $t \in Sp(A)$ implies that $f(A) \geq g(A)$ in the operator order of B(H).

We consider A, B two positive operators on a complex Hilbert space (H, < . , . >) and the following notations for operators:

$$A \nabla_\nu B = (1-\nu)A + \nu\, B, \quad \nu \in [0,1],$$

the weighted operator arithmetic mean and

$$A \sharp_\nu B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{\nu} A^{\frac{1}{2}}\ , \quad \nu \in [0,1],$$

the weighted operator geometric mean.

In addition, we enounciate the definition of the relative operator entropy $S(A/B)$ given in [15, 16] for positive invertible operators A and B, $S(A/B) = A^{\frac{1}{2}}\left(ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}$.

We recall the definition of the noncommutative perspective, $\mathcal{P}_\Phi(B,A) = A^{\frac{1}{2}}\Phi\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}$ given in [10] for continuous functions Φ defined on the interval J of real numbers, if B is a selfadjoint operator on the Hilbert space H, A is a positive invertible operator on H and $Sp\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right) \subset$ J.

The aim of this paper is to present new inequalities for functions defined by power series with real coefficients. This thing was done in Section 2 in Theorems 2, 3, 4 and 5. Then applications for some fundamental complex functions such as exponential and hyperbolic functions and also for polylogarithm function are given. Special functions and power series have many applications in engineering sciences and applied mathematics. For example, in [3, 17, 21, 34] and references therein there are many inequalities involving the polylogarithm, hypergeometric, Bessel and modified Bessel functions. In Section 3 is given in Theorem 6 a version of a Young-type inequality for the Hilbert-Schmidt norm as a generalization of Theorem 5.1 from [1] when $\lambda = n \in \mathbb{N}$.. Section 4 is devoted to trace inequalities, see Theorem 7, where a new trace inequality via a scalar Young type inequality presented in [29] is stated using the methods given in [12].

2. THE YOUNG-TYPE INEQUALITIES FOR POWER SERIES

We start by taking in Theorem 1, $\lambda = n \in \mathbb{N}^*$, $\nu = \frac{1}{p}$, $\tau = \frac{1}{p_1}$, a^p instead of a and b^p instead of b. Then we have the following inequality:

$$\left(\frac{p_1}{p}\right)^n \left[\sum_{l=0}^{n}\binom{n}{l}\frac{1}{p_1^l q_1^{n-l}} a^{pl} b^{q(n-l)} - a^{\frac{p}{p_1}n} b^{\frac{q}{q_1}n}\right] < \sum_{l=0}^{n}\binom{n}{l}\frac{1}{p^l q^{n-l}} a^{pl} b^{q(n-l)} - a^n b^n <$$

$$\left(\frac{q_1}{q}\right)^n\left[\sum_{l=0}^{n}\binom{n}{l}\frac{1}{p_1^l q_1^{n-l}}a^{pl}b^{q(n-l)}-a^{\frac{p}{p_1}n}b^{\frac{q}{q_1}n}\right]$$

where $\frac{1}{p}+\frac{1}{q}=1$ and $\frac{1}{p_1}+\frac{1}{q_1}=1$.

The following results, Theorem 2, 3, 4 and 5 are new refinements of Theorem 2 and Theorem 3 from [20] when $n\in\mathbb{N}$. Moreover, these inequalities can be obtained also for the modulus of the product of the functions f and g, see Corollary 2 (b) and several applications to special functions are presented in Corollary 3 and Corollary 4.

Theorem 2. Let $f(z)=\sum_{n=0}^{\infty}p_n z^n$ and $g(z)=\sum_{n=0}^{\infty}q_n z^n$ be two power series with real coefficients and convergent on the open disk $D(0,R), R>0$.

If the numbers $p,q,p_{1,},q_1$ have the properties $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{p_1}+\frac{1}{q_1}=1$, $1<p_1<p$, and if $a,b\in\mathbb{C},\ a,b\neq 0$ so that $|a|^{pn},|a|^{qn},|b|^{pn},|b|^{qn}\in D(0.R)$ then the following inequality takes place:

$$\left(\frac{p_1}{p}\right)^n\left[\sum_{l=0}^{n}\binom{n}{l}\frac{1}{p_1^l q_1^{n-l}}f_A\left(|a|^{ql}|b|^{p(n-l)}\right)g_A\left(|a|^{q(n-l)}|b|^{pl}\right)-f_A\left(|a|^{\frac{q}{p_1}n}|b|^{\frac{p}{q_1}n}\right)g_A\left(|a|^{\frac{q}{q_1}n}|b|^{\frac{p}{p_1}n}\right)\right]$$

$$<\sum_{l=0}^{n}\binom{n}{l}\frac{1}{p^l q^{n-l}}f_A\left(|a|^{ql}|b|^{p(n-l)}\right)g_A\left(|a|^{q(n-l)}|b|^{pl}\right)-f_A\left(|a|^{n(q-1)}|b|^{n(p-1)}\right)g_A\left(|a|^n|b|^n\right)<$$

$$\left(\frac{q_1}{q}\right)^n\left[\sum_{l=0}^{n}\binom{n}{l}\frac{1}{p_1^l q_1^{n-l}}f_A\left(|a|^{ql}|b|^{p(n-l)}\right)g_A\left(|a|^{q(n-l)}|b|^{pl}\right)-f_A\left(|a|^{\frac{q}{p_1}n}|b|^{\frac{p}{q_1}n}\right)g_A\left(|a|^{\frac{q}{q_1}n}|b|^{\frac{p}{p_1}n}\right)\right]$$

.

Proof. We start this proof taking into account that the hypothesis

$$|a|^{pn},|a|^{qn},|b|^{pn},|b|^{qn}\in D(0,R)$$

implies by calculus the following inclusions:

$$|a|^{ql}|b|^{p(n-l)},|a|^{q(n-l)}|b|^{pl},|a|^{\frac{q}{p_1}n}|b|^{\frac{p}{q_1}n},|a|^{\frac{q}{q_1}n}|b|^{\frac{p}{p_1}n},|a|^{n(q-1)}|b|^{n(p-1)},|a|^n|b|^n\in D(0,R), l=\overline{0,n}.$$

Thus, for example, $|a|^{\frac{q}{p_1}n}|b|^{\frac{p}{q_1}n}<R^{\frac{1}{p_1}}R^{\frac{1}{q_1}}=R$. We use the same method as in [20]. In this case, we consider $a=\frac{|b|^k}{|b|^j}$ and $b=\frac{|a|^k}{|a|^j}$, $j,k\in\{0,1,2,\dots,m\}$ in previous inequality and we have:

$$\left(\frac{p_1}{p}\right)^n\left[\sum_{l=0}^{n}\binom{n}{l}\frac{1}{p_1^l q_1^{n-l}}\frac{|b|^{kpl}}{|b|^{jpl}}\frac{|a|^{kq(n-l)}}{|a|^{jq(n-l)}}-\frac{|b|^{kn\frac{p}{p_1}}}{|b|^{jn\frac{p}{p_1}}}\frac{|a|^{kn\frac{q}{q_1}}}{|a|^{jn\frac{q}{q_1}}}\right]<$$

$$<\sum_{l=0}^{n}\binom{n}{l}\frac{1}{p^l q^{n-l}}\frac{|b|^{kpl}}{|b|^{jpl}}\frac{|a|^{kq(n-l)}}{|a|^{jq(n-l)}}-\frac{|b|^{kn}}{|b|^{jn}}\frac{|a|^{kn}}{|a|^{jn}}<$$

$$<\left(\frac{q_1}{q}\right)^n\left[\sum_{l=0}^{n}\binom{n}{l}\frac{1}{p_1^l q_1^{n-l}}\frac{|b|^{kpl}}{|b|^{jpl}}\frac{|a|^{kq(n-l)}}{|a|^{jq(n-l)}}-\frac{|b|^{kn\frac{p}{p_1}}}{|b|^{jn\frac{p}{p_1}}}\frac{|a|^{kn\frac{q}{q_1}}}{|a|^{jn\frac{q}{q_1}}}\right]$$

If we multiply last inequality by $|b|^{jpn}|a|^{jqn}$ then we get the following result:

$$\left(\frac{p_1}{p}\right)^n\left[\sum_{l=0}^{n}\binom{n}{l}\frac{1}{p_1^l q_1^{n-l}}|a|^{jql}|b|^{jp(n-l)}\,|a|^{kq(n-l)}|b|^{kpl}-|a|^{\frac{q}{p_1}jn}|b|^{\frac{p}{q_1}jn}|a|^{\frac{q}{q_1}kn}|b|^{\frac{p}{p_1}kn}\right]$$

$$<\sum_{l=0}^{n}\binom{n}{l}\frac{1}{p^l q^{n-l}}|a|^{jql}|b|^{jp(n-l)}|a|^{kq(n-l)}|b|^{kpl}-|a|^{jn(q-1)}|b|^{jn(p-1)}|a|^{kn}|b|^{kn}<$$

$$<\left(\frac{q_1}{q}\right)^n\left[\sum_{l=0}^{n}\binom{n}{l}\frac{1}{p_1^l q_1^{n-l}}|a|^{jql}|b|^{jp(n-l)}\,|a|^{kq(n-l)}|b|^{kpl}-|a|^{\frac{q}{p_1}jn}|b|^{\frac{p}{q_1}jn}|a|^{\frac{q}{q_1}kn}|b|^{\frac{p}{p_1}kn}\right]$$

In this point we multiply previous inequality by positive quantities $|p_j||q_k|$ and then summing over
j and k from 0 to m we obtain:

$$\left(\frac{p_1}{p}\right)^n\left[\sum_{l=0}^{n}\binom{n}{l}\frac{1}{p_1^l q_1^{n-l}}\sum_{j=0}^{m}|p_j||b|^{jp(n-l)}\,|a|^{jql}\sum_{k=0}^{m}|q_k||a|^{kq(n-l)}|b|^{kpl}-\right.$$
$$\left.-\sum_{j=0}^{m}|p_j||a|^{\frac{q}{p_1}jn}|b|^{\frac{p}{q_1}jn}\sum_{k=0}^{m}|q_k||a|^{\frac{q}{q_1}kn}|b|^{\frac{p}{p_1}kn}\right]<$$
$$<\sum_{l=0}^{n}\binom{n}{l}\frac{1}{p^l q^{n-l}}\sum_{j=0}^{m}|p_j||b|^{jp(n-l)}\,|a|^{jql}\sum_{k=0}^{m}|q_k||a|^{kq(n-l)}|b|^{kpl}$$
$$-\sum_{j=0}^{m}|p_j||a|^{jn(q-1)}|b|^{jn(p-1)}\sum_{k=0}^{m}|q_k||a|^{kn}|b|^{kn}<$$
$$<\left(\frac{q_1}{q}\right)^n\left[\sum_{l=0}^{n}\binom{n}{l}\frac{1}{p_1^l q_1^{n-l}}\sum_{j=0}^{m}|p_j||b|^{jp(n-l)}\,|a|^{jql}\sum_{k=0}^{m}|q_k||a|^{kq(n-l)}|b|^{kpl}-\right.$$
$$\left.-\sum_{j=0}^{m}|p_j||a|^{\frac{q}{p_1}jn}|b|^{\frac{p}{q_1}jn}\sum_{k=0}^{m}|q_k||a|^{\frac{q}{q_1}kn}|b|^{\frac{p}{p_1}kn}\right]$$

Taking above the limit when $m \to \infty$ we finf the desired inequality, because all the series whose partial sums are involved are convergent on the disk $D(0, R)$.

Theorem 3. Let $f(z) = \sum_{n=0}^{\infty} p_n z^n$ and $g(z) = \sum_{n=0}^{\infty} q_n z^n$ be two power series with real coefficients and convergent on the open disk $D(0, R), R > 0$.

If the numbers $p, q, p_{1,}, q_1$ are as in Theorem 2 and if $a, b \in \mathbb{C},\ a, b \neq 0$ with $|a|^{qn}, |a|^{\frac{2}{q}pn}, |b|^{pn}, |b|^{\frac{2}{p}qn} \in D(0.R)$ then one has the inequality:

$$\left(\frac{p_1}{p}\right)^n \left[\sum_{l=0}^{n} \binom{n}{l} \frac{1}{p_1^l q_1^{n-l}} f_A\left(|a|^{q(n-l)}|b|^{pl}\right) g_A\left(|a|^{\frac{2}{q}pl}|b|^{\frac{2}{p}q(n-l)}\right) - f_A\left(|a|^{\frac{q}{q_1}n}|b|^{\frac{p}{p_1}n}\right) g_A\left(|a|^{\frac{2\,p}{qp_1}n}|b|^{\frac{2\,q}{pq_1}n}\right)\right]$$

$$< \sum_{l=0}^{n} \binom{n}{l} \frac{1}{p^l q^{n-l}} f_A\left(|a|^{q(n-l)}|b|^{pl}\right) g_A\left(|a|^{\frac{2}{q}pl}|b|^{\frac{2}{p}q(n-l)}\right) - f_A(|a|^n|b|^n) g_A\left(|a|^{\frac{2}{q}n}|b|^{\frac{2}{p}n}\right) <$$

$$\left(\frac{q_1}{q}\right)^n \left[\sum_{l=0}^{n} \binom{n}{l} \frac{1}{p_1^l q_1^{n-l}} f_A\left(|a|^{q(n-l)}|b|^{pl}\right) g_A\left(|a|^{\frac{2}{q}pl}|b|^{\frac{2}{p}q(n-l)}\right) - f_A\left(|a|^{\frac{q}{q_1}n}|b|^{\frac{p}{p_1}n}\right) g_A\left(|a|^{\frac{2\,p}{qp_1}n}|b|^{\frac{2\,q}{pq_1}n}\right)\right]$$

Proof. First we check that the corresponding products of $|a|$ and $|b|$ are in $D(0, R)$ using hypothesis and then by choosing $a = |a|^{\frac{2}{q}k}|b|^j$ and $b = |a|^j|b|^{\frac{2}{p}k}$ and using the same method as in [20] we will obtain the desired inequality.

Theorem 4. Let $f(z), g(z)$ and $p, q, p_{1,}, q_1$ be as in Theorem 2, $a, b \in \mathbb{C},\ a, b \neq 0$ with $|a|^{qn}, |a|^{\frac{2}{p}qn}, |b|^{pn}, |b|^{\frac{2}{q}pn} \in D(0.R)$. Then one has the inequality:

$$\left(\frac{p_1}{p}\right)^n \left[\sum_{l=0}^{n} \binom{n}{l} \frac{1}{p_1^l q_1^{n-l}} f_A\left(|a|^{ql}|b|^{p(n-l)}\right) g_A\left(|a|^{\frac{2}{p}q(n-l)}|b|^{\frac{2}{q}pl}\right) - f_A\left(|a|^{\frac{q}{p_1}n}|b|^{\frac{p}{q_1}n}\right) g_A\left(|a|^{\frac{2\,p}{qp_1}n}|b|^{\frac{2\,q}{pq_1}n}\right)\right]$$

$$< \sum_{l=0}^{n} \binom{n}{l} \frac{1}{p^l q^{n-l}} f_A\left(|a|^{ql}|b|^{p(n-l)}\right) g_A\left(|a|^{\frac{2}{p}q(n-l)}|b|^{\frac{2}{q}pl}\right) - f_A\left(|a|^{n(q-1)}|b|^{n(p-1)}\right) g_A\left(|a|^{\frac{2}{p}n}|b|^{\frac{2}{q}n}\right) <$$

$$\left(\frac{q_1}{q}\right)^n \left[\sum_{l=0}^{n} \binom{n}{l} \frac{1}{p_1^l q_1^{n-l}} f_A\left(|a|^{ql}|b|^{p(n-l)}\right) g_A\left(|a|^{\frac{2}{p}q(n-l)}|b|^{\frac{2}{q}pl}\right) - f_A\left(|a|^{\frac{q}{p_1}n}|b|^{\frac{p}{q_1}n}\right) g_A\left(|a|^{\frac{2p}{qp_1}n}|b|^{\frac{2q}{pq_1}n}\right)\right]$$

Proof. We also check that the corresponding products of $|a|$ and $|b|$ are in $D(0,R)$ using hypothesis.

This time we replace a by $\frac{|b|^{\frac{2}{q}k}}{|b|^j}$ and b by $\frac{|a|^{\frac{2}{p}k}}{|a|^j}$ in order to obtain the inequality of the theorem.

Theorem 5. Let $f(z), g(z)$ and p, q, p_1, q_1 be as in Theorem 2, $a, b \in \mathbb{C}$, $a, b \neq 0$ with $|a|^{2n}, |b|^{qn}, |b|^{pn} \in D(0.R)$. Then the following inequality takes place:

$$\left(\frac{p_1}{p}\right)^n \left[\sum_{l=0}^{n} \binom{n}{l} \frac{1}{p_1^l q_1^{n-l}} f_A\left(|a|^{2l}|b|^{q(n-l)}\right) g_A\left(|a|^{2(n-l)}|b|^{pl}\right) - f_A\left(|a|^{\frac{2}{p_1}n}|b|^{\frac{q}{q_1}n}\right) g_A\left(|a|^{\frac{2}{q_1}n}|b|^{\frac{p}{p_1}n}\right)\right]$$

$$< \sum_{l=0}^{n} \binom{n}{l} \frac{1}{p^l q^{n-l}} f_A\left(|a|^{2l}|b|^{q(n-l)}\right) g_A\left(|a|^{2(n-l)}|b|^{pl}\right) - f_A\left(|a|^{\frac{2}{p}n}|b|^{n}\right) g_A\left(|a|^{\frac{2}{q}n}|b|^{n}\right) <$$

$$\left(\frac{q_1}{q}\right)^n \left[\sum_{l=0}^{n} \binom{n}{l} \frac{1}{p_1^l q_1^{n-l}} f_A\left(|a|^{2l}|b|^{q(n-l)}\right) g_A\left(|a|^{2(n-l)}|b|^{pl}\right) - f_A\left(|a|^{\frac{2}{p_1}n}|b|^{\frac{q}{q_1}n}\right) g_A\left(|a|^{\frac{2}{q_1}n}|b|^{\frac{p}{p_1}n}\right)\right]$$

Proof. It is easily to check that the corresponding product of $|a|$ and $|b|$ are in $D(0,R)$ using hypothesis. Then we choose $a = |a|^{\frac{2}{p}j}|b|^k$ and $b = |a|^{\frac{2}{q}k}|b|^j$ and repeat the same method as above.

Corollary 1. Let $f(z)$, $g(z), a, b$ and p, q, p_1, q_1 be as in Theorem 2. If $1 < p_1 < p$ and we take $n = 1$ in Theorem 2 then we have:

$$\frac{p_1}{p}\left[\frac{1}{q_1} f_A(|b|^p) g_A(|a|^q) + \frac{1}{p_1} f_A(|a|^q) g_A(|b|^p) - f_A\left(|a|^{\frac{q}{p_1}}|b|^{\frac{p}{q_1}}\right) g_A\left(|a|^{\frac{q}{q_1}}|b|^{\frac{p}{p_1}}\right)\right] <$$

$$< \frac{1}{q} f_A(|b|^p) g_A(|a|^q) + \frac{1}{p} f_A(|a|^q) g_A(|b|^p) - f_A(|a|^{q-1}|b|^{p-1}) g_A(|a||b|) <$$

$$< \frac{q_1}{q}\left[\frac{1}{q_1} f_A(|b|^p) g_A(|a|^q) + \frac{1}{p_1} f_A(|a|^q) g_A(|b|^p) - f_A\left(|a|^{\frac{q}{p_1}}|b|^{\frac{p}{q_1}}\right) g_A\left(|a|^{\frac{q}{q_1}}|b|^{\frac{p}{p_1}}\right)\right].$$

Corollary 2. (a) If we take $f(z) = g(z)$ in Corollary 1, then we have:

$$\frac{p_1}{p}\left[f_A(|b|^p)f_A(|a|^q) - f_A\left(|a|^{\frac{q}{p_1}}|b|^{\frac{p}{q_1}}\right)f_A\left(|a|^{\frac{q}{q_1}}|b|^{\frac{p}{p_1}}\right)\right] <$$

$$< f_A(|b|^p)f_A(|a|^q) - f_A(|a|^{q-1}|b|^{p-1})f_A(|a||b|) <$$

$$< \frac{q_1}{q}\left[f_A(|b|^p)f_A(|a|^q) - f_A\left(|a|^{\frac{q}{p_1}}|b|^{\frac{p}{q_1}}\right)f_A\left(|a|^{\frac{q}{q_1}}|b|^{\frac{p}{p_1}}\right)\right].$$

(b) Moreover, we give below the following form of the left side of the inequality from Theorem 2 and in this form appears the functions $f(z)$ and $g(z)$,

$$\left|f\left(a^{n(q-1)}b^{n(p-1)}\right)g(a^n b^n)\right| < \left(\frac{p_1}{p}\right)^n f_A\left(|a|^{\frac{q}{p_1}n}|b|^{\frac{p}{q_1}n}\right)g_A\left(|a|^{\frac{q}{q_1}n}|b|^{\frac{p}{p_1}n}\right) +$$

$$+\sum_{l=0}^{n}\binom{n}{l}\left[\frac{1}{p^l q^{n-l}} - \left(\frac{p_1}{p}\right)^n \frac{1}{p_1^l q_1^{n-l}}\right] f_A\left(|a|^{ql}|b|^{p(n-l)}\right)g_A\left(|a|^{q(n-l)}|b|^{pl}\right)$$

Similar inequalities can be given for the left side of the inequalities from Theorems 3-5. Now, taking into account that the functions $\exp(z), z \in \mathbb{C}, \frac{1}{1-z}, z \in D(0,1), \ln\left(\frac{1}{1-z}\right), z \in$ $D0,1,$ $\sinh z, z \in \mathbb{C}$ are power series with real coefficients and convergent on the open disk $D0,1$ we can rewrite the inequalities from Theorem 2, 3, 4, 5 for this functions.

Corollary 3. (a) We consider the function $f(z) = \sin(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}, z \in \mathbb{C}$ and we see that

$$f_A(z) = \sin h(z), z \in \mathbb{C}.$$

Under condition of Corollary 2 (a), the inequality becomes:

$$\frac{p_1}{p}\left[\sinh(|b|^p)\sinh(|a|^q) - \sinh\left(|a|^{\frac{q}{p_1}}|b|^{\frac{p}{q_1}}\right)\sinh\left(|a|^{\frac{q}{q_1}}|b|^{\frac{p}{p_1}}\right)\right] <$$

$$< \sinh(|b|^p)\sinh(|a|^q) - \sinh(|a|^{q-1}|b|^{p-1})\sinh(|ab|) <$$

$$< \frac{q_1}{q}\left[\sinh(|b|^p)\sinh(|a|^q) - \sinh\left(|a|^{\frac{q}{p_1}}|b|^{\frac{p}{q_1}}\right)\sinh\left(|a|^{\frac{q}{q_1}}|b|^{\frac{p}{p_1}}\right)\right].$$

(b) We take into account the function $f(z) = \exp(z) = \sum_{n=0}^{\infty}\frac{1}{n!}z^n = f_A(z), z \in \mathbb{C}$ and under condition of Corollary 2 (a), the inequality becomes:

$$\frac{p_1}{p}\left[\exp(|b|^p + |a|^q) - \exp\left(|a|^{\frac{q}{p_1}}|b|^{\frac{p}{q_1}} + |a|^{\frac{q}{q_1}}|b|^{\frac{p}{p_1}}\right)\right] <$$

$$< \exp(|b|^p + |a|^q) - \exp(|a|^{q-1}|b|^{p-1}|ab|) <$$

$$< \frac{q_1}{q}\left[\exp(|b|^p + |a|^q) - \exp\left(|a|^{\frac{q}{p_1}}|b|^{\frac{p}{q_1}} + |a|^{\frac{q}{q_1}}|b|^{\frac{p}{p_1}}\right)\right].$$

(c) Now, we consider the function, $f(z) = \frac{1}{1-z} = f_A(z),\ z \in D(0,1)$ and $a,\ b$ are complex numbers as in Theorem 2, then we get:

$$\frac{p_1}{p}\left[\frac{1}{1-|b|^p}\frac{1}{1-|a|^q} - \frac{1}{1-|a|^{\frac{q}{p_1}}|b|^{\frac{p}{q_1}}}\frac{1}{1-|a|^{\frac{q}{q_1}}|b|^{\frac{p}{p_1}}}\right] <$$

$$< \frac{1}{1-|b|^p}\frac{1}{1-|a|^q} - \frac{1}{1-|a|^{q-1}|b|^{p-1}}\frac{1}{1-|ab|} <$$

$$< \frac{q_1}{q}\left[\frac{1}{1-|b|^p}\frac{1}{1-|a|^q} - \frac{1}{1-|a|^{\frac{q}{p_1}}|b|^{\frac{p}{q_1}}}\frac{1}{1-|a|^{\frac{q}{q_1}}|b|^{\frac{p}{p_1}}}.\right]$$

Similar results can be obtained for the function $\cosh(x)$ as well.

Next we give an inequality as an application to special functions, such as polylogarithm, hypergeometric, Bessel and modified Bessel functions for the first kind, taking into account that $Li_n(z)$, ${}_2F_1(a,b,c,z), J_a(z)$ and $I_a(z)$ are power series with real coefficients and convergent on the open disk $D(0,1)$. For that it is necessary to recall the definition of polylogarithm function, $Li_n(z)$:

$$Li_n(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^n},$$

as a power series which converges absolutely for all complex values of the order n and z when $|z| < 1$.

Corollary 4. If $Li_n(z)$ is the polylogarithm function, then we have,

$$\frac{p_1}{p}\left[Li_n(|b|^p)Li_n(|a|^q) - Li_n\left(|a|^{\frac{q}{p_1}}|b|^{\frac{p}{q_1}}\right)Li_n\left(|a|^{\frac{q}{q_1}}|b|^{\frac{p}{p_1}}\right)\right] <$$

$$< Li_n(|b|^p)Li_n(|a|^q) - Li_n(|a|^{q-1}|b|^{p-1})Li_n(|a||b|) <$$

$$< \frac{q_1}{q}\left[Li_n(|b|^p)Li_n(|a|^q) - Li_n\left(|a|^{\frac{q}{p_1}}|b|^{\frac{p}{q_1}}\right)Li_n\left(|a|^{\frac{q}{q_1}}|b|^{\frac{p}{p_1}}\right)\right],$$

For any $a, b \in \mathbb{C}, a, b\ \neq 0$ under conditions of Corollary 2 (a) when $D(0,R)$ is $D(0,1)$.

3. NORM INEQUALITIES

The following result is a refinement of Theorem 5.1 from [1].

Theorem 6. Let ν, τ be two real numbers with $0 < \nu < \tau < 1$. If $A, B, X \in M_m$ with A and B positive semidefinite then the following inequality hold:

$$\left(\frac{\nu}{\tau}\right)^{2n}\left[||\sum_{k=0}^{n}\binom{n}{k}\ \tau^{n-k}(1-\ \tau)^k A^{n-k}XB^k\ ||_2^2 - ||A^{n\tau}XB^{n(1-\tau)}||_2^2\right] <$$

$$< ||\sum_{k=0}^{n}\binom{n}{k}\ \nu^{n-k}(1-\ \nu)^k A^{n-k}XB^k\ ||_2^2 - \left|\left|A^{n\nu}XB^{n(1-\nu)}\right|\right|_2^2 <$$

$$< \left(\frac{1-\nu}{1-\tau}\right)^{2n}\left[||\sum_{k=0}^{n}\binom{n}{k}\ \tau^{n-k}(1-\ \tau)^k A^{n-k}XB^k\ ||_2^2 - ||A^{n\tau}XB^{n(1-\tau)}||_2^2\right].$$

Proof. We write, as in [1], $A = Udiag(\lambda_1, \dots, \lambda_m)U^*$ and $B = Vdiag(\mu_1, \dots, \mu_m)V^*$ where U and V are unitary matrices and nonnegative λ_i, μ_i. If we set $Y = U^*XV = (y_{ij})$ then we get

$$\sum_{k=0}^{n}\binom{n}{k}\ \tau^{n-k}(1-\ \tau)^k A^{n-k}XB^k = U\left[\sum_{k=0}^{n}\binom{n}{k}\ \tau^{n-k}(1-\ \tau)^k \lambda_i^{n-k}\mu_j^k\right]y_{ij}V^*$$

and $A^{n\nu}XB^{n(1-\nu)} = U\left(\lambda_i^{n\nu}\mu_j^{n(1-\nu)}y_{ij}\right)V^*$. Taking now into account inequality from Theorem 1 for $\lambda = 2n \in \mathbb{N}^*$ we find that:

$$\left(\frac{\nu}{\tau}\right)^{2n}\left[||\sum_{k=0}^{n}\binom{n}{k}\ \tau^{n-k}(1-\ \tau)^k A^{n-k}XB^k\ ||_2^2 - ||A^{n\tau}XB^{n(1-\tau)}||_2^2\right] =$$

$$= \left(\frac{\nu}{\tau}\right)^{2n}\left[\sum_{i,j=1}^{m}\left(\sum_{k=0}^{n}\binom{n}{k}\ \tau^{n-k}(1-\ \tau)^k \lambda_i^{n-k}\mu_j^k\right)^2 |y_{ij}|^2 - \sum_{i,j=1}^{m}\left(\lambda_i^{n\tau}\mu_j^{n(1-\tau)}\right)^2 |y_{ij}|^2\right] =$$

$$= \sum_{i,j=1}^{m}\left(\frac{\nu}{\tau}\right)^{2n}\left[(\tau\lambda_i + (1-\tau)\mu_j)^{2n} - \lambda_i^{2n\tau}{\mu_j}^{2n(1-\tau)}\right]|y_{ij}|^2 <$$

$$< \sum_{i,j=1}^{m}\left[(\nu\lambda_i + (1-\nu)\mu_j)^{2n} - \lambda_i^{2n\nu}{\mu_j}^{2n(1-\nu)}\right]|y_{ij}|^2 =$$

$$= ||\sum_{k=0}^{n}\binom{n}{k}\ \nu^{n-k}(1-\ \nu)^k A^{n-k}XB^k\ ||_2^2 - \left|\left|A^{n\nu}XB^{n(1-\nu)}\right|\right|_2^2.$$

4. SOME TRACE ANALOGUE INEQUALITIES FOR A REFINEMENT OF YOUNG'S INEQUALITY

Next result is an extension for operators as in Theorem 7, of a trace inequality given in [29].

Theorem 7. Let m, M be two real numbers with $1 < m < M$ and A, B be two positive operators in $B(H)$ with $Sp(A) \subset [m, M]$, $Sp(B) \subset [m, M]$ and $P, Q \in B_1(H)$ with $P, Q > 0$. Then for any $\lambda \in [0,1]$ the following inequality takes place:

$$r\left[tr(PA)tr(Q) - 2tr\left(PA^{\frac{1}{2}}\right)tr\left(QB^{\frac{1}{2}}\right) + tr(P)tr(QB)\right] + A_1(\lambda).$$

$$.\,[tr(Q)tr(Plog^2 A) - 2tr(Qlog\, B)tr(Plog\, A) + tr(Qlog^2 B)tr(P)] \leq$$

$$\leq \lambda tr(PA)tr(Q) + (1-\lambda)tr(QB)tr(P) - tr\left(QB^{1-\lambda}\right)tr\left(PA^{\lambda}\right) \leq$$

$$\leq (1-r)\left[tr(PA)tr(Q) - 2tr\left(PA^{\frac{1}{2}}\right)tr\left(QB^{\frac{1}{2}}\right) + tr(P)tr(QB)\right] + B_1(\lambda).$$

$$.\,[tr(Q)tr(Plog^2 A) - 2tr(Qlog\, B)tr(Plog\, A) + tr(Qlog^2 B)tr(P)],$$

where $r = min\{\lambda, 1-\lambda\}$, $A_1(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{r}{4}$ and $B_1(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{1-r}{4}$ as in Lemma 1.

Proof. We use the same method as in [12], [13]. We take into account the inequality from Lemma 1, which holds for any $a, b \geq m \geq 1$ and using the functional calculus for the operator A when $1 \leq m \leq b \leq M$ is fixed, we get

$$r\left(< Ax, x > -2\sqrt{b} < A^{\frac{1}{2}}x, x > +b < x, x >\right) + A_1(\lambda).$$

$$.\,(< log^2 Ax, x > -2log\, b < \log Ax, x > +log^2 b < x, x >) \leq$$

$$\leq \lambda < Ax, x > +(1-\lambda)b < x, x > -b^{1-\lambda} < A^{\lambda}x, x > \leq$$

$$\leq (1-r)\left(< Ax, x > -2\sqrt{b} < A^{\frac{1}{2}}x, x > +b < x, x >\right) + B_1(\lambda).$$

$$.\,(< log^2 Ax, x > -2log\, b < \log Ax, x > +log^2 b < x, x >),$$

for any $x \in H$, if we denote $A(\lambda)$ by $A_1(\lambda)$ and $B(\lambda)$ by $B_1(\lambda)$.

We fix $x \in H - \{0\}$ and then by the functional calculus for the operator B for previous inequality, we have,

$$r\left(< Ax, x > ||y||^2 - 2 < A^{\frac{1}{2}}x, x >< B^{\frac{1}{2}}x, x > +< By, y > ||x||^2\right) + A_1(\lambda).$$

$$.\,(||y||^2 < log^2 Ax, x > -2 < \log By, y >< \log Ax, x > +< log^2 By, y > ||x||^2) \leq$$

$$\leq \lambda < Ax,x > ||y||^2 + (1-\lambda) < By,y > ||x||^2 - < A^{\lambda}x,x >< B^{1-\lambda}y,y > \leq$$

$$\leq (1-r)\left(< Ax,x > ||y||^2 - 2 < A^{\frac{1}{2}}x,x >< B^{\frac{1}{2}}x,x > + < By,y > ||x||^2\right) + B_1(\lambda).$$

$$.(||y||^2 < log^2 Ax,x > -2 < \log By,y >< \log Ax,x > + < log^2 By,y > ||x||^2),$$

for any $x, y \in H$,

We put now, $x = P^{\frac{1}{2}}e,\ \ y = Q^{\frac{1}{2}}f$ where $e, f \in H$ and by the above inequality we obtain,

$$r\left(< P^{\frac{1}{2}}AP^{\frac{1}{2}}e,e >< Qf,f > -2 < P^{\frac{1}{2}}A^{\frac{1}{2}}P^{\frac{1}{2}}e,e >< Q^{\frac{1}{2}}B^{\frac{1}{2}}Q^{\frac{1}{2}}f,f > + < Pe,e > < Q^{\frac{1}{2}}BQ^{\frac{1}{2}}f,f >\right)$$

$$+A_1(\lambda).(< Qf,f >< P^{\frac{1}{2}}log^2 AP^{\frac{1}{2}}e,e > -2 < Q^{\frac{1}{2}}\log BQ^{\frac{1}{2}}f,f >< P^{\frac{1}{2}}\log AP^{\frac{1}{2}}e,e > + < Q^{\frac{1}{2}}log^2 BQ^{\frac{1}{2}}f,f >< Pe,e >) \leq$$

$$\leq \lambda < P^{\frac{1}{2}}AP^{\frac{1}{2}}e,e >< Qf,f > +(1-\lambda) < Pe,e >< Q^{\frac{1}{2}}BQ^{\frac{1}{2}}f,f > - < Q^{\frac{1}{2}}B^{1-\lambda}Q^{\frac{1}{2}}f,f > < P^{\frac{1}{2}}A^{\lambda}P^{\frac{1}{2}}e,e > \leq$$

$$\leq (1-r)\left(< P^{\frac{1}{2}}AP^{\frac{1}{2}}e,e >< Qf,f > -2 < P^{\frac{1}{2}}A^{\frac{1}{2}}P^{\frac{1}{2}}e,e >< Q^{\frac{1}{2}}B^{\frac{1}{2}}Q^{\frac{1}{2}}f,f > + < Pe,e > < Q^{\frac{1}{2}}BQ^{\frac{1}{2}}f,f >\right)$$

$$+B_1(\lambda).\left(< Qf,f >< P^{\frac{1}{2}}log^2 AP^{\frac{1}{2}}e,e > -2 < Q^{\frac{1}{2}}\log BQ^{\frac{1}{2}}f,f >< P^{\frac{1}{2}}\log AP^{\frac{1}{2}}e,e > + < Q^{\frac{1}{2}}log^2 BQ^{\frac{1}{2}}f,f >< Pe,e >\right),$$

for any $e, f \in H$.

Let $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ br two orthonormal bases of H. We take in previous inequality $e = e_i,\ i \in I$ and $f = f_j,\ j \in J$ and then summing over $i \in I$ and $j \in J$, we get the following:

$$r\left(\sum_{i\in I} < P^{\frac{1}{2}}AP^{\frac{1}{2}}e_i,e_i > \sum_{j\in J} < Qf_j,f_j > -2\sum_{i\in I} < P^{\frac{1}{2}}A^{\frac{1}{2}}P^{\frac{1}{2}}e_i,e_i > \sum_{j\in J} < Q^{\frac{1}{2}}B^{\frac{1}{2}}Q^{\frac{1}{2}}f_j,f_j > + \sum_{i\in I} < Pe_i,e_i > \sum_{j\in J} < Q^{\frac{1}{2}}BQ^{\frac{1}{2}}f_j,f_j >\right) +$$

$$+A_1(\lambda)(\sum_{i\in I} < P^{\frac{1}{2}}log^2 AP^{\frac{1}{2}}e_i,e_i > \sum_{j\in J} < Qf_j,f_j > -$$

$$-2\sum_{i\in I} < P^{\frac{1}{2}}\log A\, P^{\frac{1}{2}}e_i, e_i > \sum_{j\in J} < Q^{\frac{1}{2}}\log B\, Q^{\frac{1}{2}}f_j, f_j > +$$

$$+\sum_{j\in J} < Q^{\frac{1}{2}}\, log^2\, B\, Q^{\frac{1}{2}}f_j, f_j > \sum_{i\in I} < Pe_i, e_i >) \leq$$

$$\leq \lambda \sum_{i\in I} < P^{\frac{1}{2}}AP^{\frac{1}{2}}e_i, e_i > \sum_{j\in J} < Qf_{j,}f_j > + (1-\lambda)\sum_{i\in I} < Pe_i, e_i > \sum_{j\in J} < Q^{\frac{1}{2}}BQ^{\frac{1}{2}}f_j, f_j > -$$

$$-\sum_{i\in I} < P^{\frac{1}{2}}A^{\lambda}P^{\frac{1}{2}}e_i, e_i > \sum_{j\in J} < Q^{\frac{1}{2}}B^{1-\lambda}Q^{\frac{1}{2}}f_j, f_j > \leq$$

$$\leq (1-r)\left(\sum_{i\in I} < P^{\frac{1}{2}}AP^{\frac{1}{2}}e_i, e_i > \sum_{j\in J} < Qf_{j,}f_j > - 2\sum_{i\in I} < P^{\frac{1}{2}}A^{\frac{1}{2}}P^{\frac{1}{2}}e_i, e_i > \sum_{j\in J} < Q^{\frac{1}{2}}B^{\frac{1}{2}}Q^{\frac{1}{2}}f_j, f_j > + \sum_{i\in I} < Pe_i, e_i > \sum_{j\in J} < Q^{\frac{1}{2}}BQ^{\frac{1}{2}}f_j, f_j > \right) +$$

$$+B_1(\lambda)(\sum_{i\in I} < P^{\frac{1}{2}}log^2AP^{\frac{1}{2}}e_i, e_i > \sum_{j\in J} < Qf_{j,}f_j > -$$

$$-2\sum_{i\in I} < P^{\frac{1}{2}}\log A\, P^{\frac{1}{2}}e_i, e_i > \sum_{j\in J} < Q^{\frac{1}{2}}\log B\, Q^{\frac{1}{2}}f_j, f_j > +$$

$$+\sum_{j\in J} < Q^{\frac{1}{2}}\, log^2\, B\, Q^{\frac{1}{2}}f_j, f_j > \sum_{i\in I} < Pe_i, e_i >).$$

Using the properties of the trace we find the desired inequality.

Next we take instead of B, A and instead of Q, P and then with the same conditions as in Theorem 5, we have the following result:

Corollary 5. Let m, M be two real numbers with $1 < m < M$ and A be a positive operator in $B(H)$ with $Sp(A) \subset [m, M]$ and $P \in B_1(H)$ with $P > 0$. Then for any $\lambda \in [0,1]$ we obtain:

$$2r\left[\frac{tr(PA)}{tr(P)} - \left(\frac{tr(PA^{\frac{1}{2}})}{tr(P)}\right)^2\right] + 2A_1(\lambda)\left[\frac{tr(P\ log^2 A)}{tr(P)} - \left(\frac{tr(P\ log\ A)}{tr(P)}\right)^2\right] \leq$$

$$\leq \frac{tr(PA)}{tr(P)} - \frac{tr(PA^{\lambda})}{tr(P)}\frac{tr(PA^{1-\lambda})}{tr)P\_} \leq$$

$$\leq 2(1-r)\left[\frac{tr(PA)}{tr(P)}-\left(\frac{tr(PA^{\frac{1}{2}})}{tr(P)}\right)^2\right]+2B_1(\lambda)\left[\frac{tr(P\,log^2A)}{tr(P)}-\left(\frac{tr(P\,log\,A)}{tr(P)}\right)^2\right]$$

where $r=min\{\lambda,1-\lambda\}$, $A_1(\lambda)=\frac{\lambda(1-\lambda)}{2}-\frac{r}{4}$ and $B_1(\lambda)=\frac{\lambda(1-\lambda)}{2}-\frac{1-r}{4}$ as before.

Corollary 6. If P, Q are two positive invertible operators with $P,Q\in B_1(H)$ and $Sp\left(P^{-\frac{1}{2}}QP^{-\frac{1}{2}}\right)\subset[m,M,]$ where m, M are two real numbers with $1<m<M$ then we have:

$$2r\left[\frac{tr(Q)}{tr(P)}-\left(\frac{tr(P\#Q)}{tr(P)}\right)^2\right]+2A_1(\lambda)\left[\frac{tr(\mathcal{P}_{log^2}(QP))}{tr(P)}-\left(\frac{tr(S(P/Q))}{tr(P)}\right)^2\right]\leq$$

$$\leq\frac{tr(Q)}{tr(P)}-\frac{tr(P\#_{1-\lambda}Q)}{tr(P)}\frac{tr(P\#_{\lambda}Q)}{tr(P)}\leq$$

$$\leq 2(1-r)\;\left[\frac{tr(Q)}{tr(P)}-\left(\frac{tr(P\#Q)}{tr(P)}\right)^2\right]+2B_1(\lambda)\left[\frac{tr(\mathcal{P}_{log^2}(QP))}{tr(P)}-\left(\frac{tr(S(P/Q))}{tr(P)}\right)^2\right]$$

where $r=min\{\lambda,1-\lambda\}$, $A_1(\lambda)=\frac{\lambda(1-\lambda)}{2}-\frac{r}{4}$ and $B_1(\lambda)=\frac{\lambda(1-\lambda)}{2}-\frac{1-r}{4}$ as before.

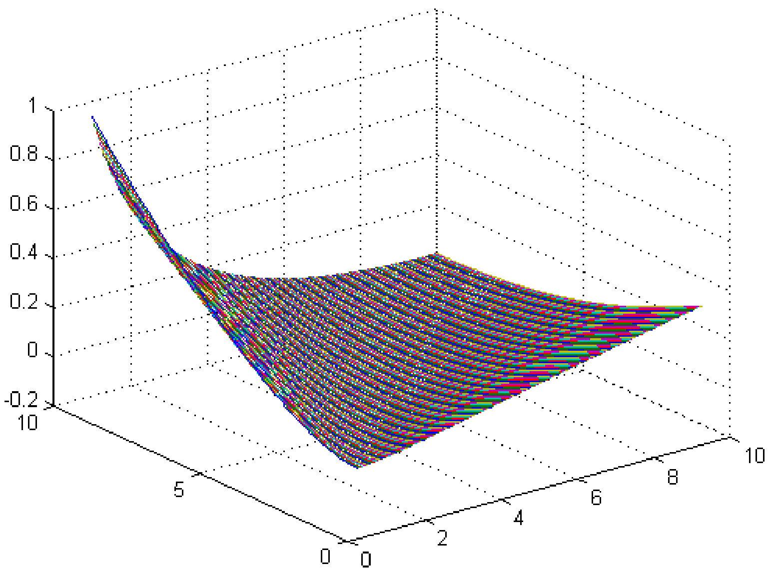

Figure 1. The function $h(a,b)$ defined on $[1,10]\times[1,10]$ when $\lambda=\frac{1}{6}$.

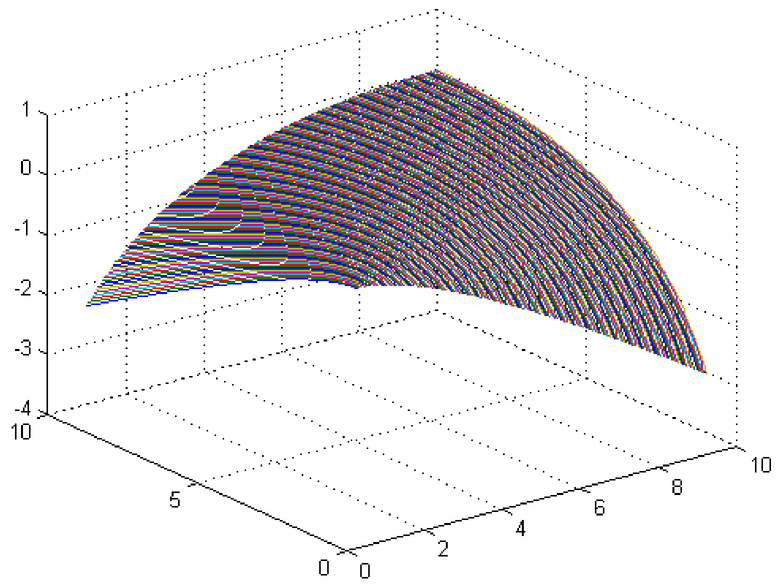

Figure 2. The function $k(a,b)$ defined on $[1,10] \times [1,10]$ when $\lambda = \frac{1}{6}$.

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